Relation between charge density and curvature of surface of charged conductor

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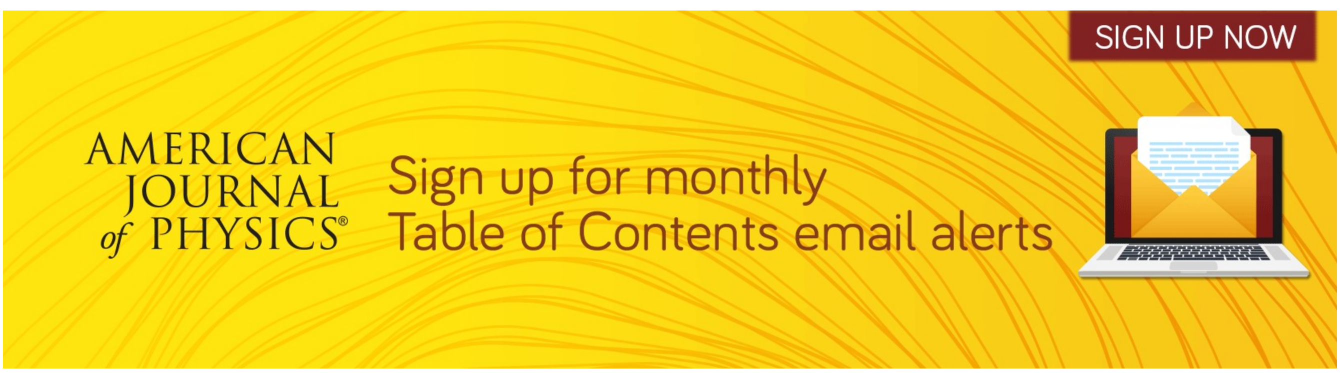

about 18 h earlier than the previous common year (24 h from the extra day in February minus the usual 6 h advance). The peak-to-peak amplitude of the oscillation in Fig. 1 is thus about 18 h. The problem with the Julian calendar, namely that the tropical year is actually shorter than 365.25 days, also is shown clearly by the graph. Over the course of time, the equinox occurs earlier and earlier in March and the year; this was precisely the problem which the Gregorian calendar reform sought to correct. Indeed the slope of the oscillatory pattern is 11 min per year, the difference between the average length of the Julian year and the tropical year. The Gregorian reform of the Julian leap year rule stipulated that century years not divisible exactly by 400 (such as the year 1900) are NOT leap years (as they would be in the Julian scheme). This effect is illustrated by the large "discontinuity" in the periodic pattern in the transition from the 19th to the 20th century; for 7 years the time of the equinox retreats further into March at the rate of about 6 h per year. Century years which are exactly divisible by 400, such as the year 2000, are normal leap years.[8] Thus, the periodic pattern (with its 11 min per year negative slope) will continue past the year 2000 without a "discontinuity" into the 21st century. Figure 1 also enables students easily to predict (by extrapolation) approximate past and future times of the vernal equinox as an interesting exercise. Toward the end of the 21st century, the date of the equinox actually will be as early as 19 March. Thus, the date of the equinox can occur as early as 19 March and as late as 21 March and remains confined between these dates. For the present and for most of the 21st century, the most frequent date is 20 March. Of course, similar patterns exist for the dates and times of the autumnal equinox and the two solstices.

The figure thus illustrates that more can be done with the Gregorian reform than memorize leap year rules and the name of Pope Gregory.

[1]Among others, see George O. Abell, *Exploration of the Universe*, 4th ed. (Saunders College, Philadelphia, 1982), pp. 128–130.

[2]G. Moyer, "The Gregorian Calendar," Sci. Am. **246**, 144 (1982).

[3]D. McNally, "The First 400 Years of the Gregorian Calendar," Irish Astron. J. **16**, 17 (1983).

[4]The gradual slowing of the rotation of the Earth due to tidal friction causes an additional small contribution.

[5]For a discussion of the precession see Bernhard M. Haisch, "Astronomical precession: A good and a bad first-order approximation," Am. J. Phys. **49**, 636 (1981).

[6]Data for the graph were secured (and reduced when necessary) from *The American Ephemeris and Nautical Almanac*, now *The Astronomical Almanac, and Planetary and Lunar Coordinates for the Years 1984–2000*, (U.S. Government Printing Office, Washington D.C.).

[7]The precise meaning of the tropical year is discussed by Reuben Benumof, "Astronomical meaning of a tropical year," Am. J. Phys. **47**, 685 (1979).

[8]The scheme subsequently has been modified slightly such that millennial years divisible exactly by 4000 (such as 4000, 8000, 12 000) will be common leap years.

Relation between charge density and curvature of surface of charged conductor

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I. INTRODUCTION

This note deals with the relation between charge density and curvature of the surface of a charged conductor and arrives at the conclusion that for the isolated charged conductors, whose surfaces are quadric, the charge density is directly proportional to the fourth root of the Gaussian curvature of the surface.

How the charge will be distributed over the surface of an isolated charged conductor is a subject of significance for discussion in electromagnetic theory. It is shown in many textbooks that it would be difficult, if not impossible, to obtain an exact analytic expression. Therefore the discussion on this topic only illustrates qualitatively that the surface density is greater at regions of large curvature and less where the curvature is small.[1,2]

By comparing the charge density and curvature of surface of many isolated charged conductors, the author has discovered that, at least for the conductors whose surfaces are quadric, there is quantitative relation between them.

II. MEASUREMENT OF DEGREE OF CURVATURE—GAUSSIAN CURVATURE

According to differential geometry, the degree of curvature of a surface is described by Gaussian curvature. Let $\mathbf{r}=\mathbf{r}(u,v)$ by a parametric equation of surface S; the Gaussian curvature is expressed as[3]

$$K=\frac{LN-M^2}{EG-F^2}, \qquad (1)$$

where

$$L = \mathbf{r}_{uu}\cdot\mathbf{n}, \quad M = \mathbf{r}_{uv}\cdot\mathbf{n}, \quad N = \mathbf{r}_{vv}\cdot\mathbf{n}\,,$$

$$E = \mathbf{r}_u\cdot\mathbf{r}_u, \quad F = \mathbf{r}_u\cdot\mathbf{r}_v, \quad G = \mathbf{r}_u\cdot\mathbf{r}_v\,,$$

with $\mathbf{n}$ the unit normal vector:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

If the surface equation takes another form; $z = z(x,y)$ then

$$K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \tag{2}$$

where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

(See Appendix.)

Now the Gaussian curvature of ellipsoids, hyperboloids of revolution, and elliptic paraboloids of revolution will, respectively, be computed as follows.

A. Ellipsoid[4]

The surface equation can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

or

$$z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2},$$

then

$$p = \frac{\partial z}{\partial x} = -\frac{c^2}{a^2}\frac{x}{z},$$

$$q = \frac{\partial z}{\partial y} = -\frac{c^2}{a^2}\frac{y}{z},$$

$$r = \frac{\partial^2 z}{\partial x^2} = -\frac{c^2}{a^2}\frac{z^2 + (c^2/a^2)x^2}{z^3},$$

$$S = \frac{\partial^2 z}{\partial x \partial y} = -\frac{c^4}{a^2 b^2}\frac{xy}{z^3}, \tag{3}$$

$$t = \frac{\partial^2 z}{\partial y^2} = -\frac{c^2}{b^2}\frac{z^2 + (c^2/b^2)y^2}{z^3},$$

$$K = \frac{at - s^2}{(1 + p^2 + q^2)^2} = \frac{1}{a^2 b^2 c^2} \times \frac{1}{[(x^2/a^4) + (y^2/b^4) + (z^2/c^4)]^2}.$$

At the three pairs of apexes we have

$$x = \pm a, \quad y = z = 0,$$

$$K_a = \frac{a^2}{b^2 c^2},$$

$$x = z = 0, \quad y = \pm b, \tag{4}$$

$$K_b = \frac{b^2}{a^2 c^2}, \tag{5}$$

$$x = y = 0, \quad z = \pm c,$$

$$K_c = \frac{c^2}{a^2 b^2}. \tag{6}$$

B. Hyperboloid of revolution of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1 \tag{7}$$

or

$$z = b\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)^{1/2}.$$

Using the same procedure as in Sec. II A, we obtain

$$K = \frac{1}{a^2 b^4}\frac{1}{[(x^2/a^4) + (y^2/b^4) + (z^2/c^4)]^2}. \tag{8}$$

At the apex of hyperboloid, $x = a, y = z = 0$,

$$K_0 = \frac{a^2}{b^4}. \tag{9}$$

C. Elliptic paraboloid of revolution

$$k^2 + 2kz = x^2 + y^2, \quad (k \text{ is a constant})$$

$$p = \frac{\partial z}{\partial x} = \frac{x}{k}, \quad q = \frac{\partial z}{\partial y} = \frac{y}{k},$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{1}{k}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = 0, \tag{10}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{1}{k},$$

$$K = \frac{k^2}{(x^2 + y^2 + k^2)^2} = \frac{1}{4(k + z)^2}.$$

At the apex of paraboloid, $x = y = 0, z = -k/2$,

$$K_0 = \frac{1}{k^2}. \tag{11}$$

III. CHARGE DISTRIBUTION OVER THE SURFACE OF AN ISOLATED CONDUCTOR

Suppose the surface equation of conductor is $f(x,y,z) = k$. Since the surface of conductor is an equipotential, it may be imagined that the field potential φ is a function of f only, i.e., $\varphi = \varphi[f(x,y,z)]$, where φ satisfies Laplace's equation $\nabla^2\varphi = 0$.

Taking derivatives with respect to x, y, z yields

$$\frac{\partial\varphi}{\partial x} = \frac{d\varphi}{df}\frac{\partial f}{\partial x}, \quad \frac{\partial\varphi}{\partial y} = \frac{d\varphi}{df}\frac{\partial f}{\partial y}, \quad \frac{\partial\varphi}{\partial z} = \frac{d\varphi}{df}\frac{\partial f}{\partial z},$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{d\varphi}{df}\frac{\partial^2 f}{\partial x^2} + \frac{d^2\varphi}{df^2}\left(\frac{\partial f}{dx}\right)^2,$$

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{d\varphi}{df}\frac{\partial^2 f}{\partial y^2} + \frac{d^2\varphi}{df^2}\left(\frac{\partial f}{\partial y}\right)^2, \tag{12}$$

$$\frac{\partial^2\varphi}{\partial z^2} = \frac{d\varphi}{df}\frac{\partial^2 f}{\partial z^2} + \frac{d^2\varphi}{df^2}\left(\frac{\partial f}{\partial z}\right)^2,$$

$$\nabla^2\varphi = \frac{d\varphi}{df}\nabla^2 f + \frac{d^2\varphi}{df^2}(\nabla f)^2 = 0,$$

$$\frac{\nabla^2 f}{(\nabla f)^2} = -\frac{d^2\varphi/df^2}{d\varphi/df}.$$

Integrating Eq. (12), the potential may be obtained

$$\frac{(d/df)(d\varphi/df)}{d\varphi/df} = -\frac{\nabla^2 f}{(\nabla f)^2},$$
$$\ln\frac{d\varphi}{df} = -\int \frac{\nabla^2 f}{(\nabla f)^2}\,df + C,$$
$$\frac{d\varphi}{df} = Ae^{-\int \frac{\nabla^2 f}{(\nabla f)^2}df}, \tag{13}$$
$$\varphi = A\int e^{-\int \frac{\nabla^2 f}{(\nabla f)^2}df}\,df + B.$$

Constants A and B depend on the boundary conditions.[5]

By using Eq. (13), one obtains the electric intensity E at the charged conductor, and then the surface density is derived. Now the surface density will be given in several cases.

A. Ellipsoid

The surface density of a conducting ellipsoid is already obtained

$$\sigma = \frac{q}{abc}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-1/2}, \tag{14}$$

where σ is the surface density and q is the total charge.[6]

At the three pairs of apexes we have

$$x=\pm a,\quad y=z=0,\quad \sigma_a = q/bc, \tag{15}$$
$$x=z=0,\quad y=\pm b,\quad \sigma_b = q/ac, \tag{16}$$
$$x=y=0,\quad z=\pm c,\quad \sigma_c = q/ab. \tag{17}$$

B. Hyperboloid of revolution of two sheets

Rewrite the surface equation as

$$\sqrt{(x+c)^2+y^2+z^2} - \sqrt{(x-c)^2+y^2+z^2} = k(k>0).$$

Relations between c, k and a,b [see Eq. (7)] are $k=2a$, $c=a+b$.

Put

$$u \equiv (x+c)^2+y^2+z^2,\quad v\equiv (x-c)^2+y^2+z^2,$$

i.e., $f=\sqrt{u}-\sqrt{v}$.

We obtain

$$\varphi = A\int e^{-\int \frac{\nabla^2 f}{(\nabla f)^2}df}\,df + B,$$

$$\varphi = \frac{A}{4C}\,\frac{2c+[\sqrt{(x+c)^2+y^2+z^2}-\sqrt{(x-c)^2+y^2+z^2}]}{2c-[\sqrt{(x+c)^2+y^2+z^2}-\sqrt{(x-c)^2+y^2+z^2}]} + B,$$

and the intensity at points just outside the surface is

$$E = \frac{A}{2ab^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{b^4}\right)^{-1/2},$$

then

$$\sigma = \frac{\epsilon_0 A}{2ab^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{b^4}\right)^{-1/2}. \tag{18}$$

At the apex of the hyperboloid

$$\sigma_0 = \frac{\epsilon_0 A}{2ab^2}\left(\frac{1}{a^2}\right)^{-1/2} = \frac{\epsilon_0 A}{2b^2}. \tag{19}$$

C. Elliptic paraboloid of revolution

$$-z+\sqrt{x^2+y^2+z^2} = k,$$

i.e.,

$$f = -z+\sqrt{x^2+y^2+z^2}.$$

We obtain $\varphi = A\ln(-z+\sqrt{x^2+y^2+z^2})+B$ and the intensity at points just outside the surface is

$$E = \frac{\sqrt{2}\,A}{\sqrt{k}\,\sqrt{k+z}},$$

then

$$\sigma = \frac{\sqrt{2}\,\epsilon_0 A}{\sqrt{k}\,\sqrt{k+z}}. \tag{20}$$

At the apex of the paraboloid

$$\sigma_0 = \frac{\sqrt{2}\,\epsilon_0 A}{\sqrt{k}\,\sqrt{k/2}} = \frac{2\epsilon_0 A}{k}. \tag{21}$$

IV. CONCLUSION

Comparing Eqs. (14), (15), (16), (17), with (3), (4), (5), (6) we obtain, for a conducting ellipsoid,

$$\sigma:\sigma_a:\sigma_b:\sigma_c = K^{1/4}: K\hat{a}^{1/4}: K\hat{b}^{1/4}: K\hat{c}^{1/4}.$$

Similarly, compare Eqs. (18), (19), with (8), (9) and (20), (21) with (10), (11); we obtain, whether conducting hyperboloid or conducting paraboloid, the following relation:

$$\sigma/\sigma_0 = (K/K_0)^{1/4},$$

i.e., for an isolated charged conductor no matter whether its surface is elliptic, revolution hyperboloid, or revolution paraboloid, the charge density is directly proportional to the fourth root of the Gaussian curvature of the surface.

The above results reflect the fact that the charge density is greater at regions of large curvature, less where the curvature is small. The same relation between the intensity at points just outside the surface and the curvature exists. However, the variation for charge density and intensity with curvature is not as quick as expected.

Results from conductors with surfaces of different shape are so consistent that it is natural for us to expect that the quantitative relation is a universal rule for conductors whose surfaces can be expressed by analytic functions.

APPENDIX

Equation (2) derived from Eq. (1):

Surface equation $z=z(x,y)$ is a particular case of

$$r = r(u,v),$$

where $u = 2$, $v = y$, then

$$E = \mathbf{r}_x \cdot \mathbf{r}_x = 1 + P^2, \quad F = \mathbf{r}_x \cdot \mathbf{r}_y = pq,$$
$$G = \mathbf{r}_y \cdot \mathbf{r}_y = 1 + q^2,$$

since

$$(\mathbf{r}_u \times \mathbf{r}_v)^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 = EG - F^2,$$

we have

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\sqrt{EG - F^2}} = \frac{(-p, -q, 1)}{\sqrt{1 + p^2 + q^2}}$$

and

$$L = \mathbf{r}_{xx} \cdot \mathbf{n} = \frac{r}{\sqrt{1 + p^2 + q^2}},$$
$$M = \mathbf{r}_{xy} \cdot \mathbf{n} = \frac{2}{\sqrt{1 + p^2 + q^2}},$$
$$N = \mathbf{r}_{yy} \cdot \mathbf{n} = \frac{t}{\sqrt{1 + p^2 + q^2}},$$

then

$$K = \frac{LN - M^2}{EG - F^2} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}.$$

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