Of lightning rods, charged conductors, curvature, and things

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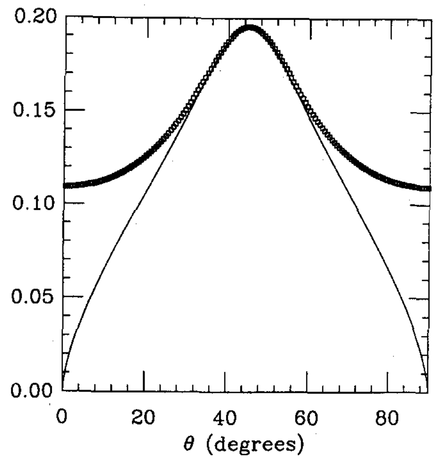

Fig. 2. Comparison of the fractional charge/length and the third root of the curvature for an infinite cylinder whose cross section is described by Eq. (13) with $d = 1$ and $e = 4$. The matrix dimension for the calculation of f is $n = 150$ and the $K^{1/3}$ function has been scaled to match the maximum of f.

Thus, as a function of θ at fixed d, $f \propto K^{1/3}$, which is reminiscent of the results of Liu.[4]

Such a relation between charge density and curvature cannot always be true.[5,6] One should see Refs. 5 and 6 for general arguments. Here, we end with a counterexample based on using for the cross section of the cylinder the generalized ellipse,

$$r(\theta) = d/(\cos^e \theta + d^e \sin^e \theta)^{1/e}. \tag{13}$$

For the specific choices of $d = 1$ and $e = 4$, a plot of both f and a scaled $K^{1/3}$ are shown in Fig. 2. There is reasonable agreement near the maximum, but K vanishes at $\theta = 0, \pi/2$, while f is everywhere finite. We found similar results for larger choices of d and e.

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[1] John B. Ross, "Plotting the charge distribution of a closed-loop conducting wire using a microcomputer," Am. J. Phys. **55**, 948–950 (1987).

[2] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1968), pp. 121–124.

[3] L. J. Adams and P. A. White, *Analytic Geometry and Calculus* (Oxford U. P., New York, 1961), p. 541 (Exercise 20).

[4] Kun-Mu Liu, "Relation between charge density and curvature of surface of charged conductor," Am. J. Phys. **55**, 849–852 (1987).

[5] M. Torres, J. M. Gonzalez, and G. Pastor, "Comment on 'Relation between charge density and curvature of surface of charged conductor,' by Kun-Mu Liu [Am. J. Phys. **55**, 849–852 (1987)]," Am. J. Phys. **57**, 1044–1046 (1989).

[6] Myriam Dubé, Mario Morel, N. Gauthier, and A. J. Barret, "Comment on 'Relation between charge density and curvature of surface of charged conductor,' by Kun-Mu Liu [Am. J. Phys. **55**, 849–852 (1987)]," Am. J. Phys. **57**, 1047–1048 (1989).

Of lightning rods, charged conductors, curvature, and things[a)]

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A recent paper by Liu[1] on the relationship between the charge density of an electrically charged conductor and its curvature has generated further investigations and discussions.[2,3] Whilst it is healthy that this paper has provoked the thoughts of others (including ourselves) there are some lessons for all of us.

Liu[1] demonstrates the relation between the charge density of a charged ellipsoid, paraboloid, and hyperboloid of revolution, and the fourth root of the Gaussian curvature of the conductor. He then suggests that perhaps this relationship between charge density and curvature holds for arbitrary smooth conductors. In fact, the relationship between the charge density of a charged ellipsoid and the Gaussian curvature has been "known" for some time. In his book, Kellogg[4] calculates the charge density of an isolated metallic ellipsoid, and in a footnote observes that this is proportional to the fourth root of the Gaussian curvature. McAllister[5] subsequently observed that Liu's result holds for an isolated conductor in the shape of any coordinate surface for general ellipsoidal or paraboloidal coordinates.

Heuristic arguments as to why the charge density is expected to be large where "the curvature" is large are to be found in nearly every text on electrostatics. However, we did find one text[6] that warns us that the relation is not exact, citing a paper by Price and Crowley.[7] In this paper the authors point out the ambiguity of the expression "the curvature" of the conductor, and then demonstrate that there is no relationship between the charge density and any function of the local curvature. They provide some interesting numerical examples. Thus, as is perhaps all too often the case, a greater awareness of the existing literature would have saved a number of people (including ourselves) some time.

In view of the above can there possibly be anything left to say on the subject? We would like to add just a little perspective on this topic. As was shown in Ref. 7 there can be no exact relationship between the charge density of a conductor and its local curvature. However, the title chosen by these authors belies the proven usefulness of such an approximate relationship. If we believe that the curvature of the conductor is a useful guide as to the approximate charge distribution (at least in many cases) then how do we argue this? We present our attempt. This involves expressing the charge density as an integral, along a field line, of the mean curvatures of a family of equipotentials surrounding the conductor.

We first give a brief introduction to the geometry of surfaces in Euclidean three-space. Let $\mathbf{x}(t)$ be a parametrized curve in R^3. Then at the point $\mathbf{p} = \mathbf{x}(t_0)$ the tangent vector is $\dot{\mathbf{x}}(t_0)$. So $\dot{\mathbf{x}}(t)$ is a vector field defined along the curve. Suppose that $\mathbf{v} = \dot{\mathbf{x}}(t_0)$ and that $\mathbf{X}$ is any vector field defined along the curve (at least). Then the derivative of $\mathbf{X}$ along $\mathbf{v}$ is defined by differentiating $\mathbf{X}$ along the curve $\mathbf{x}(t)$:

$$\nabla_{\mathbf{v}}\mathbf{X} = \frac{d}{dt}\mathbf{X}(\mathbf{x}(t))(t_0) . \tag{1}$$

It follows that $\nabla_{\mathbf{v}}\mathbf{X}$ does not depend on the details of the curve $\mathbf{x}(t)$, requiring only that $\dot{\mathbf{x}}(t_0) = \mathbf{v}$. Although we are here using the standard notation of differential geometry, another notation for $\nabla_{\mathbf{v}}$ is common in physics,

$$\nabla_{\mathbf{v}} \equiv (\mathbf{v}\cdot\nabla) . \tag{2}$$

As a special case, the derivative of the tangent field (or "velocity" field) $\dot{\mathbf{x}}(t)$ along the curve $\mathbf{x}(t)$ produces the "acceleration" $\ddot{\mathbf{x}}(t)$.

Suppose now that $\mathbf{x}(t)$ is a parametrized curve on the surface Σ. Then $\dot{\mathbf{x}}(t)$ is tangent to the surface. Conversely, if $\mathbf{v}$ is any vector at a point $\mathbf{p}$ on Σ then $\mathbf{v}$ is tangent to Σ only if $\mathbf{v} = \dot{\mathbf{x}}(t_0)$ for some curve $\mathbf{x}(t)$ on Σ with $\mathbf{x}(t_0) = \mathbf{p}$. The three-dimensional space of vectors at any point $\mathbf{p}$ of Σ may be decomposed into a two-dimensional subspace of vectors tangent to Σ, and a one-dimensional subspace of normals. A geodesic is a curve in the surface whose acceleration is everywhere normal to the surface. Thus a geodesic has zero tangential acceleration, and in this way generalizes the property a straight line in R^n has of being a parametrized curve with zero acceleration. If $\mathbf{v}$ is tangent to Σ at $\mathbf{p}$ then there is a unique geodesic $\mathbf{x}(t)$ such that $\mathbf{x}(0) = \mathbf{p}$ and $\dot{\mathbf{x}}(0) = \mathbf{v}$. This geodesic is said to "start" at $\mathbf{p}$ with initial "velocity" $\mathbf{v}$.

To study the "shape" of the surface Σ we introduce a unit normal field $\mathbf{n}$, defined at each point of Σ. Since its length is constant the derivative of $\mathbf{n}$ along any vector $\mathbf{v}$ tangent to Σ must be orthogonal to $\mathbf{n}$, that is, tangent to Σ. So if S is defined by

$$S\mathbf{v} = -\nabla_{\mathbf{v}}\mathbf{n} , \tag{3}$$

then S is a linear transformation on the two-dimensional space of vectors tangent to Σ. (Notice that S depends on a choice of orientation for Σ: We must choose one of the two possible unit normals at each point.) The operator S is called the *shape operator* (or *Weingarten map*).

To see how S measures the "shape" of Σ we take a geodesic $\mathbf{x}(t)$ starting at $\mathbf{p}$ with initial velocity $\mathbf{v}$. Since $\mathbf{x}$ is a curve in Σ then $\dot{\mathbf{x}}\cdot\mathbf{n} = 0$, and the derivative of this expression along $\mathbf{v}$ gives

$$\nabla_{\mathbf{v}}\dot{\mathbf{x}}\cdot\mathbf{n} = -\dot{\mathbf{x}}\cdot\nabla_{\mathbf{v}}\mathbf{n} .$$

But $\mathbf{v} = \dot{\mathbf{x}}(0)$ and so we can use (3) to write this as

$$\ddot{\mathbf{x}}(0)\cdot\mathbf{n} = \mathbf{v}\cdot S\mathbf{v} . \tag{4}$$

Since $\mathbf{x}(t)$ is a geodesic, its acceleration is everywhere normal, and so at any point S gives the acceleration of the surface geodesic for any given starting "velocity." The accelerations of the geodesics, which are "straight" with respect to the surface geometry, measure their departure from R^3 straightness. In this way, S measures the shape of the surface Σ. At any point $\mathbf{p}$ of Σ, the two eigenvectors of S define the principal curvature directions. The Gaussian curvature at $\mathbf{p}$ is the product of the eigenvalues, with the mean curvature function h being the average of them. Whilst it is not obvious from the above, the Gaussian curvature, unlike the mean curvature, is *intrinsic*; that is, it depends only on the geometry of the surface and not on the way that the surface is imbedded in R^3. Thus a cylinder, whose local geometry is the same as that of a flat piece of paper, has vanishing Gaussian curvature but nonvanishing mean curvature. With these conventions, an eigenvector of S has positive eigenvalue if the geodesic in its direction has R^3 acceleration toward the normal $\mathbf{n}$. Thus, for example, if we choose the outward normal for a sphere, the eigenvalues, and hence h, are negative. If r is the radius of the sphere, then both eigenvalues are $-1/r$, and so h is in this case a constant. A nice introduction to the geometry of surfaces is given by Thorpe.[8]

Consider an isolated smooth charged conductor. Then (if there are no isolated points at which the electric field vanishes) the conductor will be surrounded by a family of equipotential surfaces. Let Σ_c be the $\phi = c$ equipotential. Since each point in the exterior region lies on exactly one equipotential, a smooth function h may be defined to have the value at any point of the mean curvature of the equipotential at that point. Let $\mathbf{n}$ be a unit vector in the direction of the electric field $\mathbf{E}$, which we will write as $\mathbf{E} = E\mathbf{n}$. Thus E is the norm of the field, which will give the charge density at the surface. At any point outside the conductor we may pick an orthornormal basis $\{\mathbf{n},\mathbf{e}_1,\mathbf{e}_2\}$ where $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of the shape operator at that point. We may use this adapted orthonormal basis to write out the vacuum Maxwell equation $\nabla\cdot\mathbf{E} = 0$:

$$\nabla\cdot\mathbf{E} = (\nabla_{\mathbf{n}}\mathbf{E})\cdot\mathbf{n} + \sum_{i=1}^{2} (\nabla_{\mathbf{e}_i}\mathbf{E})\cdot\mathbf{e}_i .$$

Now

$$(\nabla_{\mathbf{n}}\mathbf{E})\cdot\mathbf{n} = (\nabla_{\mathbf{n}}E)\mathbf{n}\cdot\mathbf{n} + E(\nabla_{\mathbf{n}}\mathbf{n})\cdot\mathbf{n} = \nabla_{\mathbf{n}}E$$

since $\mathbf{n}$ is a unit vector.

Also

$$\begin{aligned}\nabla_{\mathbf{e}_i}\mathbf{E} &= (\nabla_{\mathbf{e}_i}E)\mathbf{n} + E(\nabla_{\mathbf{e}_i}\mathbf{n})\\ &= (\nabla_{\mathbf{e}_i}E)\mathbf{n} - ES\mathbf{e}_i\\ &= (\nabla_{\mathbf{e}_i}E)\mathbf{n} - \lambda_i E\mathbf{e}_i ,\end{aligned}$$

since $\mathbf{e}_i$ is an eigenvector of S with eigenvalue λ_i. Thus $(\nabla_{\mathbf{e}_i}\mathbf{E})\cdot\mathbf{e}_i = -\lambda_i E$ and so $\nabla\cdot\mathbf{E} = \nabla_{\mathbf{n}} E - 2Eh$, where the mean curvature function h is half the sum of the eigenvalues. So the vacuum equation gives

$$\nabla_{\mathbf{n}} E = 2Eh . \tag{5}$$

Establishing this equation is problem 1.11 in Jackson.[9] McAllister and Pederson[10] believe that this equation should be named after Green. Although this equation involves the mean curvature it has apparently[11] been known for a long time that this equation cannot be used to give an exact relation between the charge density and the conductor curvature.[12] The equation has been used as the basis for an approximation scheme for the electric field surrounding an isolated conductor.[13,14]

If the electrostatic potential ϕ is defined in the usual way by $\mathbf{E} = -\nabla\phi$ then we may coordinatize the exterior region with $\{\alpha, \beta, \phi\}$, where $\{\alpha, \beta\}$ are any (local) coordinates for the conductor. Using these coordinates the above becomes

$$\frac{\partial E}{\partial \phi} = -2h . \tag{6}$$

For any point P let C be the integral curve of $\mathbf{E}$ (the field line) starting at P and extending out to infinity. Then we can write

$$E(P) = \int_C 2h \, d\phi . \tag{7}$$

The above relates the surface charge density σ to the mean curvature, as $\sigma(P) = \epsilon_0 E(P)$: but not just the mean curvature of the conductor, rather the mean curvature of all the equipotentials at points pierced by the field line from P. The shape of these equipotentials is of course determined by the total charge distribution on the conductor, and not just on its local geometry. Equation (7) does *not* give a useful way of finding the charge density on the surface. To evaluate the right-hand side we would need to know explicitly the potential in order to find h, and if we know ϕ then the problem is solved. However, can we use (7) to suggest a relation between surface charge density and surface curvature? Suppose that P is a point on the conductor for which h is a maximum, and in addition let us *suppose* that the field line from P pierces each equipotential where its mean curvature is a maximum (we know of course that in general this latter supposition will not be true). Then, in this case, (7) shows that indeed the charge density will be a maximum at P. Whereas the futility of searching for an exact relation between the surface charge density and some function of the curvature has been amply demonstrated,[7] the above is sufficient to suggest that for a reasonably shaped lightning conductor the maximum charge density will occur near the point at which the mean curvature is a maximum.

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[a] This paper is a comment on Refs. 1–3.

[1] Kun-Mu Liu, "Relation between charge density and curvature of surface of charged conductor," Am. J. Phys. **55**, 849–852 (1987).

[2] M. Torres, J. M. González, and G. Pastor, "Comment on 'Relation between charge density and curvature of surface of charged conductor,' by Kun-Mu Liu [Am. J. Phys. **55**, 849–852 (1987)," Am. J. Phys. **57**, 1044–1046 (1989).

[3] Myriam Dubé, Mario Morel, N. Gauthier, and A. J. Barrett, "Comment on 'Relation between charge density and curvature of surface of charged conductor,' by Kun-Mu Liu [Am. J. Phys. **55**, 849–852 (1987)," Am. J. Phys. **57**, 1047–1048 (1989).

[4] O. M. Kellogg, *Foundations of Potential Theory* (Dover, New York, 1953) (originally published 1929), p. 191.

[5] I. W. McAllister, "Conductor curvature and surface charge density," J. Phys. D: Appl. Phys. **23**, 359–362 (1990).

[6] R. Wolfson and J. M. Pasachoff, *Physics* (Little, Brown, Boston, 1987), p. 603.

[7] R. H. Price and R. J. Crowley, "The lightning-rod fallacy," Am. J. Phys. **53**, 843–848 (1985).

[8] J. A. Thorpe, *Elementary Topics in Differential Geometry* (Springer, New York, 1979).

[9] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., p. 51.

[10] I. W. McAllister and A. Pederson, "Green's differential equation and electrostatic fields," J. Phys. D: Appl. Phys. **21**, 1823–1825 (1988).

[11] R. Cade, "The charge density near a sharp point on a conductor," Cambridge Philos. **53**, 870–877 (1957).

[12] E. Durand, *Électrostatique et Magnétostatique* (Paris, 1953).

[13] Luo Enze, "The distribution function of surface charge density with respect to surface curvature," J. Phys. D: Appl. Phys. **19**, 1–6 (1986).

[14] Luo Enze, "The application of a surface charge density distribution function to the solution of boundary value problems," J. Phys. D: Appl. Phys. **20**, 1609–1615 (1987).

Remark on Rayleigh's dissipation function

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Rayleigh's dissipation function $\mathscr{F}$ defined by[1]

$$\mathscr{F} = \frac{1}{2} \sum_i (K_x V_{ix}^2 + K_y V_{iy}^2 + K_z V_{iz}^2) \tag{1}$$

is useful to describe the equations of motion of a system of N particles submitted to conservative plus dissipative forces (denoted by $\mathbf{F}_i'$) when the latter depend linearly on the velocities, that is,

$$\mathbf{F}_{ix}' = -K_x V_{ix}, \quad \mathbf{F}_{iy}' = -K_y V_{iy}, \quad \mathbf{F}_z' = -K_z V_{iz}, \tag{2}$$

where K_x, K_y, K_z are positive constants. One has