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Citation: American Journal of Physics **53**, 843 (1985); doi: 10.1119/1.14349
View online: https://doi.org/10.1119/1.14349
View Table of Contents: https://aapt.scitation.org/toc/ajp/53/9
Published by the American Association of Physics Teachers

The lightning-rod fallacy

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(Received 24 April 1984; accepted for publication 24 August 1984)

It is accepted generally that the electric field strength on the surface of an isolated charged conductor is greatest where the surface curvature is greatest. We show here that there is, in fact, no relationship between these two maxima and that, in general, they are located at different points on the surface. Two classes of analytic examples are offered: one using conformal mapping techniques and the other involving small perturbations of a conducting spherical surface.

It is amusing how pervasive a misconception can be, even in as cut and dried a subject as electrostatics. In this paper we confront the "common knowledge" that the electric field at the surface of an isolated conductor is greatest where the curvature is greatest. It is in fact *true* that when the curvature is singular the **E** field is also singular. The coronal discharge near sharp points is exploited in lightning rods and familiar in electrostatic demonstrations. Proofs that $|\mathbf{E}|$ becomes infinite at sharp outer edges and conical apices can be found in standard textbooks.[1]

"Proofs" of the general relationship of $|\mathbf{E}|$ and curvature are rather more vague when curvature is finite. We have surveyed widely used sophomore-level physics texts and have found that the typical approach to the relationship is based on a simple example: Two spheres of different radii are connected by a long fine wire. It is demonstrated that the E-field strength on each of the spheres is inversely proportional to the radius of that sphere. The generalization that

$$|\mathbf{E}| \propto 1/R, \tag{1}$$

where R is the "radius of curvature" of the surface, is then either implied or stated explicitly. In one (older) text this method is pushed further; a sphere of radius R is matched to a point on the conducting surface and it is "proved" that in terms of the potential V (relative to infinity) of the conductor and the "radius of curvature" R at the point

$$|\mathbf{E}| = V/R. \tag{2}$$

A more satisfactory, although heuristic, approach also appears in the texts. Equipotentials very near a nonspherical conducting surface are sketched; it is noted that the equipotentials tend to become spherical at large distances. As a consequence the equipotentials are most closely spaced, and hence the **E** strength is greatest, where the curvature is greatest (see Fig. 1).

In addition to surveying the texts we have harassed colleagues and students at several universities with the question as to where the **E** strength is greatest. The response showed that the lesson in the texts has been well learned.

All this is more than sufficient motivation for us to demonstrate in this article that *on the surface of a conductor there is no general relationship between the location of the maximum of curvature and the maximum of* $|\mathbf{E}|$. Our discussions with our colleagues has forewarned us that there is a strong tendency to try to save some relationship by modifying and specializing it. We will therefore show more specifically that $|\mathbf{E}|$ is not even a *local* maximum where curvature is a *local* maximum. (We will, in fact, give an example in which $|\mathbf{E}|$ is a maximum where curvature is a local minimum.) We will also show that no relationship between the locations of the maxima can be salvaged by requiring that the conductor be convex.

The nonexistence of a relationship between maximum $|\mathbf{E}|$ and the maximum curvature is rooted in the fact that they depend in entirely different ways on the shape of the surface. Curvature depends only locally on the shape of the surface. At any point of the surface the curvature is determined by the first two derivatives, at that point, of the function specifying the surface. The curvature at that point is independent of what the surface does at points a finite distance away. The E-field strength, on the other hand, is determined by a solution of Laplace's equation for the electrostatic potential Φ. The solution depends on the specification of boundary conditions, in this instance the value of the potential *everywhere* on a closed surface. Changing the shape of the surface in one region influences the value of the potential at distant locations.

We start our set of examples with a heuristic one, which emphasizes this local dependence of curvature and nonlocal dependence of $|\mathbf{E}|$ on surface shape. The **E** field inside a closed conducting container is zero; the **E** field inside an almost-closed conducting container is almost zero. The almost-closed container shown in Fig. 2 therefore has very weak fields in the hollow that would be enclosed by the container were it not for the narrow gap at the top. At the bottom of this hollow is a small hemispherical pimple. By making the radius of the pimple sufficiently small we can guarantee that the maximum surface curvature is on the

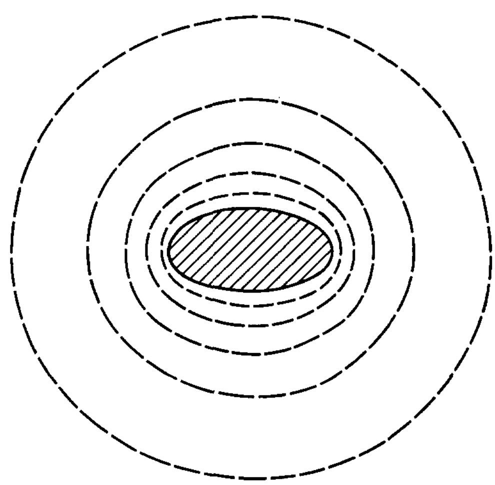

Fig. 1. Proof that $|\mathbf{E}|$ is always greatest where the curvature is greatest. The equipotentials (dashed lines) are most closely spaced, and hence the **E** strength is greatest, where the curvature is greatest.

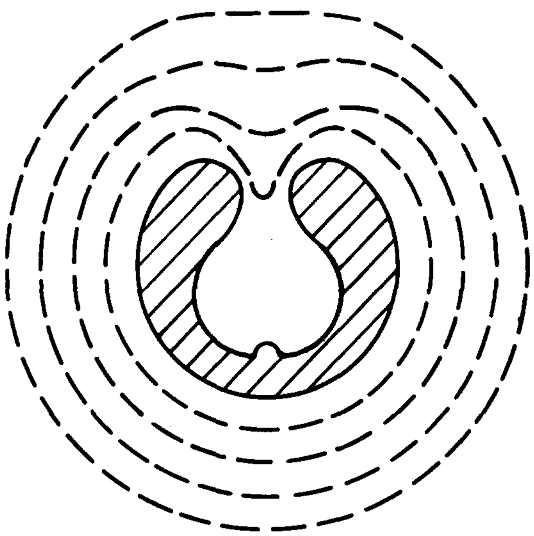

Fig. 2. Proof that $|\mathbf{E}|$ is not always greatest where the curvature is greatest. (Figure 2, like Fig. 1, represents a cross section of a solid conductor formed by rotating the figure about the vertical symmetry axis.) The curvature is greatest at the bottom of the hollow, on the small hemispherical pimple, but the **E** field there can be made arbitrarily small by narrowing the gap at the top.

pimple. But the **E** field will not or need not be a maximum on the pimple. No matter how sharp the pimple is (i.e., no matter how small its radius) we can make the $|\mathbf{E}|$-field strength on the pimple as weak as we like by making the almost-closed container more nearly closed (i.e., by narrowing the gap at the top).

If we are to exhibit quantitative examples we can no longer avoid a quantitative way of dealing with curvature. In many texts this is done with the "radius of curvature" at a point. This is presumably the radius of the sphere which approximates locally the surface at that point. Such an approximation-by-sphere is, in fact, completely explicit in one of the texts. Despite the texts, *in general a surface near a point cannot be approximated as a sphere and there is no unambiguous meaning to "radius of curvature."*

The correct quantitative description of curvature at a point P of the surface starts with a curve C, in the surface, through that point, as shown in Fig. 3. The curve is parametrized by its arc length s (from any starting point) and at any point we can compute the curve's unit tangent $\mathbf{t} = d\mathbf{r}/ds$. The outer unit normal $\mathbf{n}$ to the surface is defined along the curve, so its derivative $d\mathbf{n}/ds$ can be computed. It is fairly evident that $d\mathbf{n}/ds$ at P does not depend on any of the details of curve C, except its direction at P; any other curve with the same tanget $\mathbf{t}$ at P would lead to the same $d\mathbf{n}/ds$. At point P the scalar $\mathbf{t}\cdot d\mathbf{n}/ds$ then says something about the "bending" of the surface in direction $\mathbf{t}$.

The next truth is not self-evident, but is proved in any book on differential geometry. There are two orthogonal directions in the surface $\mathbf{t}_1$, $\mathbf{t}_2$, for which $\mathbf{t}\cdot d\mathbf{n}/ds$ is a maximum and minimum. These directions are called the principal directions; the associated values of $\mathbf{t}\cdot d\mathbf{n}/ds$ are called the principal curvatures κ_1 and κ_2:

$$\kappa_i = \mathbf{t}\cdot\frac{d\mathbf{n}}{ds}\bigg|_{\text{for } \mathbf{t}_i}, \quad i = 1,\ 2. \tag{3}$$

The reciprocals of the principal curvatures are called the principal radii of curvature[2] R_1, R_2:

$$R_i \equiv 1/\kappa_i. \tag{4}$$

For a nonprincipal direction $\mathbf{t}$, the value of $\mathbf{t}\cdot d\mathbf{n}/ds$ is calculated easily from the angle between $\mathbf{t}$ and the principal directions, and from the values of the principal curvatures, or radii. The complete characterization of the bending of a surface at a point then requires the specification of the principal directions and the associated principal curvatures, or principal radii. If the orientation of the bending is not of interest then only the principal curvatures, or radii, need be specified. The orientation-independent information about curvature is often not given as κ_1, κ_2 or R_1, R_2 but is packaged into two types of "average curvature": "Gaussian curvature,"

$$\kappa_G = \kappa_1 \kappa_2 \tag{5}$$

and "mean curvature,"

$$\kappa_M = \tfrac{1}{2}(\kappa_1 + \kappa_2). \tag{6}$$

It is of more than a little interest that the mean curvature *is* related to a property of the $|\mathbf{E}|$-field strength near a conducting surface. The fractional rate at which $|\mathbf{E}|$ decreases with distance away from the surface of a conductor is given by the mean curvature of the conductor[3]:

$$(1/|\mathbf{E}|)\mathbf{n}\cdot\nabla|\mathbf{E}| = -2\kappa_M.$$

The crucial point of the above discussion is that *two* numbers characterize the size of curvature, so that the assertion that "$|\mathbf{E}|$ is maximum where curvature is maximum" is not only wrong, it is also ill defined. It is always harder to disprove an ill-stated claim and we will need to show, with our examples, that $|\mathbf{E}|$ is neither an absolute, nor a local, maximum where κ_G *or* κ_M is maximum.

Our first class of detailed examples involves a long conductor of uniform cross section. If we idealize the conductor to be infinitely long, say in the z direction, then the electrostatic potential Φ is independent of z and Laplace's equation $\nabla^2\Phi = 0$ becomes a 2-dimensional problem, in the xy plane. We have exploited complex-variable techniques and conformal transformations to find closed-form solutions of Laplace's equation for a 3-parameter family of conductor cross sections, which more or less resemble Fig. 2 without the pimple. The details of this conformal transformation are tedious enough to be distracting here and are relegated to Appendix A. Here we concentrate on results, presented graphically, of examples for particular geometries.

The ambiguity of "radius of curvature" does not exist in this case. By symmetry the z direction is a principal direction. The corresponding principal radius of curvature is formally infinite. All the information about curvature is therefore contained in one number, the principal curvature κ in the transverse principal direction. (We could of course just as well use the principal radius of curvature, or the mean curvature.) The transverse principal curvature is determined from the curve giving the conductor cross section; it is simply the reciprocal of the radius of the osculat-

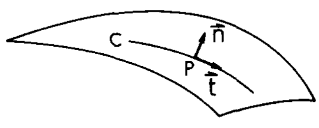

Fig. 3. Curve, unit tangent, and unit surface normal used in quantifying curvature.

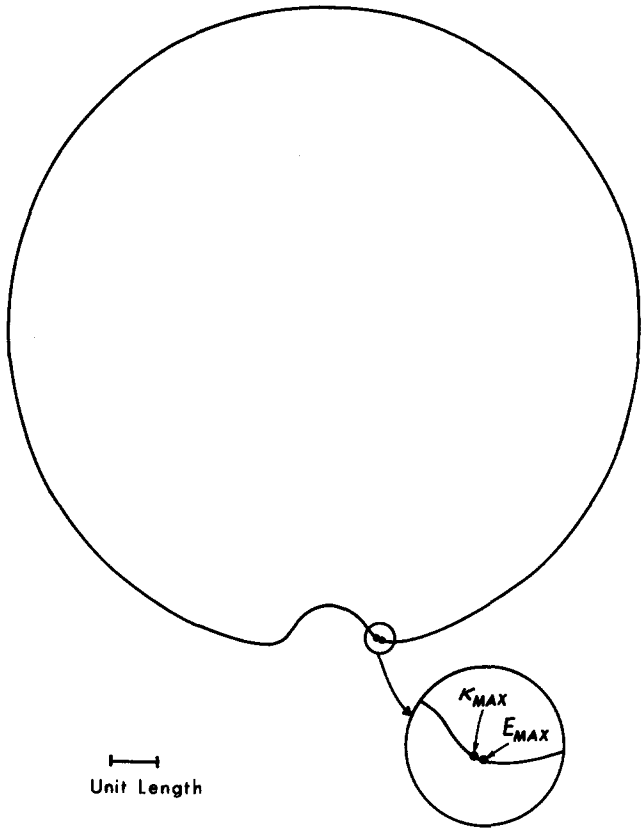

Fig. 4. Cross section of a long conductor. The enlarged detail shows the displacement of the maximum of the surface E strength from the maximum curvature.

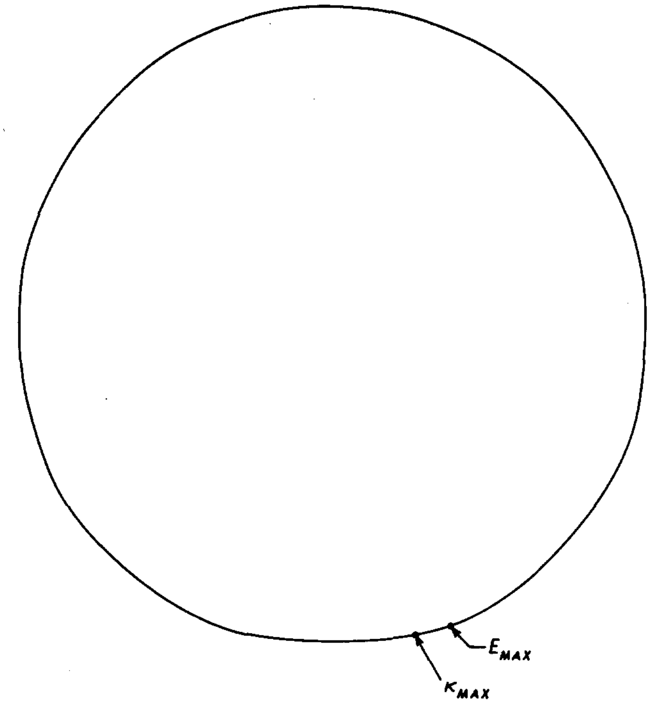

Fig. 6. Convex cross section showing the displacement of the maxima of κ and $|\mathbf{E}|$.

ing circle at any point of the curve and its calculation is straightforward. (See Appendix A for details.)

Figure 4 shows a conductor cross section for a particular choice of parameters. The greatest curvature occurs near the opening of the gap, at the point labeled κ_{max}. The arguments earlier detailed in this paper suggest that the magnitude of **E** is small in the "hollow" and the maximum of $|\mathbf{E}|$ should tend to shy away from the gap. The results presented in Fig. 4 show that this does in fact happen. The maximum of $|\mathbf{E}|$ is further from the gap than the maximum of κ.

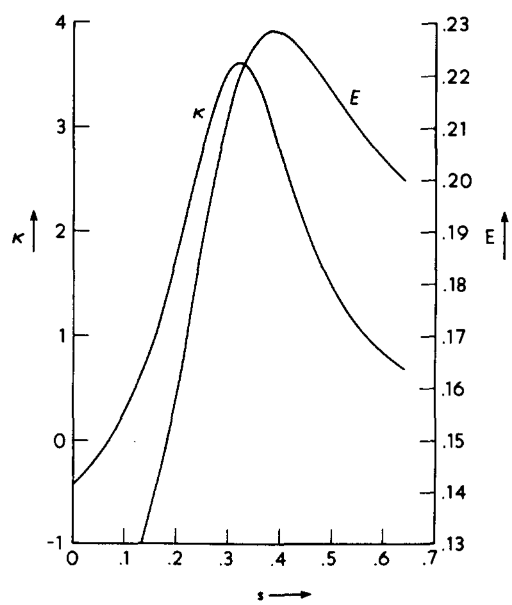

Fig. 5. The variations of curvature κ and E-field strength as functions of arc length s, along the cross section illustrated in Fig. 4.

The results are presented in a different way in Fig. 5; the values of $|\mathbf{E}|$ and κ are plotted as functions of s, the arc length along the cross section. The curvature has the physical dimension of inverse length so that numerical values for κ, as well as arc length, require a length scale for the conductor cross section. Values in Fig. 5 correspond to the "unit length" depicted in Fig. 4. The numerical values of $|\mathbf{E}|$ are in arbitrary units. (The $|\mathbf{E}|$ values *could* be related to the length scale and the charge per unit length on the conductor.)

In our discussions of this topic with others we have been accused of cheating by using a conductor which has a concave region. Rather than debate the fairness and relevance of a concave region we present a cross section which is everywhere convex. The cross section in Fig. 6 results from another choice of parameters in our conformal transformation. The convexity of this cross section is evident in Fig. 6 and is supported by the fact that the computed κ is everywhere positive. This example, like that of Figs. 4 and 5, show a displacement of the maxima of κ and $|\mathbf{E}|$.

Our second class of examples is axially symmetric first-order perturbations of a spherical surface of radius R. We take ϵ to be a very small dimensionless number and consider a surface defined by radius r as a function of polar angle θ, according to

$$r = R\left(1 + \epsilon \sum_{n=2}^{\infty} \beta_n P_n(\mu)\right), \quad \mu \equiv \cos\theta, \tag{7}$$

where P_n is the nth Legendre polynomial and the set of coefficients β_n determines the distortion of the sphere. The $n = 0$ and $n = 1$ terms in the series expansion for $r(\theta)$ correspond, respectively, to a change in the size and the origin of the perturbed sphere, and are of no interest here. It should be noted that for small ϵ all surfaces given by Eq. (7) are everywhere convex. It is very simple to solve, to first order in ϵ, for the surface E strength in terms of Q, the total charge on the perturbed sphere. The result (detailed in Appendix B) is

$$|\mathbf{E}| = QR^{-2}\left(1+\epsilon\sum_{n=2}^{\infty}\beta_n(n-1)P_n(\mu)\right). \tag{8}$$

By symmetry it is clear that the principal directions on the sphere are the ϕ direction (tangents to circles of constant r,θ) which we shall call $\mathbf{t}_1$, and the meridional direction (the "θ direction") $\mathbf{t}_2$. The corresponding principal curvatures to first order in ϵ are shown in Appendix B to be

$$\kappa_1 = R^{-1}\left[1-\epsilon\sum_{n=2}^{\infty}\beta_n\left(P_n-\mu\frac{d}{d\mu}P_n\right)\right], \tag{9a}$$

$$\kappa_2 = R^{-1}\left[1+\epsilon\sum_{n=2}^{\infty}\beta_n \times\left([n(n+1)-1]P_n-\mu\frac{d}{d\mu}P_n\right)\right]. \tag{9b}$$

Since we are working only to first order in ϵ the arithmetic mean of the principal curvatures (κ_M) and the geometric mean (corresponding to the square root of the Gaussian curvature) are the same:

$$\kappa = R^{-1}\left(1+\frac{1}{2}\epsilon\sum_{n=2}^{\infty}(n-1)(n+2)\beta_n P_n(\mu)\right). \tag{10}$$

Local extrema of κ and $|\mathbf{E}|$ will occur at the poles ($\theta=0$, π or $\mu=\pm 1$) and at the latitudes at which the derivatives, with respect to μ, of Eqs. (8) and (10) vanish. If $\beta_n=0$ for all but a single value of n, the local extrema of κ and $|\mathbf{E}|$ will coincide; in general, however, Eqs. (8) and (10) define different functions of μ and, contrary to conventional wisdom, only a fortuitous choice of the β_n will lead to coincident local maxima of $|\mathbf{E}|$ and κ away from the poles.[4] If the β_n are chosen such that the maximum of either $|\mathbf{E}|$ or κ does not occur at a pole then the absolute maximum of $|\mathbf{E}|$ and κ will not coincide.

As an interesting specific example we choose $\beta_3=6$, $\beta_5=-1$, and $\beta_n=0$ for other values of n. For this choice, depicted in Fig. 7, the sphere is perturbed with a fairly flat-topped convex hump around the $\theta=0$ axis. Because the hump is flat topped, $\theta=0$ is a local minimum of the curvature; curvature increases with θ out to $\theta\approx 14.53°$, where it is a local maximum; the absolute maximum of curvature is at $\theta\approx 111.81°$. The E-field strength is greatest at $\theta=0$, so that in this example not only is the absolute maximum of $\mathbf{E}$ far removed from the absolute maximum of κ, but it coincides with a local *minimum* of curvature!

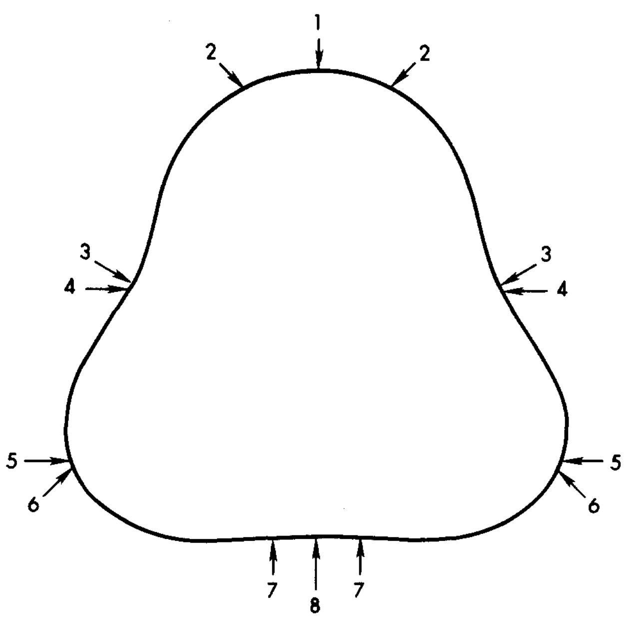

Fig. 7. A perturbed sphere with $\beta_3=6, \beta_5=-1$. Figure 7 is drawn with a large value ($\epsilon=0.05$) of the perturbation parameter to show clearly the nature of the distortion of the sphere. The numbered points correspond to the following extrema: (1) local minimum of κ, absolute maximum of E; (2) local maximum of κ; (3) local minimum of E; (4) absolute minimum of κ; (5) absolute maximum of κ; (6) local maximum of E; (7) local minimum of κ; and (8) local maximum of κ, absolute minimum of E.

ACKNOWLEDGMENTS

We would like to thank Dr. Karel Kuchar for an interesting argument, which he won, that led to the present article. We also thank Dr. Edward Kreusser for amusing suggestions in connection with Fig. 2. One of us (RHP) acknowledges gratefully the National Science Foundation for support under grant no. PHY81-06909.

APPENDIX A

Our first class of examples is based on a conformal transformation of a crescent in the complex $z=x+iy$ plane. The crescent is formed from two circular arcs, as pictured in Fig. A1, with both going through the points $y=0$, $x=\pm L$. (Here L serves as the unit length, as shown in Fig. 4.) The two parameters defining the crescent are most simply taken as l_1 and l_2, the y values of the centers of the smaller and larger circles. In terms of these we can write γ, the interior angle at the sharp corners of the crescent, as

$$\gamma = \tan^{-1}[L(l_2-l_1)/(L^2+l_1l_2)]. \tag{A1}$$

It is useful also to define the exterior angle at the corners

$$\alpha \equiv 2\pi-\gamma \tag{A2}$$

and an auxiliary angle

$$\phi_0 \equiv (\pi/\alpha)\tan^{-1}(L/l_2). \tag{A3}$$

In the formulas for γ and ϕ_0 the $\tan^{-1}$ function is to be taken in the range $(0, \pi/2)$.

For the conformal transformation we will need the complex expression

$$\xi \equiv (z+L)/(z-L). \tag{A4}$$

We define its phase to be zero on the real z axis for $x>L$ and to be continuous everywhere outside the crescent. We next define a conformal transformation from the complex z plane to the complex $W=U+iV$ plane by

$$W = (\xi^{\pi/\alpha}-e^{-2i\phi_0})/(\xi^{\pi/\alpha}-1). \tag{A5}$$

This transformation maps the exterior of the crescent in the z plane to the exterior of the unit circle in the W plane. Furthermore, it maps ∞ in the z plane to ∞ in the W plane. Finally we define the complex potential

$$F = \Phi + i\psi = \ln W. \tag{A6}$$

Since F, outside the crescent, is an analytic function of z, its real part $\Phi(x,y)$ is harmonic (i.e., $\nabla^2\Phi=0$). It is also seen easily that $\Phi=0$ on the crescent (since $|W|=1$ on the

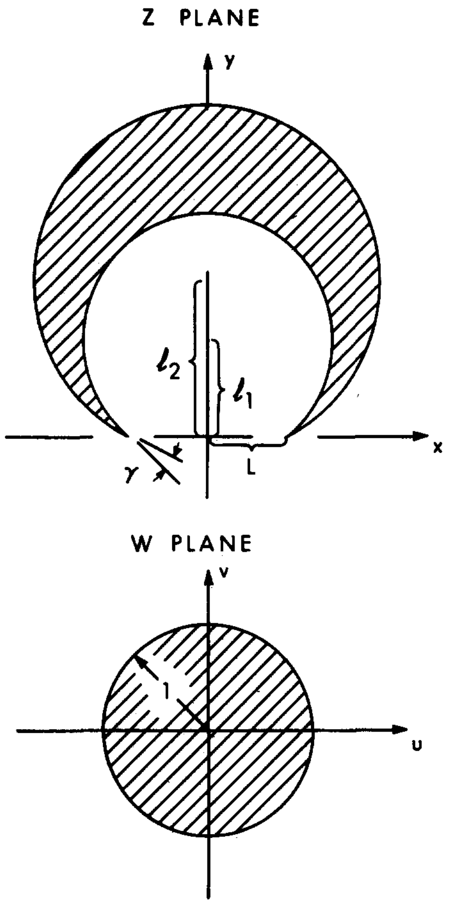

Fig. A1. Geometric parameters used in the conformal transformation, which maps the exterior of the crescent in the z plane to the exterior of the unit circle in the W plane.

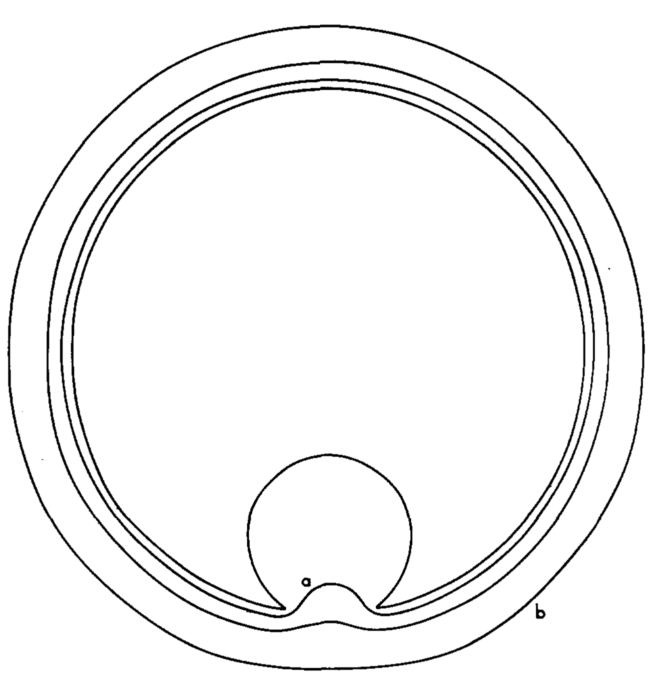

Fig. A2. Equipotentials outside a crescent, as illustrated in Fig. A1, with $l_2 = 6$, $l_1 = 2.645$, so that $\gamma = \pi/16$. Any of these equipotentials may be chosen to represent the surface of a conductor with nonsingular curvature. The equipotentials labeled a and b correspond, respectively, to $\Phi = 0.04$ and 0.22. Equipotential a is the conducting surface used in the example depicted in Figs. 4 and 5. Equipotential b is the basis for the example of Fig. 6.

mapping of the crescent) and $\Phi \propto \ln(x^2 + y^2)$ at large distances from the crescent. Clearly, $\Phi(x,y)$ then represents the electrostatic potential outside a charged 2-dimensional conducting crescent. The explicit function $\Phi(x,y)$, without reference to complex variables, is given by

$$\Phi(x,y) = \frac{1}{2}\ln\left(\frac{R^2 + 1 - 2R\cos(\phi + 2\phi_0)}{R^2 + 1 - 2R\cos\phi}\right), \tag{A7a}$$

where

$$R \equiv \{[(x+L)^2 + y^2]/[(x-L)^2 + y^2]\}^{\pi/2\alpha} \tag{A7b}$$

and

$$\phi \equiv (\pi/\alpha)\{\delta - \tan^{-1}[2yL/(x^2 + y^2 - L^2)]\}, \tag{A7c}$$

in which the principal branch $-\pi/2 < \tan^{-1} < \pi/2$ is to be taken and

$$\delta = 0, \quad \text{if } x^2 + (y - l_2)^2 > l_2^2 + L^2 \text{ and } x^2 + y^2 > L^2,$$
$$\delta = \pi, \quad \text{if } x^2 + y^2 < L^2,$$
$$\delta = 2\pi \quad \text{if } x^2 + (y - l_1)^2 < l_1^2 + L^2 \text{ and } x^2 + y^2 > L^2. \tag{A7d}$$

The crescent is, of course, an equipotential of Φ (in fact the $\Phi = 0$ equipotential) but due to its sharp corners it is not a useful choice as the conductor cross section. Any $\Phi > 0$ equipotential, however, is a smooth curve and may be chosen to represent the surface of a nonsingular conductor. The choice of the equipotential, along with the two geometric parameters l_1, l_2 defines a conducting surface we may use as an example for the relation of $|\mathbf{E}|$ and κ. Typical equipotentials are shown in Fig. A2.

The value of $|\mathbf{E}|$ on any of these surface is calculated in a straightforward manner, in principle, from

$$|\mathbf{E}| = \left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]^{1/2} = \left|\frac{dF}{dz}\right|, \tag{A8}$$

where the second equality follows from the Cauchy–Riemann relations. The formula for the radius of curvature of a plane curve can be found in elementary calculus books. Equally well we can start with the expression for the unit normal $\mathbf{n} = \nabla\Phi/|\nabla\Phi|$ and the unit tangent vector $\mathbf{t} = \mathbf{e}_z \times \mathbf{n}$ (where $\mathbf{e}_z$ is the unit vector in the z direction). Differentiation with respect to arc length can be written as

$$\frac{d}{ds} = t_x\frac{\partial}{\partial x} + t_y\frac{\partial}{\partial y}, \tag{A9}$$

so that we have

$$\kappa = \mathbf{t}\cdot\frac{d\mathbf{n}}{ds} = \left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]^{-3/2} \times\left[\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial x^2} - 2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}\right]. \tag{A10}$$

The solutions presented in the text require solving Eq. (A7) numerically for x,y pairs corresponding to the value of Φ chosen to define the conducting surface. For the x,y pairs, Eqs. (A8) and (A10) are then used to find $|\mathbf{E}|$ and κ at every point on the surface.

APPENDIX B

For our second class of examples we need to evaluate the curvature expression $\mathbf{t}\cdot d\mathbf{n}/ds$ of Eq. (3) in terms of components with respect to the orthonormal basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ for spherical coordinates. With these basis vectors a unit tangent vector $\mathbf{t}$ and the unit normal $\mathbf{n}$ are written as

$$\mathbf{t} = t^r\mathbf{e}_r + t^\theta\mathbf{e}_\theta + t^\phi\mathbf{e}_\phi, \quad \mathbf{n} = n^r\mathbf{e}_r + n^\theta\mathbf{e}_\theta + n^\phi\mathbf{e}_\phi. \tag{B1}$$

The derivative with respect to arc length ds in the direction $\mathbf{t}$ is

$$\frac{d}{ds} = t^r\frac{\partial}{\partial r} + \frac{t^\theta}{r}\frac{\partial}{\partial\theta} + \frac{t^\phi}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{B2}$$

In computing the derivative of $\mathbf{n}$ we will need the coordinate derivatives of the basis vectors:

$$\frac{\partial\mathbf{e}_r}{\partial\phi} = \sin\theta\,\mathbf{e}_\phi, \quad \frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta, \quad \frac{\partial\mathbf{e}_r}{\partial r} = 0,$$
$$\frac{\partial\mathbf{e}_\theta}{\partial\phi} = \cos\theta\,\mathbf{e}_\phi, \quad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r, \quad \frac{\partial\mathbf{e}_\theta}{\partial r} = 0. \tag{B3}$$

For an axially symmetric surface specified by $r = r(\theta)$ the outward unit normal is given by

$$n^r = r/(r^2 + r'^2)^{1/2}, \quad n^\theta = -r'/(r^2 + r'^2)^{1/2}, \tag{B4}$$

where we use prime to denote $d/d\theta$. Symmetry dictates that one of the principal directions is $\mathbf{t}_1 = \mathbf{e}_\phi$. For this direction

$$\frac{d}{ds} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{B5}$$

Since n^r and n^θ are independent of ϕ we have

$$\frac{d\mathbf{n}}{ds} = n^r\frac{d}{ds}\mathbf{e}_r + n^\theta\frac{d}{ds}\mathbf{e}_r = \frac{1}{r\sin\phi}\left(n^r\frac{\partial\mathbf{e}_r}{\partial\phi} + n^\theta\frac{\partial\mathbf{e}_\theta}{\partial\phi}\right)$$
$$= \frac{1}{r}(n^r + \cot\theta\, n^\theta)\mathbf{e}_\phi,$$

so that, for this principal direction,

$$\kappa_1 = \mathbf{t}\cdot\frac{d\mathbf{n}}{ds} = \frac{1}{r}(n^r + \cot\theta\, n^\theta)$$
$$= \frac{1 - (r'/r)\cot\theta}{(r^2 + r'^2)^{1/2}}. \tag{B6}$$

The second principal direction is given by

$$\mathbf{t}_2 = \frac{1}{(r^2 + r'^2)^{1/2}}(r'\mathbf{e}_r + r\mathbf{e}_\theta), \tag{B7}$$

for which

$$\frac{d}{ds} = \frac{1}{(r^2 + r'^2)^{1/2}}\left(r'\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\right), \tag{B8}$$

so that

$$\frac{d}{ds}\mathbf{e}_r = \frac{1}{(r^2 + r'^2)^{1/2}}\mathbf{e}_\theta,$$
$$\frac{d}{ds}\mathbf{e}_\theta = -\frac{1}{(r^2 + r'^2)^{1/2}}\mathbf{e}_r. \tag{B9}$$

We evaluate the derivatives of the components n^r, n^θ with

$$\frac{d}{ds} = \frac{1}{(r^2 + r'^2)^{1/2}}\frac{d}{d\theta}, \tag{B10}$$

where $d/d\theta$ is the total derivative $(r'\,\partial/\partial r + \partial/\partial\theta)$ along the curve $r = r(\theta)$. The result is

$$\kappa_2 = \mathbf{t}\cdot\frac{d\mathbf{n}}{ds} = \mathbf{t}_2\cdot\frac{1}{(r^2 + r'^2)^{1/2}}$$
$$\times\left(\frac{dn^r}{d\theta}\mathbf{e}_r + \frac{dn^\theta}{d\theta}\mathbf{e}_\theta + n^r\mathbf{e}_\theta - n^\theta\mathbf{e}_r\right)$$
$$= (2r'^2 + r^2 - rr'')/[(r^2 + r'^2)^{3/2}]. \tag{B11}$$

If the surface $r = r(\theta)$ specified by Eq. (7) is used and terms of order ϵ^2 and smaller are ignored, Eq. (B6) leads to the result for κ_1 given in Eq. (9a). Similarly Eq. (B11) leads to Eq. (9b) if the equation $(1 - \mu^2)\, d^2P_n/d\mu^2 - 2\mu\, dP_n/d\mu + n(n+1)\, P_n = 0$ for the Legendre polynomials is used.

The electrostatic potential everywhere outside an axially symmetric body with total charge Q can be written as

$$\Phi = \frac{Q}{r} + \sum_{n=1}^{\infty}\alpha_n\left(\frac{R}{r}\right)^{n+1}P_n(\mu). \tag{B12}$$

At the surface $r = r(\theta)$ specified by Eq. (7) the potential to first order in ϵ is

$$\Phi\Big|_{\text{surface}} = \frac{Q}{R} - \epsilon\frac{Q}{R}\sum_{n=2}^{\infty}\beta_n P_n(\mu) + \sum_{n=1}^{\infty}\alpha_n P_n(\mu). \tag{B13}$$

If Φ is to be constant on the surface the right-hand side must be independent of μ so that $\alpha_1 = 0$ and

$$\alpha_n = \epsilon(Q/R)\beta_n, \quad n \geqslant 2, \tag{B14}$$

and outside the surface, to first order in ϵ,

$$\Phi = \frac{Q}{r} + \epsilon\frac{Q}{R}\sum_{n=2}^{\infty}\beta_n\left(\frac{R}{r}\right)^{n+1}P_n(\mu). \tag{B15}$$

The electric field intensity is given by

$$|\mathbf{E}| = \left[\left(\frac{\partial\Phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Phi}{\partial\theta}\right)^2\right]^{1/2}. \tag{B16}$$

The $\partial\Phi/\partial\theta$ term contributes to $|\mathbf{E}|$ only to order ϵ^2 and higher, so that to first order in ϵ

$$|\mathbf{E}| = \left|\frac{\partial\Phi}{\partial r}\right|$$
$$= \frac{Q}{r^2} + \epsilon\frac{Q}{R}\sum_{n=2}^{\infty}(n+1)\beta_n\frac{R^{n+1}}{r^{n+2}}P_n(\mu). \tag{B17}$$

The $\mathbf{E}$-field strength just outside the surface is found by using Eq. (7) in Eq. (B17). The result, to first order in ϵ, is given as Eq. (8).

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[1] See, for example, Secs. 2.11 and 3.4 in J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.

[2] If the two principal radii are equal at a point then, at that point, $\mathbf{t}\cdot d\mathbf{n}/ds$ is independent of $\mathbf{t}$. At such a point there is only one value of κ and R and the surface *can* be approximated near the point by a sphere. The points on the axis of any surface of revolution have this property of possessing a unique radius of curvature.

[3] This is given, for example, as Prob. 1.11 in Ref. 1. This shows that, despite the local nature of curvature, and the nonlocal nature of solutions of Laplace's equation, there can be *some* relationships between the two.

[4] The same argument can be applied to Eqs. (8) and (9). The functional form for $|\mathbf{E}|$ is different from that of either Eq. (9a) or (9b), so that there can be no general relation between the maximum of $|\mathbf{E}|$ and the maximum of either principal curvature.