Computational Mechanics featuring *Matlab*

DRAFT EDITION $Revision: 1.42 $

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Contents

0.1 Introduction – for the student

0.1.1 How to read this book

In this age of hyperlinks and non-linear thinking, I believe learning can still be best served by an engaging story that starts at the beginning and progresses logically to the end. Hyperlinks are good when you are already an expert in an area and you know just what you are looking for. The story is best when exploring a world that is new to you; you do not yet know what you are looking for. While some of the students taking this course may have computer programming experience, none is expected in advance.

The best way to read a physics (or math, or engineering text) is with pencil and paper, so that you repeat (or fill in) all the details of derivations and examples that are given in the book. Psychologists have long known that "active learning" is the only way to keep your mind engaged in a new and difficult subject. Just scanning the words on the page is fine for a novel, but not for learning a technical subject. When the subject is explicitly computer programming, one should have, in lieu of pencil and paper, an open terminal, or in this case, and open *Matlab command line*.

This book in no way tries to be exhaustive in its coverage of the *Matlab* language itself. In fact *Matlab* is huge, and it grows every year. Since it is a commercial program, it is in the interests of the company that supports it that it should add features every year. Fortunately, the core language is relatively untouched over the past 20 years, and I expect it will remain so. This book focuses on the core language only. It has been my experience in teaching this course that toward the end, students are comfortable enough with what they need to know that they can effectively search the extensive on-line and built-in documentation associated with *Matlab*.

Because I try to get on with using *Matlab* for physics as quickly as possible, I do not have an excessive number of examples of programming concepts. Thus, it is important that you carefully read each one, and, as suggested, type each example into the command line (or as a script) as you read. This should leave you in good shape to do the assigned end-of-chapter problems.

0.1.2 Extra credit for corrections!

What you hold in your hands at the moment is the "Zeroth Edition". It has been used to teach a course just once – so it is sure to still be full of typos and non-sequiturs. Please forgive me. I hope your professor will give you extra credit if you are the first to e-mail me errors that you find.

0.1.3 Why computer programming in physics

I teach the class for which this book is designed. Some students are enthusiastic about the class from the start, but others ask "Why do we need to learn to program a computer? If I wanted to do that, I would be a CS major." Here are my answers to this question.

First, programming can help you learn the physics. Before you can program a problem, you have to really understand the problem. Debugging the program when it seems to produce nonsensicial results ALSO helps you learn the physics. You have to ask yourself repeatedly "What did I expect to happen?" and "What are test cases that I understand?". It also enables the solution of many (analytically) intractable problems, from quadratic drag to the orbital motion of multiple bodies in astrophysics.

Second, computers are now a tool to be used by every scientist and engineer in the routine pursuit of their work. While many work with professional tools written by professional programmers, there is often a need for data analysis, or the application of a simple model in a new area for which no professional tools exist. If you work in industry, you may have an idea which you can test with a computer model and learn a lot in a week, or even a day, if you are comfortable with basic computer programming. New ideas often do not work, at least not right away, and they are particularly hard to "sell" to your boss, particularly if you want them to give you budget or assign a programmer to help you test out your idea. However, if you can test it yourself, you either save embarassment, or, if successful, help your company take your idea to the next level. If you work in government or academia, budgets are always tight. Even if you can afford to hire a programmer, you will give them much better direction if you have tried to code the problem (however crudely) yourself.

Third, averaging over all parameters of interest, processor speed, memory, hard-drive space, and graphics performance; computers have improved by factors of 1000 to 10,000 over the past thirty years. No other aspect of our society has improved by even a factor of two in the past thirty years. If you do not know how to fully take advantage of a computer, you are cutting yourself off from the most incredible advance of our age.

Fourth, CS majors in many cases study computational theory and not its applications to real world problems. Some CS professors no longer even program themselves. If you came to college with an interest in programming, you might actually prefer to be a computational physics major, rather than a CS major!

Finally, computer programming is fun. Writing a program is like building a machine, but it is generally a lot faster and easier than cutting, drilling, screwing, and soldering.

0.1.4 Other references

There are a plethora of books that aim to teach *Matlab*, the language, and many of them do it well, and inexpensively. In this book, I am constantly interleaving

language instruction with physics so that you the student can always see "the point" of the new language feature being learned. I find that reading a book that merely describes the language often leaves beginners unable to bridge the gap between knowing the language features and using it to solve problems. I wrote this book carefully so that a student who has read from the beginning and done the homework will always have the tools they need to do the next assignment. Still, human brains vary, and some may find it frustrating to have to leaf through the book to find information on some language feature. To those I say "use the index", I included all the commands. If that is not enough feel free to buy a *Matlab* specific reference. My view is that the online documentation is so extensive and of such high quality that a specific reference book is not needed – you can use the electronic documentation.

0.2 Introduction – for the instructor

I assume the potential instructors have read my messages to the students, so I will not repeat those.

0.2.1 To the instructor who is new to *Matlab* or programming

I applaud you for stepping up to the challenge of offering this course. You will learn a lot. I believe that my text will educate you as well as your students. I hope you can get three chapters ahead. This ought to provide you with sufficient buffer to be able to help the students. Clearly, you need to do the homework problems yourself. Then you will have a good chance of being able to help the students through their lab periods. My experience, having written the material and with extensive experience with programming and *Matlab*, is that task switching between twenty struggling students is still a challenge, so I urge you not to try to wing it.

During lectures, students often ask questions about the format and capabilities of various matlab commands. If you do not yourself have this mastered, I recommend an experiential approach. Have a matlab command line projected on a screen on the side of the classroom. Go experiment with various commands in front of the class. It will answer their question and will drive home the point that *Matlab* is meant to be used interactively in this way. If you are not sure how something works, experiment! You cannot "break" anything by hacking.

0.2.2 Intended audience

This book is designed for a lower-division physics course with three goals.

1. To give students a second look at the crush of traditional topics presented in a typical single semester course on classical mechanics.

2. To give students a first look at the more advanced material and optional material in classical mechanics, such as projectile motion with air-resistance, fluid mechanics, and center of mass.

3. To teach a programming language, *Matlab* and to specifically apply it to the topics of classical mechanics, in hopes that students can carry this new tool on into more advanced classes and research.

0.2.3 Course design

I have been using pieces of this book for the past four years to teach this course and so it has been classroom tested. At our school, computational mechanics was a one semester, 3-unit course with two 50 minute lectures and one two-hour "lab" or "recitation" per week. It should be emphasized that the "lab" portion of the course is quite important.

0.2.3.1 Lecture portion

In the lecture portion of the course, I explain concepts and work examples, either physics examples or programming examples. I used a traditional classroom and a blackboard. If you have access to a smart classroom with a computer running matlab and a projector, that would also be good. When I used a projector, I increased the default font size in the Matlab editor to make it more legible. To make it easier to go over code details without a projector, I provide, at the end of the early chapters, a final section called "Code examples used in this chapter". This allows you to refer to particular lines of code or constructions in your lecture without needing to write them rapidly (and usually incorrectly) on the blackboard and without needing to excessively jump around in the book.

0.2.3.2 Lab portion

This is a hands on course, so the lab is critical, particularly for students with little/no programming background. This is where students need to get helped over the first several roadblocks before they begin to become confident that they can debug their own work. I strongly urge you to run the lab sections yourself, at least for the first few times that you teach the course. Students make all sorts of programming mistakes that you cannot imagine until you see them. If you do not attend the lab yourself, you will be inevitably disconnected from the real problems your students are having with the programming part of the course. One year I ran the course with an experienced TA running the labs in my place. While students survived, the course was less effective than when I ran the lab myself. The TA was somewhat overwhelmed by the students difficulties.

For each lab, I assign several homework problems and then circulate while the students work. Sometimes I use a half-hour to demonstrate matlab techniques (particularly debugging) while students type the same thing at their keyboard. I do not assign a pre-lab.

0.2.3.3 Facilities required

I used a traditional classroom (chairs, blackboards and no technology) for lectures and a computer lab (50 desktop computers and 50 monitors with an instructors workstation and an LCD projector) for labs.

One might imagine teaching the entire course in a computer lab, but I recommend against this. Labs full of desktop computers tend to be noisy, and the computers are a welcome distraction to the students. Very often computer labs have inadequate blackboards, and the students are further apart because of the space taken up by the cmoputers.

The combination of reduced intimacy, noise and distractions result in relatively ineffective lectures and discussions relative to traditional classrooms.

If you find that you have a computer enabled classroom where students have individual small and quiet computers (e.g. laptops, iPads) and where the instructor has a tablet which can be projected in real time, you might be able to do all sessions in it.[1]

0.3 Features of this book

Before getting to the details of each chapter, the broad features of this book are as follows:

0.3.1 General features

Universality The goal is to teach programming. Matlab is just a first language. Over-reliance on Matlab-specific features is avoided.

Physics integration Programming ideas are always tried out on familiar physics. This reinforces the physics and gives students a handle to hang their programming knowledge on.

Structured programming Before object-oriented programming (OOP), there was structured programming. Matlab is a fine language for teaching the structured programming approach (break your jobs up into functions, let execution proceed in an orderly way from inputs to outputs with as few tortuous branches as possible, use local variables rather than globals, document what you are doing). Beginners do not have to start with OO abstractions. OOP can be taught once students have some experience under their belt (and I use Matlab's "handle graphics" as a way to introduce objects without going overboard.)

[1] Matlab does not run on an iPad, but since it has always been client/server based, you can have a window on an iPad to a matlab session running in your computer lab or in the cloud.

Bias toward the command line Students are raised in a point-and-click world, and *Matlab* supports this. You can customize data plots through a GUI (graphical user interface). I eschew these methods at every turn. *Matlab* allows programming the appearance of figures and I think it allows for higher consistency and scientific productivity to write programs to automate, the analyis, display, and documentation of data rather than encouraging a lot of handwork on each and every data plot.

New physics Numerical methods can solve many problems that are not solvable analytically. It is motivating to students to see their new skills put to work to solve interesting problems.

0.3.2 Course timing

The book is designed in the usual way on the assumption that you will cover a chapter a week. Experienced instructors know that one sometimes needs to slow down. In particular, expect to spend 3 weeks on chapter 5/6 together and another 3 weeks on chapters 7/8. Thus, for a one quarter course, the material provided is just the right length. For schools on the semester system, this book is a little short. I know this to be true, and will be adding more material in the second edition. What I did (and will continue to do in the 2nd edition) is to get through the 8 chapters in roughly 10 weeks, and then cover other pure physics topics in class while giving students time to explore the more advanced material in chapter 8 in labs and to work on independent projects.

0.3.3 Projects

I had students working on these projects for the last several labs, and shifted my focus to getting them through the homework to getting them to make progress on their projects. It was quite civilized and allowed for some fairly ambitious projects. Most of the projects on the list I provide have been attempted and completed successfully. Some of the best projects were those in which students came up with their own, sometimes by poking around on Youtube. The descriptions I provide are intentionally pretty broad, and too ambitious in some cases. Refining a project to something that is achievable in the time given is a skill being taught via the "deep-end" method. It is your job as instructor to help the student refine/define the project and not drown. My requests for Preproposals and Proposals were intended to get the students focused on what they could really achieved. In some cases they worked. There were of course students discovering in the final days of the course that they had to start over. That is educational too!

0.3.4 Additional topics

After covering these 8 chapters, I devoted the last six weeks of the course to three additional areas: free-body diagrams and dynamics, statics, and collisions in one and two dimensions. You might choose to cover different areas. These areas fit nicely in the context of this book and will be included explicitly in future editions. I did not include them in the first edition because there is such a wealth of material in other books covering these topics. This course was not intended as a first exposure to these topics, but rather as a second exposure to allow some degree of mastery.

0.3.5 Chapter by chapter content

Here is a summary of the features of each chapter.

Chapter 1 – Hello world! This gets the students started running *Matlab*, and introduces the excellent built-in help system immediately. In addition to doing routine arithmetic at the command line, I introduce formatted output. This may seem an advanced topic, but it gets students thinking about data formats and types of numbers right away. Even though *Matlab* is forgiving about the difference between characters, integers, and floats, pointing out the difference makes students aware of what is going on inside the computer and provides them with background useful for any future programming language they might learn.

Chapter 2 – Loops & conditionals We get right to the meat of programming with **for** loops, **while** loops, **if** statements and boolean logic. Only single variables (no arrays) are used in this chapter, as I try to provide an approach that will port nicely as students move to other languages. We get to physics right away, using the familiar equations of projectile motion to illustrate looping. Conditionals are introduced as a way to figure out when the projectile has returned to ground (and to stop looping). Output to a text file is also covered, so that by the end of chapter 2 students can create data files based on simple equations or sequences.

Chapter 3 – Arrays, Matrices & Functions Arrays are a universal programming construct, and *Matlab* has particularly powerful array/matrix capabilities. We link to physics by pointing out that a 3-element position or velocity vector can be represented as an array, then extend to showing that an array is an excellent way to contain a time-series or other scientific data. We compare operating on an array element-by-element in a loop to the parallel array operations built into *Matlab*. Though *Matlab* will tolerate a programmer who does not first declare an array, it is bad practice and disastrous in other languages. I introduce **zeros()** as a way to define an array and think about its needed size. Functions are also introduced, as we want to get the student learning to break the problem into small chunks that can be handled by functions. Students write functions to calculate the

mass of a sphere of known density and the gravitational attraction between two masses at arbitrary distance. This also allows understanding how to pass arrays in and out of functions, a subtle concept worth some discussion in class.

Chapter 4 – Working with scientific data Since students know about arrays, it is time to read files of data and plot it. Labeling your axes and making your plots self-documenting is emphasized and demonstrated. I provide data for a weather-balloon flight. Your school doubtless has resevoirs of data that students can practice on. Scientific data is usually imperfect. It has noise, gaps, missing or corrupted data. It is educational to see real data as early as possible, and the ability to crunch real data gives students an employable lab skill. Numerical differentiation is introduced as an example of a **function** that is non-trivial.

Chapter 5 – Projectile motion with drag More is said about data plotting and Matlab's "handle graphics" features are introduced. The formal DSMV method (Define, Setup, Model, View) is introduced to encourage students to view their program as an organized and modular process. Animation is introduced, both because it is exciting and because it makes the point that you might have the same data output from your model but want to view it either as a static plot or an animation. Finally, all of this is applied to projectile motion with linear drag. Linear drag is analytically solvable, so students can practice their differential equations and plot something real.

Chapter 6 – Quadratic drag & Euler method Quadratic drag is not mathematically tractable, so it gives an excuse to introduce our first numerical method for solving differential equations, the Euler method. Despite its limited accuracy, the Euler method leads students through all the steps of solving a differential equation numerically. Students use the Euler method on quadratic and linear drag and can compare their analytical solutions from the previous chapter with numerical solutions.

Chapter 7 – Gravitation By this time, all the basic programming concepts students will need have already been introduced. There is no new programming introduced in this chapter. This gives the students time to absorb the lessons of the first six chapters, and the instructor time to fill in the physics that will be needed for Chapter 8. We cover all of Kepler's laws and use orbital motion as an example of the importance of the center-of-mass frame. We also use orbits to remind students about angular momentum and potential energy. Finally we derive (by direct integration) the beautiful result that spherical masses may be treated as point masses. This introduces spherical coordinates and reinforces calculus.

Chapter 8 – Runge-Kutta method & orbital simulation This is the chapter where all the previous work comes together. The 2nd and fourth-order Runge-Kutta methods are introduced and explained. The two methods are applied to quadratic drag, and then RK2 is applied to orbital mechanics.

The student extends the orbital mechanics model to RK4 for homework, and is then ready to tackle a variety of problems, from two-body eccentric orbits to multibody problems and such applications as transfer orbits and Rutherford scatttering. Students tend to get excited as they realize they can program a working model of the inner solar system, animate it, and watch the planets whizzing around. This chapter is the jumping off point for independent projects (See appendix).

Chapter 1

Hello World!

1.1 Steps to "Hello World"

Brian Kernighan and Dennis Ritchie[1] started their famous book on *C* with a program that typed "Hello world". They did this for good reason, because in *C* you need to install a "compiler" and a "linker". You need to write your code with something called a text editor. Then you needed to compile it, link it, and finally, run it. So there was a lot of overhead just to get to "Hello world!".

With *Matlab* and other modern computing languages like *Python* life is much easier. *Matlab* does not require a compiler or a linker, and it has two modes. The first is *Matlab command line*, at which you can just type any single command or group of commands, and they are immediately executed. This is very helpful for prototyping; about which more will be said later. The second mode is more like traditional programming languages. It consists of putting a bunch of *Matlab* commands into a special file (called an *m-file* or *script*). In general, these commands are executed in the same way as if you had typed them at the *Matlab command line*. In what follows, I will call any set of commands in a file a *script*. In *C*, "Hello world" is a six-line program, but in *Matlab* it is just a one line script.

1.2 "Hello World"

1. Create and edit the file *hello.m* with the built-in Matlab editor.
2. Your program consists of just this line:
   `disp 'Hello World'`

[1] programming gods at Bell Laboratories in the 1970s who defined *C* and also had a lot to do with *Unix*

3. Save the file *hello.m*

4. Run the program:
   ```
   >> hello.m
   ```
   ??? Undefined variable "hello" or class "hello.m".

5. Oops! Leave out the *.m*
   ```
   >> hello
   Hello World
   ```

You just wrote and ran your first program. Congratulations!

1.3 Getting help

Though we have barely begun, it is worth mentioning how to get help in using *Matlab*, because its excellent built-in documentation is one of its best features. Of course there is a web page at *http://www.mathworks.com* at which you can look up the full user manual, see many worked examples, and participate in user forums at *Matlab Central*. However *Matlab* predates the web, and so a lot of support is built into the program itself. It is often far faster to use this built-in help while hacking away than to go surf the web.

Specifically, there is help at the *Matlab command line*. For example:

```
1 >> help log
2  LOG     Natural logarithm.
3    LOG(X) is the natural logarithm of the elements of X.
4    Complex results are produced if X is not positive.
5
6    See also log1p, log2, log10, exp, logm, reallog.
7
8    Reference page in Help browser
9       doc log
```

Note the definition of `log` given on line 2 and the extension to complex numbers given on line 4. Most helpfully, line 6 lists related functions. Surely you can guess which of these is base 10 logarithm, and of course you could rapidly check your guess by typing `help log10`. Finally, on line 9 is a reference to the Help browser. If you try this yourself you will see that the reference is actually a hyperlink to the local documentation, which provides more detail, and typically examples of usage of the command.

The main trick in using the command line help is to know the name of the command you are interested in. Usually you can make an educated guess, and then *See also ...* will help you.

1.4 Browsing Help

At first, you will not be familiar enough with *Matlab* to even begin to know what commands to look up in the command-line help facility. *Matlab* has extensive on-line introductions, tutorials, demos, and of course complete and detailed documentation. To begin to use it, either press the F1 function key or select *Product Help* from the pull-down menus. You will see screens like those shown in figures 1.1 and 1.2.

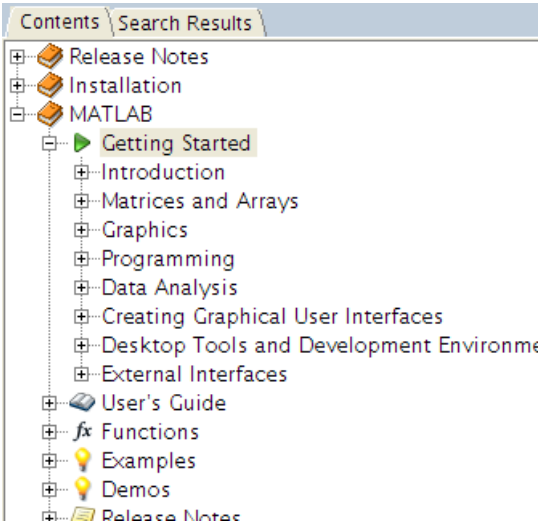


Figure 1.1: The *Getting Started* documentation does what it says. When you are ready for more detail, the *User Guide* provides it.

1.5 *Matlab* interactive mode

Matlab has a special mode that allows you to write and run programs one line at a time. This mode of *Matlab* is called the "interactive mode", and the symbols `>>` that show up when you are in this mode are called the "matlab command prompt", or just the "command prompt"[2]

Let us now trying entering *Matlab* commands at the prompt. So far we only know one command. Here it is again:

```
>> disp "Hello, Kitty!"
??? disp "Hello, Kitty!"
                        |
```

[2] "Prompt" is another computing word. In *Windows*, the prompt is C:\ > or C:\Windows> and in *Linux* or *Darwin* it is something like **you@computer:**

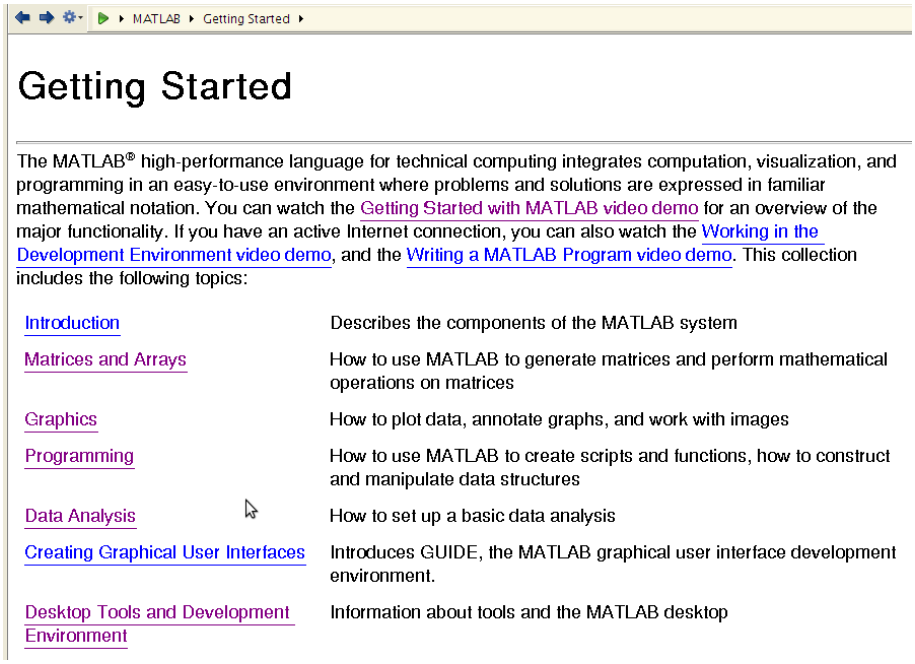


Figure 1.2: *Getting Started* tells you everything you might need about *Matlab*. The point of this textbook is to tell you *why* you might want to use these features.

```
Error: Unexpected MATLAB operator.
>> disp "Hello, Kitty"
??? disp "Hello, Kitty"
                      |
Error: The input character is not valid in MATLAB statements or
expressions.

>> disp "Hello Kitty"
??? Error using ==> disp
Too many input arguments.

>> disp 'Hello, Kitty!'
Hello, Kitty!
```

Note I used double quotes instead of single-quotes and thus got three different error messages! Computers are notoriously finicky about things that normal humans ignore. The advantage of prototyping commands in interactive mode before putting them into scripts is that you rapidly fix these inevitable slips of computer grammar.

We can also print the results of a calculation. Here is a simple example:

```
>> fprintf('The answer to life, the universe and everything is %d.\n',6*7)
The answer to life, the universe and everything is 42.
```

The example above demonstrates formatted output which will be discussed further in section 1.8.

1.6 Operators and how to calculate

The familiar symbols $+, -, \times$, and $\div$ are referred to by the fancy name of "mathematical operators", or just "operators". Of course, *Matlab* includes other functions you would expect, such as exponentials, and logarithms (as mentioned in section 1.3). Trigonometric functions are accessed as you would imagine, `sin(x), csc(x), etc.`. Further, there are other, less common operations in *Matlab* that are still quite useful. So "operators" is a useful piece of vocabulary to add to our hacking jargon.

In most computer languages, $+, -, \times$, and $\div$ are replaced by $+, -, *$, and /. The $*$ is used for multiplication because $\times$ is too easy to confuse with the variable "x". The / is used for division because few computer keyboards include a $\div$.

You can do calculations directly at the *Matlab* "prompt". Follow along at your computer as we do so:

```
>> 2**3
??? 2**3
      |
      Error: Unexpected MATLAB operator.

>> 2^3

 ans =

   8.0000e+000
```

Note again the error message for `2**3`. In this case *Matlab* did not recognize the operator `**`. It does recognize `*`, however, so the error statement shows a vertical line pointing at the second star. *Matlab* tries to be as specific as possible about what it does not understand to try to help you recognize the error and fix it.

Having fixed our error, we can notice that `"2^3"` means 2^3, and that `3^(0.5)` means $\sqrt{3}$.

From math classes you know about "order of operations". The order of operations in computer programming are the same as in mathematics. Just as in math, if you find a particular formula ambiguous, use parenthesis to clarify. See the examples in the exercises below.

1.6.0.1 Exercise 2A

What is the surface area of the Earth?

```
>> 4*pi*6400E3^2
5.14e+014
```

1.6.0.2 Exercise 2B

There are 99 bottles of beer on the wall. Each bottle contains 16 ounces. Before you arrived, someone took 9 bottles down, and passed them around. There are currently 74 people at the party. Assuming all share equally, how many ounces does each person get of the remaining beverage?

```
>> (99-9)*16/74
19.4595
```

1.6.0.3 Exercise 2C

A projectile is launched with an initial speed of 25 m/s at an angle of 30° above horizontal. What is its component of velocity parallel to the ground?

```
>> 25*cos(30*pi/180)
21.6506e+000
```

Note that no parentheses were needed in Exercise A because exponentiation occurs before multiplication. Note also that π is built into *Matlab*. Finally, note that the **cos()** function we used just above expects angles to be expressed in radians. You can easily convert degrees to radians as we did, or you can use the alternate functions **cosd()** and **sind()**, which accept arguments in degrees.

1.7 Variables and Memory

You can also assign the result of a calculation to a variable. Computer languages like *Matlab* use variables much like they are used in algebra, except that computer variables can (and usually should) have names that are longer than just a single letter.

```
1 >> x=3+4;
2 >> deltaX=3*5-1
3 deltaX =
4      14.0000e+000
5
```

```
6  >> cubeof2=2^3;
7  >> y=2^4;
8  >> [x,deltaX]
9  ans =
10        7.0000e+000      14.0000e+000
11
12 >> y
13
14 y =
15       16.0000e+000
16
17 >> z=x+y-3*deltaX
18
19 z =
20      -19.0000e+000
```

Notice that the results of the calculations were assigned to variables, then the values of these variables were printed just by typing the name of the variable. Why was the value for *deltaX* on line 3 printed right away, but for *y* on line 9 you had to again type *y* on line 12? *Matlab* makes debugging easy by giving you the value of any variable as soon as it is assigned or changed. This is often not what you want, though, so you can suppress this printout by adding a ";" to the end of the line.

Computers have "memory", which is where all the numbers used in any program are kept. For example, on line 1: `x=3+4`. The numbers 3 and 4 are in memory, and line 1 tells the computer to calculate their sum and stick it in another part of its memory. It names that part of its memory "x", so when you use "x" later in the program, the computer checks its memory and fetches the number you saved.

Notice also the statement on line 17: `z=x+y-3*deltaX`. The computer has already put numbers in memory locations called "x", "y", and "deltaX" from the statements on previous lines. It fetches the numbers back, does the calculation, and puts the result in a new memory area "z". This is really all you need to know about how variables work, but it is a fascinating subject, and if you want to know more about variables and memory, read section 1.11.

1.7.1 "=" does not mean "Equal"

The meaning of the equal (=) sign in computing is similar but not identical to that in mathematics. This is most clearly illustrated with the statement `x=x+1`. In math, you would "solve for x" and discover that there were no values of x to satisfy the equation. In computing, the convention is that this statement is interpreted beginning to the *right* of the "=" sign. The value in `x` is retrieved, the number 1 is added to it, finally the result is assigned back in the memory location labeled by `x`. For this reason, "=" is sometimes referred to as the *assignment* operator. For more details, see Section 1.11.

1.7.2 Variable Types

We just said that variables are locations in memory where numbers are kept. Computers tend to distinguish between different kinds of numbers. In *Matlab*, the three most important kinds of variable are "integer", "float", and "string". Integers in computing are defined as in math. They correspond to the whole numbers, and do not have decimal points. Thus -72, 3, and 11348934432 are integers, but 1.0 is not. Floating point numbers, or "floats" correspond to the "real numbers"[3] in math class. Floats can be recognized by their decimal points. The number 1.000 is a float, as is 22.44572435. The computer actually saves floats in memory in a different format than integers. Languages like *C* take all this very seriously. *Matlab* is a bit more casual about it, but it is important to know the difference now, as you will likely use other languages in your career that are more "strongly typed". With *Matlab* you can mostly ignore the difference between integers and floats, but it is worth knowing that there is a difference, and it can sometimes be important. (For example, *Matlab* array subscripts must be integers, and you will get an error if you use a float. We will talk about arrays later!)

"Strings" are stored in memory as integers, but when either printed or used in calculations, these numbers are interpreted as letters.[4] One character strings are 'a', 'D', '&', '(', '8', etc. You can make sequences of one character strings, like 'Hello world'.

In summary `'42'`, the string, `42` the integer and `42.0` the float are all stored differently by the computer.

1.7.3 Scientific Notation

While discussing floats, we should mention that *Matlab* has scientific notation built in. We already used scientific notation in example 1 for the radius of the Earth. In 2004, Cornell scientists weighed a single E-coli bacterium. Its weight was 6.65 femtonewtons, or $6.65 \times 10^{-15} N$. In *Matlab* this number is written as either 6.65E-15 or 6.65e-15.

1.8 Using formatted output

Here are some examples of scientific calculations with *Matlab*. You can either prototype this code by typing it a line at a time at the *Matlab* prompt, or you can use the *Matlab* editor to create a file called *ly.m* that contains the following

[3] To be more precise, floats correspond to the "rational numbers" in math. Remember that rational numbers have a finite number of decimal places, or can be expressed by a fraction of two integers. Real numbers can have an infinite number of decimal places. Clearly a computer cannot store an infinite number of decimal places. In practice, *Matlab* works with up to 16 digits of precision.

[4] See section 1.11 for examples of how numbers are interpreted as letters.

lines: (Note: Either way you do it, you can omit typing lines 1-5, 9, and 12. Lines that start with a % are called "comments" and are there to provide information for the programmer. They are extremely important for your future programming success, but the computer ignores all these lines. It as if they were blank.)

```
1  %DOCUMENT
2  %script ly.m
3  %Calculates the number of meters in a lightyear
4  %USAGE:  >> ly
5  %DEFINE
6  c=2.99792458E8; %speed of light in m/s
7  secs_per_hour=3600;
8  days_per_year=365.25;
9  %DERIVE
10 secs_per_year=secs_per_hour*24*days_per_year;
11 lightyear=c*secs_per_year;
12 %DISPLAY
13 fprintf('There are %f meters in a lightyear \n', lightyear)
14 fprintf('There are %e meters in a lightyear \n', lightyear)
15 fprintf('There are %10.2e meters in a lightyear \n', lightyear)
16 fprintf('There are %13.2f meters in a lightyear \n', lightyear)
17 fprintf('There are %10.0f meters \t in a lightyear \n', lightyear)
18 fprintf('There are %x meters in a \t lightyear \n', lightyear)
```

Lines 13-18 introduce a more general and powerful way to display numerical results. After **fprintf()**, you can type whatever text you like, [5] There is something new in this string; constructs such as %f, %e, %10.2f, or %x. (These new things are collectively called "format control characters"). Following the string and the format control characters is a comma, and then a variable. The output of these **fprintf()** statements should suggest how it all works.

```
There are 9460730472580800.000000 meters in a lightyear
There are 9.460730e+15 meters in a lightyear
There are   9.46e+15 meters in a lightyear
There are 9460730472580800.00 meters in a lightyear
There are 9460730472580800 meters        in a lightyear
There are 9.460730e+15 meters in a       lightyear
```

The % expression is replaced by the value of whatever variable follows the comma. (We just saw the % character used to represent a comment, but this use is different.) The general name for this is "formatted output". Some of the formatting characters, and their uses, are tabulated below. It should be noted that **fprintf()** in *Matlab* is almost identical to **printf()** in *C*. Likewise, this entire section on control characters and formatted output carries almost without alteration directly over to *C*, *Java*, *Fortran*, *Python*, and several other languages.

[5] Programmers often call a group of letters used like this in code a *text string* or just a *string*.

Letter	Meaning	Use
g,G	General	*Matlab* chooses best way to show the number
e,E	Exponential	Number is shown in scientific notation, even if it is "2"
f	Fixed point	Number shown with no exponent but with fixed number of decimal places
i,d	Decimal	Integer or decimal notation
o	octal	Integer - in base 8
x,X	hex	Integer - in base 16
s	string	String - for printing groups of letters
c	char	Character - for printing single letters

Table 1.1: Format control characters for displaying numerical or character results in different ways. For further details, type `>> help fprintf.`

The `\n` is also a special control character. It tells **fprintf()** to skip to the next line. Try leaving it out and see what happens! Another useful control character is `\t` which inserts a tab. You can see its effect on lines 5 and 6 of the output.

Line 15 of the script contains `%10.2e`. The number 10.2 means leave 10 spaces for the answer. You can see the effect of the 10 spaces on line 3 of the output. The number itself only takes up 8 spaces, so two blank spaces are left before it. This .2 after the 10 means to show two decimal places in scientific notation. On line 16, `%13.2f` means show two decimal places after the decimal point, but do not use scientific notation. Again the 13 means allow 13 spaces. Note on line 4 of the output that the answer actually takes up 19 spaces. *Matlab* pads short numbers with spaces, but if the number is longer than the space requested, it still shows the entire number.

In section 1.6, you did some basic calculations with *Matlab*. Now you can do calculations that use variables to make the calculations clearer. You can also use **fprintf()** statements to create output that explains the result instead of just giving a number. Try these exercises, then look at the solutions to see how to use variables and **fprintf()** statements to make each calculation very clear.

Ex. 1-A — Two spherical students are sitting 50 centimeters apart. What is the force of gravitational attraction between them? Assume one student weighs 120 pounds and the other 180 pounds.

Answer (Ex. 1-A) — 1.19×10^{-6} N

```
1 %script studentAttraction.m
2 %Calculates the attractive force between two students
3 %USAGE:  >> student_attraction
4 G=6.673E-11;  %m^3 / kg s^2
5 m1=120*0.4536;
```

```
6  m2=180*0.4536;  %453.6 g / pound
7  r=0.5;
8  F=G*m1*m2/r^2;  %Newton's Universal law of gravitation
9  fprintf('Student masses are %4.1f and %4.1f kg.',m1,m2)
10 fprintf(' Student separation is %3.1f m. \n',r)
11 fprintf('Interstudent attractive force is %5.2e N. \n',F);
```

Ex. 1-B — How small would the friction coefficient of the student with the ground need to be for the students to slide together owing to their gravitational attraction to one another?

Answer (Ex. 1-B) — $\mu < 2.1 \times 10^{-7}$

```
1 %script miniMu.m
2 %Calculates the friction coefficient to allow student coalescence.
3 %USAGE:  >> miniMu
4 studentAttraction;
5 g=9.8;
6 mu=F/(m1*g);
7 fprintf('Friction coefficient needs to be < %5.2e. \n',mu);
```

1.9 Combining short scripts

Note that Exercise 1-B made use of exercise 1-A. This is the first example of a very powerful programming technique which we will use extensively. Rather than write one long script, it is often much more productive to write several short scripts, debug them individually, and then combine them. The script *miniMu.m* in its first line of code calls the script *studentAttraction.m*. The power in this is that *miniMu* does not need to recalculate the force of attraction between students, it can use the result already calculated by *studentAttraction*.

Some more explanation is needed about why this works. In *Matlab*, ordinary scripts, which are just lists of *Matlab* commands, can share variables with eachother and are available to you at the *Matlab command line*. A new computing definition is in order. Variables that may be shared between multiple programs without any special effort are called *global* in scope. The power of *global* variables comes at a cost, which will be discussed later. Suffice it to say that not all variables in *Matlab* are *global*. If a *script* starts with the special command **function**, then its variables are only valid inside itself. They are called *local* variables. We will discuss more about *global* and *local* variables, as well as *workspaces* later.

1.9.1 *script* and *m-file*

All the files we will create to use in matlab end with `.m` . In this chapter, the terms *script* and *m-file* are synonomous. In the next chapter we will see that there is a second common type of *m-file* called a *function*.

1.10 A first interactive program

Section 1.2 told you how to run the same script more than once. However, if you get the same result every time it is not interesting!

With one more command, `input` we can write a program that it worth running more than once, even if it is not fascinating. Google will tell you that a furlong per fortnight is the same as 0.000166 m/s. Likewise 1 mile per hour is 0.447 m/s. So we can write a script to convert speed in miles per hour to speed in furlongs per fortnight.

```
1 %Script: furlongs.m    date: 2009/08/20    author: Sonnenfeld
2 meter_per_s_to_fur_per_fort_conversion = 0.000166309;
  %m/s
3 mph_to_meter_per_s_conversion = 0.44704 ;   %m/s
4 mph_to_fur_per_fort_conversion = ...
5     mph_to_meter_per_s_conversion/meter_per_s_to_fur_per_fort_cor
6 mph_speed=input('Please enter your speed in miles per hour\n');
7 fur_fort_speed=mph_speed*mph_to_fur_per_fort_conversion;
8 fprintf('Your speed is %6.1f furlongs per fortnight \n',fur_fort_
```

Note that `input` allows the user to enter a number, and gives them a message telling them what to enter. Try running this program a couple of times, entering different numbers. Now try entering some text instead of a number. You will see that *Matlab* gives an error message because it, like most computer languages, treats text and numbers differently.

1.11 *Variables, Memory, and binary codes

We consider again a simple expression like: `x=3+4`. Here is what the computer is actually doing:

1. Move the binary number 0011 (3) from memory to the ALU[6]
2. Move the number 0100 (4) from memory to the ALU
3. Execute the instruction `add`, and get 0111 (7).
4. Stick this result in a new location ("address") in memory.

[6] ALU – Arithmetic Logic Unit

5. Use the letter "x" as a shortcut for this "address".

If this seems slow, remember that each of the steps above takes place in the time of one cycle for the microprocessor (approximately). For a "slow" 1 GHz processor, a cycle is a nanosecond, so the above statement could be completed in 5 nanoseconds.

As you may have heard, inside a computer there really is nothing but numbers. The pictures that we love are encoded as numbers, and there are numerical codes that are converted to text characters. For example in ASCII[7] The letter Q corresponds to the decimal number 81 or the hexadecimal number [8] 51 or the binary number 01010001.

To translate from binary (base 2) to decimal in your head, add the appropriate power of two for every digit in a binary number that is a "1", and ignore every "0". It is just a "place-holder" (the same as in base 10).

For example: 11_2 means $2^1 + 2^0 = 2 + 1 = 3$.
Another example: 10110_2 means $2^4+2^2+2^1 = 16 + 4 + 2 = 22$.
You have probably figured out by now that the subscripted 2 is a way of clarifying the base so that you know that 11_2 means "three", while 11 means "eleven".

The electrical engineers at New Mexico Tech wear the following T-shirt: "There are only 10 kinds of people. Those that know binary, and those that do not." Enough said!

Ex. 1 — Write the binary numbers from 1-16

Ex. 2 — Convert to base 10: 101_2

Ex. 3 — Convert to base 10: 10101_2

Ex. 4 — Convert: 11101_2

Ex. 5 — Convert: 1010001_2

Answer (Ex. 1) — 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010 1011, 1100, 1101, 1110, 1111, 10000

Answer (Ex. 2) — 5

Answer (Ex. 3) — 21

Answer (Ex. 4) — 29

Answer (Ex. 5) — 81

[7] ASCII – American Standard Code for Information Interchange
[8] hexadecimal, or "hex", is base 16 – we'll discuss!

1.12 Review of commands introduced in this chapter

For more information on each command type `>>help cmdname` at the *Matlab command line.*

cos() Cosine of argument in radians.

disp() disp(X) displays variable X without printing its name. Otherwise it is same as leaving the semicolon off and typing X. If X is a string, the text is displayed.

input() Prompt for user input.

fprintf() Write formatted data to text file.

%s, %d, %e, %i Formatting strings. Strangely, *Matlab* help refers you to a *C* manual for these.

strings Character strings, arrays of text (`help strings`).

log() Natural logarithm.

`+, -, *, /, ^` Arithmetic operators (`help arith`).

types of numbers *Matlab* has integers, floats, and strings, (and variations on these)(`help datatypes`).

1.13 End of Chapter Problems

(1) The universe is about 15 billion years old. The radius of the observed universe is 15 billion light-years. Using this fact and the observation that there are only about one million atoms in every cubic meter of space (on average), write a script to calculate the total number of atoms in the universe. Use a **fprintf()** statement to give your numerical answer with a complete sentence. [The result of this calculation, by the way, is about the largest possible number that represents a number of objects. There are many larger numbers used in mathematics and science, but they represent ideas or probabilities, not real things.]

(2) In England and Ireland people still give their weight in "stones". Write an interactive script that asks you your weight in pounds and tells you your weight in stones.

(3) Write an interactive script that first asks you the speed of a projectile, then the launch angle in degrees, and returns to you the x and y components of the velocity, as follows:

```
Launch angle = 30°, speed=10 m/s, vx= 8.6 m/s, vy=5 m/s
```

(4) Write an interactive script that first asks you your name, then asks you your weight, then tells you roughly how many moles of hydrogen, oxygen and carbon you contain. You can approximate that the human body is 75% water by weight, and the rest is carbon. The output of the script should include your name and atomic composition. For example:

```
Hello Richard, you contain 7,500 moles of hydrogen, 3750 moles of
oxygen and 1875 moles of carbon.
```

hint #1 By default, the **input()** command expects you to enter a number. If you want to enter text, you need to use the alternate form of **input()** which allows the entry of strings. *Matlab* will explain it to you if you type `>> help input` .

hint #2 Among other things, you have been asked to print back the name of the user using **fprintf()**. To print text in *Matlab* (and many other computer languages), you use the `%s` format control sequence (refer to Table 1.1 above

hint #3 In case you forgot how moles work, a mole of (for example) Helium weighs four grams, because the atomic weight of Helium is four Atomic Mass Units (AMU).

Chapter 2

I Will not Throw Massive Projectiles in Class

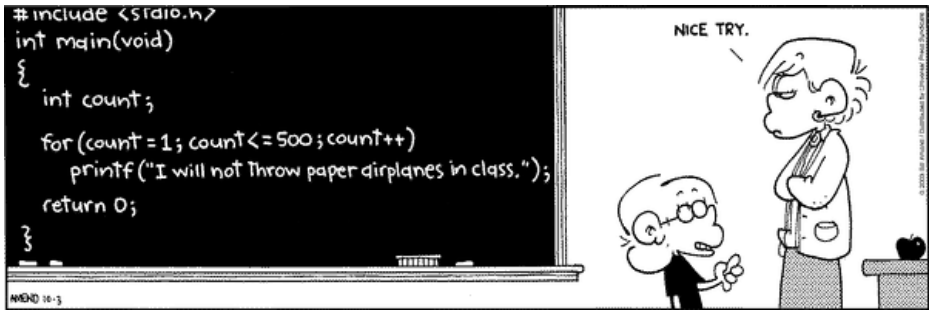


Figure 2.1: Jason Fox makes short work of a writing assignment in *C*, but *Matlab* is even quicker!

2.1 Getting homework done quickly with loops

Computers are supposed to save you work. Notice that Jason Fox has been assigned to write "I will not throw paper airplanes in class", five hundred times. Naturally, he found a way to have the computer *save* him a lot of tedious work. Let us see how to do that. We need the programming concept of a "loop".

```
1 for k=1:10
2   fprintf('%i.  I''ll not throw airplanes in class\n',k)
3 end
4 % ==========  end of paperairplanes.m  ================
```

Try saving this code as a script and running it. Next type it line by line at the commmand line. You will see that *Matlab* will not come back with a `>>` prompt after you have entered the first line at the command line. It will not come back after the second line either. However, once you have typed **end** to end the loop, it will spit out the ten line punishment all at once. That's a big time saver! Three lines of code generated ten lines of a writing assignment. Try changing the first line to `for k=1:200`, and you will see you have become even more efficient!

This illustrates *loops*, another universal programming construct. We will dissect the first line of code: `for k=1:10`. This line can be read aloud as follows: *For k runs from one to ten.* The variable **k** is often called the *loop index* or the *loop counter*. The number 1 is the *lower limit* or *initial value* of the loop, while the number 10 is the *upper limit* or *final value*. As you might imagine, the loop counter could be called anything, **x, index, jane, voldemort**. Likewise the initial value does not need to be 1. It can be any number, and need not be an integer. After the **for**, there can be any number of lines of code. The loop must end with **end**, which tells *Matlab* that it should go back to the *top of the loop*, increment the loop index, and continue. Given your math experience, you have perhaps realized that `for k=1:200` is similar to:

$$\sum_{k=1}^{k=200} k \tag{2.1}$$

except that you can do anything you want with **k**. You do not have to sum it.

Finally we note that, the loop can count down as well as up, and it does not need to count by 1. For example, the statement `for k=3:-0.1:1.2` sets k to 3, then decreases it by 0.1 every time through the loop until it gets to 1.2. The number `-0.1` in the previous sentence is called a *step size*.

The concepts and terms of *loop*, *loop index*, *lower limit*, *upper limit*, and *step size* exist in essentially all computer languages, including *C* and *Python*.

2.2 Using loops to make a table

Before computers and electronic calculators, engineers did their jobs by looking up important quantities in tables. A loop lets you make your own table very quickly. Let's say you want to convert between degrees Celsius and Fahrenheit.

```
1
2 %Makes a conversion table between C and F.
3 for degC = 0:6
4     degF=(9.0/5.0*degC)+32;
5     fprintf ('%2d C = %4.1f F\n', degC, degF)
6 end
7 % 0 C = 32.0 F
8 % 1 C = 33.8 F ...
```

```
% 6 C = 42.8 F
% ========== end of tempTable.m ====================
```

Clearly, it would be trivial to extend this table from 0 to 100 C simply by changing the limits in the **for** loop.

2.3 Projectile motion for heavy projectiles

The study of projectile motion can also be called "Two-dimensional kinematics under constant force". It is the study of how things move in two dimensions when there is only a constant force directed along one of the dimensions. You will recall that the equation for motion under constant acceleration is, in vector form:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \tag{2.2}$$

The vector definition of instantaneous velocity is:

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{2.3}$$

Considering only the x, and y components, equation 2.2 becomes:

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \qquad y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \tag{2.4}$$

Where we have taken positive y to be upwards and where the constant acceleration due to gravity is directed down. Since $a_x = 0$ and $a_y = -g = -9.8$ m/s^2, equations 2.5 become

$$x = x_0 + v_{0x}t \qquad y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \tag{2.5}$$

Taking the derivative of both components, we get

$$v_x = v_{0x} \qquad v_y = v_{0y} - gt \tag{2.6}$$

These simple equations sum up motion of heavy projectiles. We specify *heavy* projectiles of course in order to say that they are not much affected by air resistance. For projectiles that are affected by air-resistance, there is an additional acceleration term in y that is proportional to the projectile velocity, and also an acceleration term in x. In fact, equation 2.2 simply is incorrect for *light* projectiles, and we must obtain our solutions by solving differential equations or by numerical methods. Fortunately *Matlab* is excellent for numerical methods, and we are building up to precisely this application.

2.3.1 Using loops to make a table of projectile motion

Having recalled our equations for projectile motion, we can readily make a table of position and velocity for just the y-component of projectile motion.

```
1  %DOCUMENT
2  %Makes table of positions & velocities for proj. motion
3  %USAGE:  >> projectile
4  %DEFINE
5  g= 9.81; %g in m/s^2
6  y0=10; %m
7  v0y=22; %m/s
8  fprintf('Time \t Velocity \t  Altitude \n')
9  fprintf(' sec \t    m/s   \t     m    \n')
10 fprintf('==== \t ======== \t  ======== \n')
11 %DERIVE
12 timestep=0.2;
13 for t=0:timestep:10
14     y = y0 + v0y*t - (1/2)*g*t^2;
15     vy = v0y- g*t;
16     %DISPLAY
17     fprintf('%4.1f \t %6.1f \t %7.1f \n',  t, vy, y)
18 end
19 % ==========  end of projectile.m  ====================
```

Note that line 13 of the above script uses a variable step size, incrementing time `t` by `0.2` seconds on every loop instead of by one second.

The program above gives altitude and speed vs. time for a projectile thrown straight up. How might one figure out at what time a projectile would land given some initial velocity v_{y0} and displacement y_0? We could certainly solve the quadratic equations in 2.4 for the time at which y takes on the value y_0. However, once the effects of air-resistance are included equation 2.4 will no longer apply and might not allow for an analytical solution. A general way to find the landing time is to use a *conditional* statement.

2.4 Conditional statements

2.4.1 Using conditional statements to stop a loop

A very important part of computer programming is allowing the computer to make choices of what to do next based on the calculations it has already done. Statements which do this are called conditional statements. Every programming language known supports a conditional execution. Most languages use a syntax similar to "if – else " to implment conditional execution. Conditional statements usually need logical (or relational) operators. (Section 2.6 discusses what a logical

operator is and explains many of the logical operators that *Matlab* supports. Once you read section 2.6, you will see that there are a number of other logical operators available.)

Below is an excerpt from the projectile script of section 2.3.1, to which a conditional statement has been added.

```
1  g= 9.81; %g in m/s^2
2  y0=10; %m
3  v0y=22; %m/s
4  fprintf('Time \t Velocity \t  Altitude \n')
5  fprintf('==== \t ========== \t  ========== \n')
6  timestep=0.2;
7  for t=0:timestep:10
8      y = y0 + v0y*t - (1/2)*g*t^2;
9      vy = v0y- g*t;
10     fprintf('%4.1f \t %6.1f \t %7.1f \n',  t, vy, y)
11     if (y < 0)
12         break
13     end
14 end
15 % ==========  end of projectileBreak.m  ==============
```

Lines 11-13 are the added code. The $<$ on line 11 is a relational operator. Every time through the loop, the **if** statement on line 11 is executed and the expression $y < 0$ is evaluated. If y is less than zero, then the statement $y < 0$ evaluates to **true**. (In many computer languages, including *Matlab*, **true** is a built in variable with a value of 1. **false** is also built in, and has a value of 0). If y is greater than or equal to zero, the expression $y < 0$ evaluates to 0 (**false**). When a the condition following an **if** is true, the lines immediately following the **if** are executed. If the condition is false, all the lines between **if** and the next **end** are skipped. In the script at hand, there is only one line to execute between the **if** and the **end**. That command on that single line is **break**. The keyword **break** tells the computer to jump to the statement following the **end***and* to stop looping. This is exactly the behavior we want. Our table is generated until $y < 0$, then **break** is executed and the loop which generates the table stops.

Please type the script above into *Matlab* and see for yourself that the table stops on the first point after the projectile is lower than zero altitude. Having verified that the script works, change line 2 so that $y0 = -10$ instead of $y0 = 10$? Run the script and notice that the table stops printing after the first line. Why? Of course we started our projectile with $y < 0$, so now the condition is true immediately. Our code is working properly. It is our logic that is incorrect. What we really want is to stop the script once the projectile gets below 0 if it is *falling*. We can easily check if it is falling by also checking that $vy < 0$. It is only necessary to change line 11 of the script as follows:

```
if (y < 0) && (vy < 0)
```

```
% ========== end of projectilebreakexcerpt ==========
```

The above conditional is only **true** when both $y < 0$ and $vy < 0$. This demonstrates that you can check more than one condition in a single **if** statement.

2.4.2 Using conditional statements to allow choices

Conditional statements (**if** commands) are broadly useful to programmers. For example, conditional code can allow a user to make choices. Below is a simple temperature conversion script that allows the user to choose what units to convert temperature to.

```
1  %Converts Fahrenheit temperatures to other selected units.
2  F=input('Enter a temperature in Fahrenheit (e.g. 32) ');
3  outUnits=input('Scale to convert to? (C, K, or R) ','s');
4  outUnits=upper(outUnits);
5  absZeroK=273.16;
6  C=(5/9)*(F-32);
7  if outUnits=='K'
8      Tout=C+absZeroK;
9  elseif outUnits=='C'
10     Tout=C;
11 elseif outUnits=='R'
12     Tout=F+459.57;
13 else
14     error(['Undefined temperature scale ',outUnits])
15 end
16 fprintf('%6.2f Fahrenheit is %6.2f %s.\n',F,Tout,outUnits)
17 % ========== end of temperatureConvert.m ==========
```

Note that the **if** extends from lines 7–15. If the selected units are `K`, it executes the equation on line 8, then jumps to line 15. If the selected units are not `K`, it jumps to the **elseif** on line 9 and checks if they are `C`. If so, line 10 is executed and execution jumps again to line 15. The final check is for equality to `R` (Rankine units, fairly obscure). Please note the **else** on line 13. The program jumps to line 14 for *ALL* inputs that are not `K, C, or R`. The **error()** function is built in to *Matlab* to allow the programmer to define their own errors besides those already built into the language. It is always good practice to *trap* errors. That is to say, you always expect a certain input from the user, but the program should fail gracefully if the user input (or the input from your measuring instruments, or all the other surprises you get when a computer interacts with things like people and instruments) differs from what is expected.

A few more subtleties are illustrated in the temperatureConvert script. The function **upper()** on line 4 allows the user to enter either a lower or upper-case letter and converts it to upper-case. This makes the **if** statements simpler. In the absence of line 4, line 7 would be written as follows: `if (outUnits=='K' || outUnits=='k')`

One might think that line 5 is superfluous, as the number `273.16` could have been used on line 8 without first being set to a variable. One might even argue that this wastes space and computer memory. While this is true, and might still matter for code written on a small embedded microprocessor, the average scientific computer is so well provisioned with memory and speed that we can choose to obey the general programming guideline *no magic numbers*, which emphasizes clarity over performance. A *magic number* is a programming term for any constant (other than 0, 1 or pi) that appears in computer code to make something work. By assigning the number first to a variable, the programmer has an opportunity to remind him/herself why that number is there, as it is not always as obvious as it is in this example.

2.4.3 Using conditional statements in a while loop

While you can accomplish almost any conceivable loop task using only **for**, **if**, and **break**, *Matlab* (and *C*) provide an alternative construction called **while**. A **for** loop has to be told a starting limit and an ending limit. A **while** loop is told only a condition which needs to be true for it to keep running. It runs forever so long as its test condition is true. (In fact, if your test condition is always true, either on purpose or by error, you have created the famous *infinite loop*). Below the projectile script is rewritten illustrating how to replace the **for** loop with a **while** loop.

```
1  g= 9.81; y0=10; v0y=22;
2  fprintf('Time \t Velocity \t  Altitude \n');
3  timestep=0.2;
4  t=0; y=y0;
5  while ~(y<0)
6      y = y0 + v0y*t - (1/2)*g*t^2;
7      vy = v0y- g*t;
8      fprintf('%4.1f \t %6.1f \t %7.1f \n',  t, vy, y);
9      t=t+timestep;
10 end
11 % ==========  end of projectile2while.m  ============
```

Several changes had to be made to go from **for** to **while**. The **if** test that lead to a **break** if *True* is no longer needed. The test condition is incorporated directly into the **while** statement, with one change. Comparing the test condition on line 5 of the **while** loop above with that on line 11 of the previous script using a **for** loop, we see that we added a `~` operator. Thus the **while** loop operates while `y is NOT less than 0`. Alternately, we could have changed the test condition to `y>=0`; the two approaches are logically equivalent. Other changes also had to be made. The **for** loop automatically defines and increments t. The **while** loop does not, so we must set a starting value for **t** on line 4. (This is a common requirement and the programming term is that we *initialized* the variable **t**.) Also, the **while** loop checks the value of **y** so we need also to initialize **y** before the first time through. Finally the **for** loop increments **t** every time through,

whereas we must do it manually (see line 9) for the **while** loop. In general, if you know how many times you want to go through a loop, a **for** command is better. If you want to exit the loop on a condition, you can either use the **if** and **break** construction, or use **while**.

2.5 Creating ASCII data files

So far we have written data tables to the screen, but to be really useful, the data tables need to be in a file. *Matlab* has lots of facilities for writing and reading data files, including the functions **load()** and **save()**. While these commands work well and are easy to use (and we will use them) you do not have so much control of what the file looks like when you use these *high-level* commands. Also, this course is intended to prepare you to use languages which are not quite as supportive of novice programmers as *Matlab* (*Python*, *C*, *Java*) and all of these languages use an alternate mechanism for writing files very similar to the one you are about to learn.

```
1  % Makes a table of projectile motion in 1-D
2  g= 9.81; y0=-10; v0y=22;
3  fid=fopen('projectile_table.txt','w');
4  fprintf(fid,'%% Time \t Velocity \t  Altitude \n');
5  fprintf(fid,'%% ==== \t ========= \t  ========= \n');
6  for t=0: 0.2 :10
7    y = y0 + v0y*t - (1/2)*g*t^2;
8    vy = v0y- g*t;
9    fprintf(fid,'  %4.1f \t %6.1f \t %7.1f \n',  t, vy, y);
10   if (y < 0) && (vy < 0)
11     break
12   end
13 end
14 fclose(fid);
15 % ==========  end of projectile2Writefile.m  ==========
```

To our previous projectile script we have added **fopen()** (*file open*) on line 3, and **fclose()** (*file close*) on line 14. **fopen()** takes as a first argument the name of the file you want to open, and as a second argument what you want to do with the file. The `'w'` means we want to write to the file, as well as erasing what might have already been in that file. Other options are appending to the file (`'a'`) and reading data from the file (`'r'`). The **fopen()** function returns *fid*, a file id, which is then used by **fprintf()** so it knows where to put its data. A more subtle change to the script is on line 9. Note that the **fprintf()** now starts with the fid, followed by the text you want to write. This is very convenient. Once you know how to write to the screen, just add an *fid* and you are writing to a file. This is not an accident, by the way. In the *Unix* tradition, from which *Matlab* grows, all input/output devices are considered to be files, including the screen and the keyboard. Once you finish writing data, you need to *close* the file. If

you do not, you will not be able to open it next time around, and you may lose some of your data as well.

One final comment regarding lines 4 and 5. Note the **fprintf()** starts with a `%%`. This is *not* required. The data would be written without the `%%`. We do this because we want each of the first two lines to start with a `%`. Why type `%%` when we only want `%`? Recall that the formatting strings `%s, %f, %d`, etc. all start with `%`. Thus `%` has a special meaning. You have to put `%%` to tell *Matlab* that you are not writing an erroneous formatting string, but that you really want it to print a `%`. This is a quirk of special characters in computing languages, not limited to *Matlab*.

2.6 Logical or Relational operators

Logical operators (also called relational operators) are extremely important because they are used in almost all conditional statements. "Logic" was a branch of philosophy taught in ancient Greece alongside grammar and rhetoric. At that time, logicians considered what conclusions could properly be deduced from given facts. Here is an example of a logical proposition.

```
IF an animal has fur AND produces milk for its newborn offspring,
THEN we call it a mammal.

A platypus has fur.
A platypus produces milk.

THEREFORE a platypus is a mammal.
```

A platypus also lays eggs. Nonetheless, since it fulfills the requirements of having fur and giving milk, logic tells us that a platypus is a mammal.

In the early 20^{th} century, Bertrand Russell and others replaced some of the words in logical statements with symbols. Here is how they would talk about platypi.

$$\texttt{if (animal } A \texttt{ has fur)} \wedge \texttt{(}A \texttt{ produces milk)} \Rightarrow \texttt{(}A \texttt{ is a mammal)}$$
$$\texttt{(Platypus } B \texttt{ has fur)} \wedge \texttt{(Platypus } B \texttt{ produces milk)} \Rightarrow \texttt{(A platypus is a mammal).}$$

Because the word "and" was replaced by the symbol $\wedge$, and the words "then", or "therefore" were replaced by the symbol $\Rightarrow$, Russell called his creation "symbolic logic". For philosophy, it might have been a bit too fancy, but, when computers were created in the 1950's, it was recognized that symbolic logic was a very good way to describe their operation. When talking about computers today, we usually refer to just "logic", but we mean "symbolic logic".

In addition to arithmetic, computers can do logic. The following "logical operators" are some of those defined.

Symbol	Name	Truth values
==	Equivalent to	a==b is *True* when a and b are identical
∼=	Not Equiv. to	a∼=b is *True* when a and b are different
<	Less than	a<b is *True* when a is a smaller number than b
>	Greater than	a>b is *True* when a is a larger number than b
&&	'and'	a && b is *True* when a is *True* AND b is *True*
||	'or'	a || b is *True* when either a or b is *True* (and when BOTH are *True*.)

Here are some examples.

```
>> 4==5
0
>> 4==4
1
>> 4~=5
1
>> -5 < 47
1
>> x=-5
>> y=47
>> x > y
0
```

The above examples might seem silly – but they are not when you plug truth values into variables and use them in `if` statements, as you did in section 2.4.1.

2.7 More built-in functions

Programming languages can do much more than add, subtract, multiply, and divide through the use of additional built-in functions. In chapter one we used **log()** and **cos()**. In fact, **fprintf()** is also a built-in function. Anytime you see a pair of parenthesis, you are looking at a function. The *Matlab* function syntax should be easy to remember, as it is exactly the way functions look in mathematics and pencil-and-paper physics. Below we introduce four functions useful in error calculations, factoring, and rounding numbers.

2.7.1 Checking the size of an error – Absolute value

The function **abs()** gives the absolute value of its argument. This can be useful when doing a least-squares fit or any time you are trying to calculate an error

between two quantities and do not care if the error is positive or negative.

2.7.2 Checking for divisibility – fix() and mod()

Consider the division problem $14 \div 5$. In math class, you would write this either as 2.8, $2\frac{4}{5}$ or "2 remainder 4". There are reasons in programming when you might want to know the integer part of the quotient ("2") separately from the remainder ("4").

```
1  >> 14/5.
2   2.8
3  >> fix(14/5).
4   2
5  >> mod(14,5)
6   4
7  >> 14/5-fix(14/5)
8   0.8
9  >> fix(8/3.2)
10  2.0
11 >> mod(8,3.2)
12  1.6
```

Line 1 shows ordinary division, with an ordinary result. Line 3 introduces **fix()**, the "truncation" function, which only returns the part of the quotient before the decimal point. Likewise, line 5 introduces **mod()**, the "modulus" function, which only returns the remainder. Line 7 gets perhaps excessively clever and subtracts the truncated quotient from the actual quotient to get the "remainder", as a decimal Lines 9 and 11 show that the truncation and modulus functions also make sense for non-integer division.

Why do you care about this? Truncation and modulus functions are useful if you want to check divisibility of two numbers.

2.7.3 Rounding functions

As long as you are working with the integer part of numbers and their decimal parts, you may need to round a number either up or down to an integer. The function **floor()** rounds any number down to the nearest integer closer to -infinity. As you might then guess, the function **ceil()** rounds any number up to the nearest integer closer to +infinity. Of course **round()** rounds a number to the nearest integer depending on whether its decimal part is greater or less than `0.5`.

2.8 What's next?

In this chapter we concentrated on things you could do with single variables and built-in functions. In the next chapter we will introduce *arrays*, which are sets of

related variables. We will also see how to extend the built-in functions in *Matlab* by writing our own user-defined functions.

2.9 Review of commands introduced in this chapter

For more information (and to see a list of related commands) on each command type `>>help cmdname` at the matlab command line.

abs Absolute value.

for Repeat statements a specific number of times.

while Repeat statements while a logical condition is **true**.

if Conditionally execute statements.

`==, <, &&, ||` Relational operators (`help relop`).

:, colon J:K is the same as [J, J+1, ..., K] (`help colon`).

break Terminate execution of for or while loop.

end Terminate scope of if, for, while, switch, and try statements.

error Display message and abort function.

fopen, fclose Low-level commands to open a file to read or write, and close it when you are done.

load, save General high-level *Matlab* data saving and data loading commands.

ceil, floor Round a floating point number up down toward the next higher lower integer.

round Round a floating point number to nearest integer.

fix Truncates the decimal part of a floating point number, creating an integer.

mod Modulus after division.

2.10 End of Chapter Problems

(1) Write a script that writes a file called `trig.txt` which consists of a table of trigonometric functions in increments of 4 degrees between -90° and 90°. The first column of the table should be the angle θ in degrees. The second column should be the $sin(\theta)$ with two digits after the decimal, the third column should be $cos(\theta)$ and the fourth column should be $sin^2(\theta)+cos^2(\theta)$. The first line of the table should read
`% x cos(x) sin(x) sin^2(x)+cos^2(x)` Show the results in the fourth column to 18 decimal places! The results are not exactly 1. Why?

Hint: In math, the notation $sin^2(x)$ means the $sin(x)$, squared. *Matlab* does not know this notation. You need to write `sin(x)^2` to actually do that calculation.

(2) Write a script with a **for** loop that counts backwards by 2 from 50 to 0.

(3) Generate a table of approximations to π by summing and printing the following series.

$$\pi = 4\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} \tag{2.7}$$

This is a very slowly converging series. How many terms does it take to get to within 0.01 of 3.14?

(4a) Generate π by summing the slowly converging series again. This time do not bother with a table. Let the loop run and break out of it using a conditional statement when the approximation to pi gets within 0.000001 of π. Print how many terms were needed and what the approximation was at that point.

Hint: It takes enough terms that your script will be running for a noticable amount of time. Recommend testing it with a target error of 0.01 and a loop count of 1000. When you think it works, fix the target and increase the loop count as needed.

(4b) Repeat the previous problem, but accomplish the task using a **while** loop instead of a **for** loop.

Hint: You may find that the **while** loop takes one more or one fewer iterations to complete than the **for** loop. You do not have to fix this. However, if this happens to you, insert a comment line at end of your script explaining why you think the results differ.

(5) Write a program to generate the Fibonacci series and print a table of Fibonacci numbers and the ratio of successive Fibonacci numbers. Stop the loop when the ratio ϕ between successive numbers has changed by < 0.00001. At this point ϕ approximates something called the golden ratio which appears throughout art and nature. It also has the interesting property that $1/\phi = \phi - 1$.

(6) Your script should ask the user for any two integers and test if the first is divisible by the second. It should respond with statements like the following:

```
26226 is divisible by 423.
21371 is not divisible by 592
```

Hint: This problem makes an excellent application for the **if-else** construction.

(7a) Write a script that writes a table of projectile motion in two dimensions to a file called `projectile.txt`. Let the launch velocity be 30 m/s and the launch angle be 60° above horizontal. The table should stop once the projectile has come back to zero altitude. The first two lines of the file should read as follows:

```
% v0=30 m/s, theta=60 degrees
% Time (s)  x(m)   y(m)   vy(m/s)
```

(7b) Modify the script from part a so that it only gives the range of the projectile. (For convenience, I recommend that you also change it so that it prints its results to the screen rather than the file it used in part a.) Keep the test conditions the same as part "a" with the following exception: The projectile is allowed to *land* on a cliff of arbitrary height. The script should ask you the cliff height and print something like this:

```
cliff height=   28.0, range= 57.00
```

Test your script for valleys/cliffs ranging from -30 m to 30 m.

2.11 Code examples for this chapter

```
for k=1:10
  fprintf('%i.  I''ll not throw airplanes in class\n',k)
end
% ==========  end of paperairplanes.m  ================
```

```

%Makes a conversion table between C and F.
for degC = 0:6
    degF=(9.0/5.0*degC)+32;
    fprintf ('%2d C = %4.1f F\n', degC, degF)
end
% 0 C = 32.0 F
% 1 C = 33.8 F ...
% 6 C = 42.8 F
% ==========  end of tempTable.m  ====================
```

```
%DOCUMENT
%Makes table of positions & velocities for proj. motion
%USAGE:  >> projectile
%DEFINE
g= 9.81; %g in m/s^2
y0=10; %m
v0y=22; %m/s
fprintf('Time \t Velocity \t  Altitude \n')
fprintf(' sec \t    m/s   \t      m    \n')
fprintf('==== \t ======== \t  ======== \n')
%DERIVE
timestep=0.2;
for t=0:timestep:10
    y = y0 + v0y*t - (1/2)*g*t^2;
    vy = v0y- g*t;
    %DISPLAY
    fprintf('%4.1f \t %6.1f \t %7.1f \n',  t, vy, y)
end
% ==========  end of projectile.m  ====================
```

```
g= 9.81; %g in m/s^2
y0=10; %m
v0y=22; %m/s
fprintf('Time \t Velocity \t  Altitude \n')
fprintf('==== \t ======== \t  ======== \n')
timestep=0.2;
for t=0:timestep:10
    y = y0 + v0y*t - (1/2)*g*t^2;
```

```
9       vy = v0y- g*t;
10      fprintf('%4.1f \t %6.1f \t %7.1f \n',  t, vy, y)
11      if (y < 0)
12          break
13      end
14  end
15  % ==========  end of projectileBreak.m  ===============
```

```
1   if (y < 0) && (vy < 0)
2   % ==========  end of projectilebreakexcerpt  ==========
```

```
1   %Converts Fahrenheit temperatures to other selected units.
2   F=input('Enter a temperature in Fahrenheit (e.g. 32) ');
3   outUnits=input('Scale to convert to? (C, K, or R) ','s');
4   outUnits=upper(outUnits);
5   absZeroK=273.16;
6   C=(5/9)*(F-32);
7   if outUnits=='K'
8       Tout=C+absZeroK;
9   elseif outUnits=='C'
10      Tout=C;
11  elseif outUnits=='R'
12      Tout=F+459.57;
13  else
14      error(['Undefined temperature scale ',outUnits])
15  end
16  fprintf('%6.2f Fahrenheit is %6.2f %s.\n',F,Tout,outUnits)
17  % ==========  end of temperatureConvert.m  ============
```

```
1   g= 9.81; y0=10; v0y=22;
2   fprintf('Time \t Velocity \t  Altitude \n');
3   timestep=0.2;
4   t=0; y=y0;
5   while ~(y<0)
6       y = y0 + v0y*t - (1/2)*g*t^2;
7       vy = v0y- g*t;
8       fprintf('%4.1f \t %6.1f \t %7.1f \n',  t, vy, y);
9       t=t+timestep;
10  end
11  % ==========  end of projectile2while.m  =============
```

```
1   % Makes a table of projectile motion in 1-D
2   g= 9.81; y0=-10; v0y=22;
3   fid=fopen('projectile_table.txt','w');
4   fprintf(fid,'%% Time \t Velocity \t  Altitude \n');
5   fprintf(fid,'%% ==== \t ========= \t  ========= \n');
6   for t=0: 0.2 :10
```

```
7     y = y0 + v0y*t - (1/2)*g*t^2;
8     vy = v0y- g*t;
9     fprintf(fid,'  %4.1f \t %6.1f \t %7.1f \n',  t, vy, y);
10    if (y < 0) && (vy < 0)
11      break
12    end
13  end
14  fclose(fid);
15  % ==========  end of projectile2Writefile.m  ==========
```

Chapter 3

Arrays, Matrices and Functions

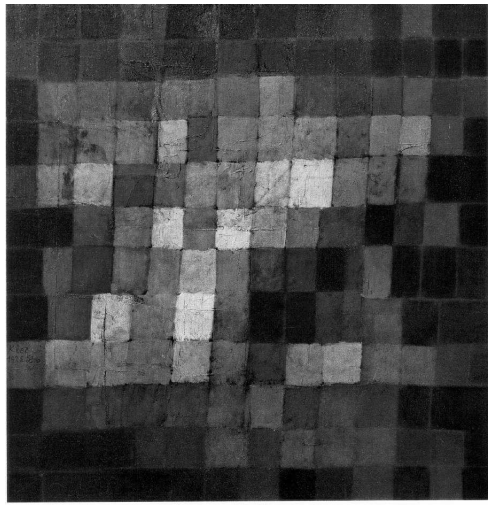


Figure 3.1: Everything in *Matlab* is a matrix.

3.1 Arrays and Matrices

The name *Matlab* is a shortened form of "Matrix Lab". The base code from which *Matlab* originally grew was a NASA *Fortran* library for handling matrices. Thus, everything in *Matlab* is a matrix. An ordinary variable like `x` (what we would call a *scalar* in physics) is considered by *Matlab* to be a `1x1` matrix. *Matlab* also allows what are called *arrays* in other languages. A 10-element array would be considered a `1x10` matrix in *Matlab*.

3.1.1 Arrays

We learned in section 1.11 that a variable in a programming language provides a human-friendly name for a binary location in computer memory where a number will be stored. When an array is defined, the computer assigns the same name to a set of memory locations, often memory locations that are adjacent to eachother in address space. All programming languages support arrays in one form or another. An array in programming is much like a sequence in mathematics.

3.1.1.1 Defining and referencing arrays

The interactive session below demonstrates defining and using arrays. Try entering these commands as you read.

```
1  >> b=[2:5]
2  b =  2.0000e+000     3.0000e+000     4.0000e+000
3       5.0000e+000
4
5  >> b=[2:0.1:2.3]
6  b =  2.0000e+000     2.1000e+000     2.2000e+000
7       2.3000e+000
8
9  >> b(1)
10 ans =  2.0000e+000
11 >> b(3)
12 ans =  2.2000e+000
13
14 >> b(7)
15 ??? Index exceeds matrix dimensions.
16 >> b(0)
17 ??? Subscript indices must be real positive integers or
18      logicals.
19
20 >> b=linspace(2,2.3,4)
21 b =  2.0000e+000     2.1000e+000     2.2000e+000
22       2.3000e+000
23 >> b=linspace(2,5,4)
24 b =  2.0000e+000     3.0000e+000     4.0000e+000
25  5.0000e+000
26 % ========== end of arrays1.txt ======================
```

On line 2 above, the variable **b** is seen to contain 4 numbers. On line 5 above, **b** has been redefined; now it contains four different numbers, from 2.0 to 2.3. The syntax of lines 1 and 4 should be somewhat familiar. You used the colon character in **for** loops in chapter 2. `2:5` means all the integers between 2 and 5 (inclusive). In analogy to **for** loops, the construction `2:0.1:2.3` on line 4 creates an array of numbers between 2 and 2.3 in increments of 0.1. Another way to

define arrays is with the **linspace** function. Line 17 is equivalent to line 4, and line 19 accomplishes the same function as line 1.

In ordinary mathematical notation, one could define the set **b** of all integers from 2 to 5 inclusive and it would produce the same result as line 2 above. In mathematics, if you wanted to refer to the third element of the sequence **b**, you would probably use the notation b_3. Programming languages are the same, though most (including *Matlab*) refer to b_3 as b(3). You can see on lines 7 and 9 that b_1 and b_3 have the values you would expect based on lines 4 and 5. On line 12 and 14, we ask for b_7 and b_0. Neither of these are defined for the array **b**, resulting in error messages on lines 13 and 15.

3.1.1.2 Using the diary() function to log your exploration

One of the advantages of *Matlab* is that you can experiment at the command line before needing to write a *script* or a *function*. You can even create a log of your experiments using the **diary()** function. To create a diary called *'experiment1.txt'* type `>> diary('experiment1.txt')`. You will see the file appear in your current directory. Thereafter, anything you type at the command line, as well as all the results that the computer prints as a result of what you have typed, will appear in the file *experiment1.txt*. When you are finished with your experiment, type `>> diary off` and the experiment will be saved. You can easily edit the diary file if you want to delete the mistakes you made during your interactive session. In general computing terms, this sort of file is called a *session log*, but *Matlab* calls it a *diary*.

3.1.2 Row and Column Arrays (1-D Matrices)

Note that the error message on line 13 above uses the word "matrix". As previously stated, *Matlab* treats arrays like **b** as one-dimensional matrices. In *C* and many other languages, one can define an array of N numbers, but it does not make sense to ask if it is a row or a column array. However, since *Matlab* is aware of matrices, it distinguishes between a row of numbers ($1 \times N$ matrix) and a column of numbers ($N \times 1$ matrix). For almost all of what we do in this course, it will not matter much whether the arrays we use are row or column matrices. However, if row and column arrays are mixed together in the same expression, one either gets errors or surprises.

In addition to defining arrays using a colon or using **linspace**, they can be defined directly. A one row array is defined with spaces or commas between elements as follows:

```
1 >> a = [1  17  -3445  2*pi];
2 >> a
3 a =  1.0000e+000      17.0000e+000      -3.4450e+003      6.2832e+000
4
5 >> a(3)
6  -3.4450e+003
```

A one column array is defined using semi-colons between the elements like this:

```
1  >>  a = [1; 17; -3445; 2*pi];
2  >> a
3  a =
4       1.0000e+000
5       17.0000e+000
6       -3.4450e+003
7       6.2832e+000
8
9  >> b = a'
10 b =  1.0000e+000    17.0000e+000    -3.4450e+003    6.2832e+000
```

Note that *Matlab* prints row arrays in a row, and column arrays in a column. One can turn row arrays into column arrays with the transpose operator. In line nine above, a single quote in the expression `b = a'` sets **b** equal to the *transpose* of matrix **a**. *Transpose* is a standard term in matrix math; it means exchanging the rows for the columns of a matrix.

3.1.2.1 Managing matrices with the size() function

```
1  >> r1=[1 2 3];r2=[4 -3 7];
2
3  >> size(r2)
4  ans =  1    3
5
6  >> r3=r2'
7  r3 =      4
8           -3
9            7
10
11 >> size(r3)
12 ans =  3    1
13
14 >> r1*r2
15 ??? Error using ==> mtimes
16 Inner matrix dimensions must agree.
17
18 >> r1*r3
19 ans =     19
20 % ==========  end of vectors3.txt  ====================
```

As before arrays r_1 and r_2 are defined. The built-in **size()** function gives the size of the array r_2 as 1 row by 3 columns. On line 6, a new array r_3 is created by taking the *transpose* of r_2. *Matlab* shows it as a column array when it prints it out on lines 7:9. The **size()** function on line 11 confirms that r_3 is now 3 rows by 1 column. Whereas `r1*r2` fails with an error, `r1*r3` evaluates to

the same value as a dot product. (By the rules of matrix multiplication, a row-matrix multiplied by a column-matrix *is equivalent to* a dot product.) It is worth mentioning here that whereas **size()** tells you rows and columns of a matrix or vector, the command **length()** gives you the largest dimension of an array. Thus a `1x10` and a `10x1` array would both have a length of 10.

3.1.3 Using Arrays to represent vectors

A familiar physics application in which you might use an array would be to store the `(x,y,z)`-components of a vector. In this case you would use a 3-element array. A quick session log will demonstrate key features of vector handling using position vectors.

```
1 >> r1=[1 2 3];r2=[4 -3 7];
2 >> delta_r=r2-r1
3 delta_r = 3 -5 4
4
5 >> r1+r2
6 ans = 5 -1 10
7
8 >> 2*r1+r2
9 ans = 6 1 13
10 % ========== end of vectors.txt ======================
```

Notice that lines 1-3 of the script are equivalent to the physics notation: $\vec{r}_1 = 1\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{r}_2 = 4\hat{i} - 3\hat{j} + 7\hat{k}$, $\Delta r = \vec{r}_2 - \vec{r}_1$. Lines 5, 6 show that you can add vectors with the same notation we already used for scalars. Lines 8, 9 show you can multiply vectors by constants.

```
1 >> r1=[1 2 3];r2=[4 -3 7];
2 >> dot(r1,r2)
3 ans = 19
4
5 >> cross(r1,r2)
6 ans = 23 5 -11
7
8 >> norm(r1)
9 ans = 3.7417
10
11 >> r1_hat=r1/norm(r1)
12 r1_hat = 0.2673 0.5345 0.8018
13
14 >> norm(r1_hat)
15 ans = 1
16
17 >> r2_hat=r2/norm(r2)
18 r2_hat = 0.4650 -0.3487 0.8137
19
```

```
20 >> r3_hat=cross(r1_hat,r2_hat)
21 r3_hat =     0.7146     0.1553    -0.3418
22
23 >> norm(r3_hat)
24 ans =     0.8072    %Why is this not 1?
25 % ========== end of vectors4.txt ======================
```

On lines 2 and 5 we see that *Matlab* has built-in dot-product and cross-product functions. The function **norm()** (line 8) calculates the magnitude of a vector. Thus we can easily turn $\vec{r}_1$ into $\hat{r}_1$ (line 11), and as one would expect, the **norm()** of a hat vector is 1 (line 15). On line 24 we take the **norm()** of $\hat{r}_1 \times \hat{r}_2$ and see that it is less than 1. Brief reflection should help you explain this little puzzle.

3.1.3.1 Finding the angle between vectors of arbitrary dimension

In crystallography it is not unusual to need to find the angle between two vectors that are hard to visualize. Consider first $\vec{r}_1 = 2\hat{i}$ and $\vec{r}_2 = 1\hat{i} + 1\hat{j}$. Clearly the angle between these vectors is 45°. Now let us keep $\vec{r}_1$ the same but add a third dimension so that $\vec{r}_2 = 1\hat{i} + 1\hat{j} + 1\hat{k}$. Immediately it becomes difficult or impossible to visualize the geometry and calculate the angle between the vectors. The dot-product can help us.

$$\begin{aligned} \vec{r}_1 \cdot \vec{r}_2 &= |r_1||r_2|cos(\theta) \\ \vec{r}_1 \cdot \vec{r}_2 &= x_1x_2 + y_1y_2 + z_1z_2 \\ \therefore cos(\theta) &= \frac{x_1x_2 + y_1y_2 + z_1z_2}{|r_1||r_2|} \end{aligned}$$

We see that combining the two equivalent definitions of dot product allows calculating the cosine of the angle (and thus the angle) between two arbitrary vectors, even if we cannot visualize them. This trick may be generalized to an arbitrary number of dimensions. Imagine two vectors in six-dimensional space with components `(x,y,z,p,q,r)`. The equations above become.

$$cos(\theta) = \frac{x_1x_2 + y_1y_2 + z_1z_2 + p_1p_2 + q_1q_2 + r_1r_2}{|r_1||r_2|}$$

To solve for θ, know that the inverse cosine function in is called **acos()**.

3.1.4 Using Arrays to represent sets of measurements

We just used a three-element array to represent a vector, but vectors in physics are not necessarily limited to three elements, and arrays in computer languages are definitely not limited. You can, and probably will, create an array with a million elements. Physicists running programs on supercomputers might create arrays

of a billion, or even a trillion elements, up to the memory limit of the computer. What would you do with such a large array? Usually it does not represent a vector, instead the array elements often represent a set of measurements, or calculations. Imagine you had measured the x-position of an object once per second between zero and eight seconds. You would probably want to store those nine positions in a nine-element array called **x**, and the nine times at which you measured position in a separate array **t**. Here is an example with the familiar kinematic formula for a ball thrown straight up in the presence of gravity. In physics we would write $y = \frac{1}{2}gt^2 + v_0t + y_0$. In *Matlab* we do almost the same thing:

```
1  >> t=0:8; g=-9.8; v0=40; y0=20; y=0.5*g*t.*t+v0*t+y0
2
3  y =
4     Columns 1 through 5
5      20.0000    55.1000    80.4000    95.9000   101.6000
6     Columns 6 through 9
7      97.5000    83.6000    59.9000    26.4000
8
9  >> t=0:8; g=-9.8; v0=40; y0=20; y=0.5*g*t*t+v0*t+y0
10 ??? Error using ==> mtimes
11 Inner matrix dimensions must agree.
12
13 >> t=0:8; g=-9.8; v0=4; x0=1; y=0.5*g*t.^2+v0*t+y0
14 y = 1.0000     1.4714     1.7429     1.8143     1.6857
15     1.3571     0.8286     0.1000
16 % ==========   end of vectors2.txt   ====================
```

Notice that on line one we define **t**, **g**, **v0**, **x0**, and **x** all at once. This was mostly just to save space in the book, and it would be better practice to have taken five lines, but *Matlab* allows you to put multiple statements on the same line, so long as they are separated by semi-colons. Because *Matlab* operates on all elements of the array at once, the result for **y** on lines 3–7 are the altitudes of the ball corresponding to each time 0–8 seconds. There is something else to notice on line 1, the construction `t.*t` . We are trying to do t-squared, but what does the period in the expression mean? The period is there because, while addition and subtraction of arrays is unambiguous, multiplication and division are less clear. If adding two arrays **r** and **s**, *Matlab* assumes you want to add the first element of **r** to the first element of **s**, the second element of **r** to the second of **s** and so forth. (The arrays **r** and **s** need to be the same length for this to work, and *Matlab* gives an error if they are not.) Turning to multiplication; `t*t` might mean **t** dotted with itself. It might mean a cross product. It might mean matrix multiplication, or it might mean to multiply the first element of **t** by itself, then the second, and so forth, as is done with addition. To clarify the situation, the `.*` notation is used. It explicitly tells *Matlab* to do element-by-element multiplication (or division, or exponentiation).

On line 9, the period is intentionally left out. *Matlab* now interprets the multiplication sign to mean matrix multiplication. The error comes because matrix-multiplication requires one of the participants to be a row-vector, while the other needs to be a column vector.

We defined times between zero and eight seconds via `t=0:8` . What if we want more times, and we do not only want integers? Line 13, we solve the same problem another way. In this case, we are interested in times between zero and one second. The notation `t=0:1` is not helpful in this case as **t** would only be zero and one. The **linspace** function allows one to specify the start and stop time, and the third argument requests an array of 8 equally spaced points between these two times. Also, line 13 uses the notation `t.^2`, which is equivalent to `t.*t`.

3.1.5 Comparing vectorized calculations to element-by-element calculations

In Chapter 2 we discussed **for** and **while** loops. These can be used to do calculations on arrays, and have the advantage that the **for** construction is almost universal in computer languages. However, *Matlab*, a language designed to work with arrays (data sets) provides generally much shorter and simpler ways of doing calculations on arrays (what I call *vectorized* methods). Some examples should make this clear.

3.1.5.1 Initialization

It is important when defining a new variable to give it a value before you use it (even if you may change it later). This prevents errors or unexpected results. When defining an array, it is also important to set every value of an array to a number (for example, zero). This process is called *initialization*. The **for** loop below sets every element of a 10-element array to zero.

```
1 %element by element initialization;
2 for i=1:10
3   a(i)=0;
4 end
5
6 %vectorized initialization;
7 a=zeros(1,10);
```

In contrast to the element-by-element approach of the **for** loop, the vectorized **zeros()** function sets the entire array **a** to zero at once.

3.1.5.2 Assignment

It is common to set one array equal to another (or to some part of another). In any language this can be done in an element-by-element fashion, as below:

```
1 %element by element assignment;
2 k=4
3 for i=1:7
4   a(i)=b(i+k-1);
5 end
6
7 %vectorized assignment;
8 a=b(4:11);
```

Note that the first time through the loop, **a**(1) is set to **b**(4), next time through **a**(2) is set to **b**(5), and finally **b**(11) is assigned to **a**(7). Meanwhile, the vectorized form of assignment on line 8 accomplishes the same task in a single line of code.

Imagine having three arrays **a**, **b**, and **c**. The goal is to multiply each element of **b** by the corresponding element of **c**, then add the correct element of **a** to each term, and finally to take the square root of the result. First is shown the element by element approach, then the vectorized approach.

```
1 %element by element calculation;
2 for i=1:length(b)
3   d(i)=sqrt(a(i)+b(i)*c(i));
4 end
5
6 %vectorized calculation;
7 d=sqrt(a+b.*c);
```

Note on line 2 the loop upper-limit is `length(b)`. That is a nice trick to tell the loop how many times to run without having to explicitly type in the length of the array **b**. For the vectorized approach, *Matlab* knows the vector lengths and (so long as they are the same length) accomplishes the job without any extra work. Note also that many of the functions in *Matlab*, (e.g. **sqrt()**, **cos()**, **exp()**) can operate on an array all at once without being told how to do it.

Having seen that *Matlab* supports a vectorized approach, you might wonder why you would ever do things another way? First, as stated, the element by element approach works in any computer language. Second, there are types of calculations (beyond arithmetic) that are involved enough that the vectorized method either will not work or might be harder for you to figure out how to code.

3.1.6 More documentation about arrays and matrices

I have provided an introduction, and hope I have showed you how arrays in particular may be useful in physics, but more detail is needed. I recommend that you open the *Matlab* Getting Started documentation and scan the section called *Matrices and Arrays*.

3.2 User defined functions

Here we introduce functions that you can create yourself. These are a feature of most programming languages and make the language "extensible". That is to say, you can extend it to meet your needs. In fact, many of the functions (like **fprintf()**) that seem to be built in to *Matlab* are actually user-defined functions that were written back at the Mathworks and included in the *Matlab* distribution. [1]

From the *Matlab Editor*, if you select *New File / Function* a window will open and you will see the following:

```
function [ output_args ] = Untitled( input_args )
%UNTITLED Summary of this function goes here
%   Detailed explanation goes here
end
```

Matlab is helping you set up a function so that you get the syntax correct. Now let us write our own function that does something slightly useful and physics-related. Note that I do not have to explain what the function does, because all the comments explain it for me. This is proper procedure when writing code.

The function *mass()* takes two arguments. As you might imagine, you can define the function to have as many arguments as you like. You may also use arrays and matrices as arguments. *Matlab* does not care. You do not have to do anything special to declare arrays or matrices. The function *returns* one value, **m**. Note that in the *USAGE* section I assign mass to a variable called *spheremass*. I could just as well have called it *joe*. *Matlab* does not care what you call either the input or output arguments when you call the function. It recognizes by position on the line which value is what. Of course, within the function, if you want to return **m** you must calculate **m** in the function.

```
function [ m ] = mass( radius, density )
%Calculates mass of a sphere of known radius & density.
%  radius (meters), density (kg/m^3)
%  mass (kg)
%USAGE spheremass=mass(3.7E-2,1000)
% >>  spheremass = 0.2122

% This comment line will not appear as help because it
% follows a blank line.
volume=4/3*pi*radius^3;
m=density*volume;
end
% ==========  end of mass.m  ============================
```

[1] This is true of other languages like *C* as well. *C* uses function libraries such as *stdio* and *math* in order to be portable between different types of computer

3.2.1 Help on User Defined functions

Matlab helps you take advantage of your own documentation. If you type "help mass" you will see that all of the comments you wrote in the file are shown to you. (Note that once you leave a blank line, the following comments are assumed to be only comments, and not help, and will not be shown.).

Also, the very first line of a file (after the function definition) is called by *The Mathworks* the *H1-line*. When you click on a function in the *Matlab integrated development environment*, you will see the function name, its arguments, and the H1-line.

3.2.2 Difference between a script and a function

Having defined a function, it is important to mention the subtle differences between a *script* and a *function*. There is one obvious difference in syntax; a *script* does not take any arguments (there are no parenthesis). Also, a *function* starts with the keyword *function*, whereas a *script* just starts with its first command (or a comment).

This obvious syntax difference indicates a more subtle difference in use, which gets to the universal programming concept of variable *scope*. We could create ten different functions that used the variable **v** (which might mean velocity) and setting a value for **v** in one function would not affect its value in any other function. The variables used inside a function are said to have *local scope*.

The fact that the variables inside one function cannot affect another is very powerful and greatly reduces confusion. Imagine if you did have 10 different functions that used **v**. You would have to be very careful that you called them in the right order and kept track of how each one was changing **v**.

If you really do intend for the variable **v** to be shared between different programs, you can write a *script* rather than a *function*. If you have ten different scripts that use **v**, they all refer to the same variable **v**. Changing **v** in one script results in a different value when the next script is called. In scripts, ALL the variables used have *global scope*. The advantage of a script is that you do not have to explicitly return values you want to use. This is also the disadvantage of a script. For very simple problems, or for variables that have unusual names and are unlikely to be used twice, you can use a *script*. As soon as you work on a more complicated problem, a *function* is safer.

One trick that works reasonably well is to have a *Master* script that calls a bunch of different functions. This is reasonably safe (unless you decide to create two master scripts and run them in the same *Matlab* session.)

A final difference between a *script* and a *function* is that you can test every line of a *script* at the *Matlab command line* before making an *m-file*. You can test almost every line of a *function*, but you cannot test the *function* command itself. This is related to the *local scope* of a *function*. Since it's variables cannot exist

at the *Matlab command line*, it cannot exist either. Try creating a function at the *Matlab command line*, and you will see the error it causes.

3.3 Review of commands introduced in this chapter

For more information (and to see the related commands not discussed here) on each command type `>>help cmdname` at the *Matlab command line*.

acos() Inverse cosine, for angle in radians.

dot() dot(a,b) gives dot product of vectors **a**, and **b**. **a** and **b** must be equal length.

cross() Vector cross product

function() Add new function.

length() Gives length of longest dimension of an array or matrix.

linspace() Linearly spaced vector.

norm() Gives magnitude of a vector.

size() Size of array or matrix.

transpose() Interchanges row and columns of a matrix.

zeros() Sets an array (or matrix) of defined size to zero.

3.4 End of Chapter Problems

(1) Here you will experiment with *Matlab* at the command line and record your experiments in a file called *problem1.txt* using the **diary()** function (See section 3.1.1.2). Experiment all you like. Please clean up your diary file before you submit it so that it contains *only* the requested experiments in the order given below.

(a) Create a row array **a** of the integers between -42 and -35

(b) Add 10 to every element of **a**.

(c) Create a column array **b** of the same integers in **a**.

(d) Create a row array **c** of 32 numbers between pi/4 and pi/2 with **linspace**.

(e) Create **d** consisting of the numbers from 3 to 6 counting by 0.3

(f) Set **e**=**d**(5) and **f**=**d**(7)

(g) Set **e**= the 3rd through the 6th elements of **d**.

(h) Set **f**= the 5th through the 8th elements of **d**, and add it to **e**.

(i) Look up the random number function and use it to generate a row vector of 10 random numbers.

(2) Here are more command line experiments which will go in the diary *problem2.txt*

(a) Create $\vec{r}_1 = 5\hat{i} + 12\hat{j}$ and get its magnitude with the **norm()** function.

(b) Get the magnitude of $\vec{r}_1$ by multiplying it by its transpose and taking the **sqrt()** of this result.

(c) Repeat the previous but put the transposed vector first. You will get a very different result. Matrix multiplication is usually not commutative.

(d) Create $\vec{r}_2 = 0\hat{i} + 2\hat{j} + 0\hat{k}$. $\vec{r}_3 = \hat{i} + \hat{j}$. Calculate $\vec{r}_2 \cdot \vec{r}_3$ using the **dot()** function and using vector multiplication.

(e) Calculate $\vec{r}_2 \times \vec{r}_3$.

(f) Calculate the cosine of the angle between $\vec{r}_2$ and $\vec{r}_3$ using technique from section 3.1.3.1.

(g) Calculate the angle between $\vec{r}_2$ and $\vec{r}_3$ in degrees using appropriate *Matlab* commands.

(h) Repeat the previous two steps to find the angle between $\vec{r}_1$ and $\vec{r}_3$.

(3) Here is the first line of a function to calculate `F`, the force of attraction between two masses.
function F = gravity(m1,m2,R).
Finish the function, including the documentation. Rewrite example 1-A in section 1.8 to make use of your function.

(4) Rewrite the **gravity()** function so that it can handle *m1* and *R* as arrays and give a result which is *F* as an array. Also write a test script that calculates your weight in pounds from your mass in kilograms simultaneously on Earth, Mars, Jupiter, and the Sun. (Jupiter and the Sun do not really have solid surfaces, but they do have nominal radii and masses, which are easy to look up. Ignore "engineering problems" like the lack of solid surface on Jupiter and the sun and the rapidity with which you will vaporize on the sun.)

(5) Write a function to calculate the angle (in degrees) between any two arbitrary vectors using the technique from the section 3.1.3.1. Also write a test script which tests the function on the following:

(a) [2 0 0] and [1 1 0]

(b) [2 0 0] and [1 1 1]

(c) Two randomly generated 1x10 vectors

Note: The randomly generated vectors must be randomly generated (and different) every time you run the script.

(6a) Here is the definition of a function to calculate whether number **a** is divisible by number **b**:

```
function result = isDivisible(a,b)
%Example: isDivisible(69,23)
%           ans = 1
%Example: isDivisible(69,22)
%           ans = 0
```

Write the "guts" of the function. (Divisibility checking was discussed in section 2.7.2). Test the function with a calling script that asks the user for numbers **a** and **b**, then contains the following construction:

```
if isDivisible(a,b)
      fprintf('%d is divisible by %d',a,b)
else
      fprintf('%d is NOT divisible by %d',a,b)
end
```

(6b) Ask the user for a number and state whether or not it is prime. To do the divisibility checking, your script should call the function **isDivisible()** which you wrote in the previous problem. Your script *must complete, in under 10 seconds* for all of the following numbers, 2, 3, 7, 9, 19, 6151, 6153, 7919, 987654321, 987643211, 98765432111.

Note: *Matlab* has a built-in primes function. You may refer to it or use it to check your work, but you cannot use it in this exercise.

Hint: The blind way to do this is to check the number N by all possible divisors from 1 to N-1. Clearly this is not necessary. For example, there is no point checking for divisibility by even numbers. Further, you can stop checking far before you get to N-1. If you think about the greatest possible factor for a number (other than itself) you will see how far you need to go.

(7) Write the guts of a function with the following header and help.

```
function time = timeAloft(v0,theta)
%timeAloft returns the time in seconds that a projectile is
%airborne for a given v0 and theta.  It is assumed the
%projectile leaves from the ground and lands on the ground.
%v0=Initial speed (m/s)
%theta=launch angle (degrees above horizontal)
```

Write a script that calls this function and prints the results for launch velocities of 5 and 50 m/s and launch angles of 10°, 40°, 60° and 80°. Solve the problem analytically for the 50 m/s and 40° case and check that your function works.

(8) Write the guts of a function with the following header and help.

```
function time = timeAloft(v0,theta,zi,zf)
%timeAloft returns the time in seconds that a projectile is
%airborne for a given v0 and theta.  It is assumed the
%projectile leaves from the ground at altitude zi and lands on gr
%with altitude zf.
%v0=Initial speed (m/s)
%theta=launch angle (degrees above horizontal)
%zi, zf=z_initial and z_final (m above sea-level)
```

Write a script that calls this function and prints the results for launch velocities of 5 and 50 m/s and launch angles of 10°, 40°, 60° and 80°. Solve the problem analytically for the 50 m/s and 40° case with zi=100 m and zf=140 m.

3.5 Code examples for this chapter

```
>> b=[2:5]
b =  2.0000e+000   3.0000e+000   4.0000e+000
     5.0000e+000

>> b=[2:0.1:2.3]
b =  2.0000e+000   2.1000e+000   2.2000e+000
     2.3000e+000

>> b(1)
ans =  2.0000e+000
>> b(3)
ans =  2.2000e+000

>> b(7)
??? Index exceeds matrix dimensions.
>> b(0)
??? Subscript indices must be real positive integers or
    logicals.

>> b=linspace(2,2.3,4)
b =  2.0000e+000   2.1000e+000   2.2000e+000
     2.3000e+000
>> b=linspace(2,5,4)
b =  2.0000e+000   3.0000e+000   4.0000e+000
5.0000e+000
% ========== end of arrays1.txt  ======================
```

```
>> r1=[1 2 3];r2=[4 -3 7];

>> size(r2)
ans =  1   3

>> r3=r2'
r3 =     4
        -3
         7

>> size(r3)
ans =  3   1

>> r1*r2
??? Error using ==> mtimes
Inner matrix dimensions must agree.

```

```
>> r1*r3
ans =     19
% ==========  end of vectors3.txt  ====================
```

```
>> r1=[1 2 3];r2=[4 -3 7];
>> delta_r=r2-r1
delta_r = 3 -5  4

>> r1+r2
ans =   5  -1  10

>> 2*r1+r2
ans =   6  1  13
% ==========  end of vectors.txt  ======================
```

```
>> r1=[1 2 3];r2=[4 -3 7];
>> dot(r1,r2)
ans =  19

>> cross(r1,r2)
ans =     23      5    -11

>> norm(r1)
ans =     3.7417

>> r1_hat=r1/norm(r1)
r1_hat =     0.2673     0.5345     0.8018

>> norm(r1_hat)
ans =      1

>> r2_hat=r2/norm(r2)
r2_hat =     0.4650    -0.3487     0.8137

>> r3_hat=cross(r1_hat,r2_hat)
r3_hat =     0.7146     0.1553    -0.3418

>> norm(r3_hat)
ans =     0.8072    %Why is this not 1?
% ========== end of vectors4.txt  ====================
```

```
>> t=0:8; g=-9.8; v0=40; y0=20; y=0.5*g*t.*t+v0*t+y0

y =
   Columns 1 through 5
    20.0000   55.1000   80.4000   95.9000  101.6000
   Columns 6 through 9
```

```
    97.5000   83.6000   59.9000   26.4000

>> t=0:8; g=-9.8; v0=40; y0=20; y=0.5*g*t*t+v0*t+y0
??? Error using ==> mtimes
Inner matrix dimensions must agree.

>> t=0:8; g=-9.8; v0=4; x0=1; y=0.5*g*t.^2+v0*t+y0
y =  1.0000    1.4714    1.7429    1.8143    1.6857
     1.3571    0.8286    0.1000
% ==========  end of vectors2.txt  ====================
```

```
function [ m ] = mass( radius, density )
%Calculates mass of a sphere of known radius & density.
%  radius (meters), density (kg/m^3)
%  mass (kg)
%USAGE spheremass=mass(3.7E-2,1000)
% >>  spheremass = 0.2122

% This comment line will not appear as help because it
% follows a blank line.
volume=4/3*pi*radius^3;
m=density*volume;
end
% ==========  end of mass.m  ============================
```

Chapter 4

Working with Scientific Data

Figure 4.1: An instrumented weather balloon provides real-world position vs. time data

4.1 Working with real data

4.1.1 Reading and Writing Data files

In discussing arrays in the previous section it was pointed out that they are very useful for working with sets of scientific data. Thus it is time to work on scientific data. You have already learned how to write data to a file, the hard way, using **fprintf()**. It was worth learning because it is the technique that provides the most control about how the data will look and is also pretty general and works in a similar way for other programming languages. However *Matlab* provides several simpler ways to read and write data files. Among these are **load()** and **save()**. Both of these methods are vectorized, meaning they work on the whole data set at once rather than line-by-line as did **fprintf()**.

Scientific data often comes in the form of a table of text. Below is the beginning piece of a data file taken from the custom balloon-based instrument you see being launched in the cover photo for this chapter.

```
1  %GPS Data: 2003-07-25 balloon flight from Langmuir Lab
2  %   LONG        LATITUDE     ALT     TIME
3  %   (deg)         (deg)      (m)      (s)
4   -107.188606  33.982315  03249.5  1708
5   -107.188408  33.982296  03302.0  1717
6   -107.188431  33.982269  03306.7  1718
7   -107.188477  33.982182  03351.9  1727
8   -107.188416  33.982212  03357.1  1728
9   -107.188194  33.981983  03407.1  1737
10  -107.188217  33.981918  03411.6  1738
```

Our initial interest is in the altitude (column 3) and the time (column 4). The table also has latitude and longitude giving the location of the balloon above the earth. For now, this is extraneous data. Scientific data often contains extraneous data, and sometimes, missing or erroneous data. It is the task of the data analyst to figure out how to exclude the invalid data.

The script below reads the data file in, selects the variables of interest and writes it back out.

```
1 %selectData.m (selecting and saving two columns of data)
2 a=load('bflight1_20030725_mod.pos');
3 time=a(:,4);
4 alt=a(:,3);
5
6 clean_data=[time alt];
7 save('bflight1_20030725_cln.txt','clean_data','-ascii');
8 save('bflight1_20030725_cln','clean_data');
```

The second line reads the file and loads all the data into the matrix **a**. Every column of the data file becomes a column of the matrix. Line 3 assigns the fourth

column of the matrix to an array called `time`. Line 4 assigns the third column to the array `altitude`. (A fuller explanation of the notation used in lines 3 and 4 is provided in section 4.1.2.)

Line 6 takes the two column-arrays `time` and `altitude` and slaps them back together to make a new matrix (The name `cleandata` was chosen to describe the matrix; it contains only the data of interest and nothing extraneous). The **save()** on line 7 saves the data to a file (the filename is given first). The second entry, `cleandata` tells **save()** which data we want to save. The entry `'-ascii'` tells the **save()** function to write the data as a text file readable by humans.

Line 8 saves the same data in a different format, called a `mat-file`. One does not generally need to save the same data twice. It was only done here to illustrate two different ways to save data and to allow you to compare the data files created. The `ls -lh bflight*` below stands for
`list -Long -Humanreadable all files starting with bflight.` [1]

```
1 %============ls_datafiles.txt=============
2 >> ls -lh bflight*
3 -rw-r--r--  18K 2011-09-09 3:25 bflight1_20030725_mod.pos
4 -rw-r--r--  17K 2011-09-09 9:57 bflight1_20030725_cln.txt
5 -rw-r--r-- 2.7K 2011-09-09 9:57 bflight1_20030725_cln.mat
```

Note in the list of files that the original data file on line 2 was 18-kilobytes long. The *clean* data file is 17-kilobytes long, while the `mat-file` occupies a mere 2.7-kBytes. The reason for the difference in size is that the `mat-file` is stored in a compact binary format.

The text files use ASCII code, which is specifically a code for representing letters/numbers requiring 1-byte per letter or digit. The *clean* data file shown below has 31 characters on a line (spaces count as characters in ASCII).

```
1 %== bflight1_20030725_clean_excerpt.txt    ==
2     1.3970000e+03    3.2338000e+03
3     1.3980000e+03    3.2339000e+03
4     1.4070000e+03    3.2344000e+03
5     1.4080000e+03    3.2345000e+03
6     1.4170000e+03    3.2351000e+03
```

In addition, it has 1 or 2 invisible characters that separate the lines (The invisible line-break character is the same one that you represent as `\n` in **printf()** commands.) There were 514 lines, times 32 bytes per line, giving almost 17000 bytes, which is how big the file appears to be.

Contrast the appearance of the text file with the first few lines of the `mat-file`.

```
1 %=====bflight1_20030725_clean.mat========
2 MATLAB 5.0 MAT-file, Platform: GLNX86, Created on: Fri
3 Sep  9 10:40:50 2011  ^@^AIM^O^@^@^@ @  ^C^@^@x<9c>-<94>{
4 x<8e>u^\  _                                V     B  <85><9c><8a><8c>.  T
```

[1] Though ls is a native *Unix* command, *Matlab* also supports it on *Windows*.

```
5 4 <9c>I^X^]<8c> X ^ Z 6 <9c><86>( Z 3 ( 0 m h ^D<9c><83>
6 p^M^<87> 7 (^D<8c> 0 <84>^ N F ^ P 6 ^ B a g <82> 7
7 ^X<93>p$<8e> ^ N N ^Op^ Z l <9c><83>^ Y ^P<97> 2 u 5
8 X<88> q ^ O C x ^ X 4 ^ K ^ G V ] d ^ T ^TM5(<86> P ] j L M %
9 ^TG^] ^[ u ^D C <83>(<91><92> i (^ M 1 4 <96> Q
```

In sum, `mat-files` have the advantage of being small, while ASCII files have the advantage of being readable by humans. With modern computers, any file less than a megabyte could be considered small and thus worth the convenience of using ASCII. As files get into the Gigabyte range, the relative compactness of binary is worth the inconvenience of having to work with raw binary data.

4.1.1.1 About Metadata

The **load()** function is simple to use, but it is easily upset. Note the first three lines of the original data file contain useful information explaining what the columns of data mean. In science, this is referred to as *meta-data*, or the data that tells you what the numbers mean. Meta-data is exceedingly important; only a couple of decades past, scientists recorded their data in lab-books and explained adjacent to the log entries what the data meant. Now that almost all scientific data is acquired by and kept on computers, it is important that the file itself explain the data. Thus the first three lines of the file that is our current study are important, as are the `%` signs that begin them. Try deleting the `%` signs on any of the lines of metadata and using the **load()** function again; note that it complains loudly:

```
??? Error using ==> load
Number of columns on line 1 of ASCII file
/home/richard/work/mat_mechanics/bflight1_20030725_mod.pos
must be the same as previous lines.}
```

Without the `%` signs, *Matlab* thinks you want it to interpret the metadata text, and it does not know how. The `%` signs tell *Matlab* that your explanatory text is just comments and should be ignored. You might recall that in tables you generated for homework, your lines of text started with `%`. Now you know why. Doing so allows *Matlab* to read your tables back in, while also reminding you what they contain.

4.1.2 Working with matrices

When you use **load()**, you generally end up with a matrix. It is thus necessary to learn a little bit about how to manipulate matrices. We will practice on the first six lines of ballooning data, shown again below for easy reference.

```
1 %=== practice.txt (sample ballooning data) =====
2  -107.188606  33.982315  03249.5  1708
3  -107.188408  33.982296  03302.0  1717
4  -107.188431  33.982269  03306.7  1718
5  -107.188477  33.982182  03351.9  1727
6  -107.188416  33.982212  03357.1  1728
7  -107.188194  33.981983  03407.1  1737
```

First we will learn how to address individual matrix elements, or groups of elements.

```
1 % ======matrix1.txt (selecting matrix elements)=========
2 >> a=load('practice.txt');
3 >> a(2,3)
4 ans = 3302
5
6 >> a(1,1:3)
7 ans =    1.0e+03 *    -0.1072     0.0340     3.2495
8
9 >> a(1,[2,4])
10 ans =    1.0e+03 *     0.0340     1.7080
11
12 >> a(1,[2:4])
13 ans =    1.0e+03 *     0.0340     3.2495     1.7080
14
15 >> a(1,1:5)
16 {??? Index exceeds matrix dimensions.}
17
18 >> a(2,:)
19 ans =   1.0e+03 *   -0.1072    0.0340    3.3020    1.7170
```

We can address any single matrix element as on lines 2 and 3 above: `a(2,3)` means 2nd row, 3rd column. The expression on line 6 means *first row, elements 1 through 3*. Note that *Matlab* jumped to scientific notation on line 7. The `1.0+e03` multiplier applies to the whole row of numbers[2]. Line 9 requests only the 2nd and 4th elements on the 1st row of the matrix, whereas line 12 requests the 2nd *through* the 4th elements, reminding us that minor changes in punctuation give significantly different results. On line 15, the first 5 elements of the first row are requested; and the error on line 16 says there are only 4 available. It is often the case that we want an entire row or column of a matrix. Line 18 shows a shortcut. `a(2,1:4)` would have given us the entire 2nd row, but requires knowing that the row is 4 elements wide. The shortcut `a(2,:)` tells *Matlab* to give you the entire 2nd row without having to specify how many elements wide the row might be.

Below we use the `:` shortcut to cut up and graft together matrices at will.

[2] *Matlab* sometimes makes the decision to show results in scientific notation at the command line. This is automatic. You could override it if you used **fprintf()**.

```
% ==================matrix2.txt ========================
% Demonstrate selecting/reassembling matrix rows/columns
>> long = a(:,1)
long = -107.1886
       -107.1884
       -107.1884
       -107.1885
       -107.1884
       -107.1882

>> lat = a(:,2);

>> latandlong=[lat' ; long']
latandlong =   33.9823   33.9823   33.9823   33.9822
               33.9822   33.9820
              -107.1886 -107.1884 -107.1884 -107.1885
              -107.1884 -107.1882

>> size(latandlong)
ans =     2     6

>> length(latandlong)
ans =     6

>> latandlong2=[lat  long]
latandlong2 =
   33.9823 -107.1886
   33.9823 -107.1884
   33.9823 -107.1884
   33.9822 -107.1885
   33.9822 -107.1884
   33.9820 -107.1882

>> size(latandlong2)
ans =     6     2

>> length(latandlong2)
ans =     6

>> latandlong3=[lat ; long]
latandlong3 =
   33.9823
   33.9823
   33.9823
   33.9822
   33.9822
```

```
47      33.9820
48    -107.1886
49    -107.1884
50    -107.1884
51    -107.1885
52    -107.1884
53    -107.1882
54
55  >> length(latandlong3)
56  ans =     12
```

On line 3 we assign the first column of the matrix to the variable `long` (longitude). The 2nd column was assigned to `lat`. On line 13, `lat` and `long` are transposed into rows and then stitched together top to bottom using the `;`, resulting in a `2x6` matrix called `latandlong`. On line 25, `lat` and `long` are stitched together side-by-side using a space character (or a `,`), resulting in a `6x2` matrix (`latandlong2`). Finally, the arrays are stitched together end-to-end on line 40 to form `latandlong3`. **length()** and **size()** functions can be used to keep track of matrices. Knowing the size of a matrix is primarily important because when you want to combine matrices or vectors by adding or multiplying them (for example) they need to have the same sizes.

4.2 Plotting data with *Matlab*

4.2.1 Basic plotting

Since we have something moderately interesting to plot, the path of a weather balloon, let us learn how to plot data with *Matlab*. Any modern programming language has facilities to plot data, but *Matlab* is designed to make plotting particularly easy, specifically because it allows operations to be performed on entire arrays, as previously discussed, without the need for **for** loops. Consider the following:

```
1  %======== plotting.m (basic plotting demo) ============
2  d=linspace(1,10,25);
3  f=d.^(0.5);
4  subplot(2,1,1)
5     %Set up 2 plots stacked vertically.  This is first.
6  plot(f,'+')
7     %Plot sqrt of numbers from 1-10 vs. index 1-25
8  subplot(2,1,2)
9     %This is 2nd of two vertically stacked plots.
10 plot(d,f,'s')
11    %Plot sqrt of 1-10 vs. the numbers from 1-10
```

Line 2 uses **linspace** to create an array of 25 numbers evenly spaced between 1 and 10 and assigns it to the variable **d**. Line 3 creates **f**, which contains the square root of each element of **d**. Line 6 plots all values of **f** on the y-axis, and the x-axis is the index of **f**. Since **f** has 25 elements, the x-coordinates run from 1 to 25. While this form of the **plot()** may be useful for looking at an array, it is not what is usually wanted in scientific work. Usually a scientific graph shows a plot of one variable against another; this is what the form of the **plot()** function on line 10 does. The second plot looks similar to the first, except that the x-axis runs from 0 to 10 (rather than 1 to 25). Note also the '+', and the 's'. These specify to plot using plus-signs and squares, respectively. Try replotting the data using the symbols '*', '.', 'h' and 'p' rather than '+' and 's'.

Below is a script that plots altitude vs. time for the balloon.

```
1 %=============== a20030725_ZvsT.m ====================
2 % Loads & plots a weather balloon trajectory.
3
4 a=load('bflight1_20030725_mod.pos');
5 time=a(:,4);
6 alt=a(:,3);
7 plot(time,alt,'.');
8 xlabel('Time (s)');
9 ylabel('Altitude (m)');
10 title('July 25, 2003 telemetry test flight');
```

The plot generated by this script is the left panel of figure 4.2. As discussed in section 4.1.2, lines 5-6 pull arrays out of the matrix of data which are plotted on line 7. One of the advantages of working with an array for each type of data is that *Matlab* can plot them whether they are row or column (so long as they are both the same). The **xlabel()** and **title()** commands take text strings as arguments. While both plots in figure 4.2 show the same data, the right panel is far easier to read. It demonstrates enhancements possible using built-in functions to adjust fonts and font-sizes, which are the topic of the next section.

4.2.2 Improving the appearance of basic plots

The default fonts used by *Matlab* are too small for many purposes. For example, the labels on the left figure are almost too small to read. Also, it is common at scientific conferences for inexperienced speakers to used the default settings of their data analysis software and project scientific graphs whose meaning is indetectable by their audience because only those in the front row can read the titles/labels.

Fortunately, you can change the default font settings anytime you use the **xlabel()**, **title()**, or **text** functions as is illustrated on lines 9–12 below. Note that we have changed the font style to Times, and increased the size (in points) of the labels and title. Line 13 introduces the **text** command, which allows one to put explanatory text anywhere on the figure. The numbers `400, 1700` are the

coordinates on the graph of the lower left corner of the explanatory text. To properly place the text, you have to first look at the graph and decide where it belongs. If you wish to do this interactively, use the **gtext** function, which waits for you to click on where you want the extra text.

The right panel of figure 4.2 shows the actual data points overlaid on a line joining the data points. It is often beneficial to show experimental data this way. You want to show the actual data, but it is often helpful to show a line fitting the data to "guide the readers eye". This is achieved on line 7 which plots the same data twice, first with red dots using the `r.` control character, then again with a blue line (the `b-` control sequence). The `MarkerSize` parameter increases the size of the dots, while the `LineWidth` parameter adjusts the linewidth property for the figure.

In the next section we will fully explain what line 14 means; for now we just say that it is a magic invocation that increases the size of the numerical labels on both axes to a more readable value.

```
1  %========== a20030725_ZvsTfancy.m ====================
2  % Loads and plots a balloon trajectory.
3  % Demonstrates changing fonts.
4  % ("listfonts" command gives you font options)
5  a=load('bflight1_20030725_mod.pos');
6  time=a(:,4); alt=a(:,3);
7  plot(time,alt,'r.',time,alt,'b-','MarkerSize',22,...
8       'LineWidth',2);
9  xlabel('Time (s)','Fontname','Times','FontSize',18);
10 ylabel('Altitude (m)','FontName','Times','FontSize',18);
11 title('Telemetry test flight','FontName','Times', ...
12      'FontSize',22);
13 text(400,1700,'July 25, 2003','FontSize',16)
14 a=gca; set(a,'FontName','Arial','FontSize',16)
```

4.2.3 Saving figures as images

Now that you know how to draw figures, you will want to save them as images. The *pointy-clicky* way to do this is to pick the save option in the figure window and choose a file type. The default `.fig` file-type you see first is not the right choice for this. A *Matlab* `.fig` file is not an image at all. Rather it is a binary file of commands and data that allows *Matlab* to recreate your figure. It has some value if you are continuing to work on a figure and want to save it, come back to it and be able to keep tweaking it, but is not for finished work. Also, if you are sending it to someone else, it cannot be opened without a copy of *Matlab* running.

Matlab supports many actual image types. I recommend either a `jpeg` or an `eps`. The `jpeg` format has the significant advantage of being small and universally supported, as it is the most common format for digital cameras. The `eps` format

has the advantage of giving the highest quality output, as it is a form of post-script. Not all computers have `eps` viewers, but you can install `ghost-script` on *Windows*, *Apple*, or *Linux*. While the *pointy-clicky* method works for selecting these alternate image formats, you can also do this at the command line as follows.

```
>> print(1,'-djpeg','homework2_2.jpeg')
>> print(1,'-djpeg','-r300','homework2_2.jpeg')
>> print(1,'-deps','homework2_2.eps')
```

The first argument in the **print()** command is the figure number. (You can have multiple figures up at once, and they have numbers. So far our figures are all numbered 1.). The second argument is the type of image you want to save preceded by `-d`. The third argument is the filename you want your image to go to. Alternately, you can insert another argument (e.g. `-r300`) which gives the figure resolution in dots-per-inch. You will note that the default, which is what you get with the *pointy-clicky* method, is relatively low resolution, and does not look all that clear with a *.jpeg*.

4.2.4 Advanced plotting: Objects and handle graphics

To further customize plots in *Matlab*, one needs a deeper understanding of the inner workings of plotting. *Matlab* plots provide a concrete example of a fancy-sounding technique called *Object Oriented Programming*. Software objects have properties, just like real objects. Before computer plotting software was common, scientists drew graphs of data on paper with rulers, french-curves, and colored pencils. Thus, *Matlab* uses the analogy of a piece of paper with axes and data points sketched onto it to structure its software for plotting. When you draw a graph on a piece of paper, that graph has many properties. First consider the paper itself. It has a length and a width. It has a color. It has a position on your desk. If it is in a book or part of a report, the paper might have a page number on it. The *figure* object is the top-level object in a *Matlab* plot. Like a piece of paper it has a size, a color, a number, and a location on your computer screen. The first thing you do with a blank piece of paper is to draw an `x-y` axis on it. Similarly, the next level down from figure in *Matlab* is *axis*. The *axis* has properties like size, location on the paper, color, tick-marks, etc. Once you have drawn axes on paper, labeled them, and put a title, you can finally start plotting your data. Likewise, the third level object in *Matlab* is a *plot* object. Among its properties is the set of data points it contains, the size and shapes of marker you use to plot those points, whether you join your points with lines, etc.

When you issue the **plot()** command, *Matlab* automatically opens a new *figure* window, sets up an *axis* on that figure, and plots data on the axes. After this, you can use your script to customize the properties of the *figure*, the *axis* or the *plot* itself. Some of these adjustments are so common that they are made easy. *Matlab* provides commands for **title()**, **xlabel()**, **ylabel()**, **legend()**, and explanatory text (**text**). Other adjustments, such as tick-mark location, tick-labels, data-marker size, etc, require you to find (using **get()**) and change (using **set()**) the appropriate property.

Since 2002 at least, *Matlab* also allows the adjustable properties of plots to be edited directly using a GUI. While a GUI is a quick way to customize a single plot, it is advantageous to learn how to adjust plots under program control. Adjusting plots under program control allows one to generate evolving versions of similar figures quickly, rather than having a long point and click customization session every time a new figure is generated. The GUI is helpful to easily explore what may be customized. It is also possible to deduce from the GUI how to customize a feature under script control.

Let us learn by example how to adjust plots under script control:

```
1  %===== handles1.txt (demonstrates get command) ========
2  >> x=1:5:360;
3  >> p=plot(x,sind(x))
4  p = 159.0414e+000
5
6  >> g=gca
7  g = 158.0409e+000
8
9  >> get(g)
10           Color = [1 1 1]
11           FontAngle = normal
12           FontName = Helvetica
13           FontSize = [10]
14           FontUnits = points
15           FontWeight = normal
16           LineStyleOrder = -
17           LineWidth = [0.5]
18           TickDir = in
19         ...
20           XLim = [0 400]
21           YLabel = [162.041]
22           YAxisLocation = left
23           YLim = [-1 1]
24           YScale = linear
25
26           Children = [159.041]
```

Line 3 is the usual **plot()** command, but in this case a variable **p** is set by the plot command. The value 159.0414 has a meaning to *Matlab* (the exact value might differ when you type these commands). The number is called a "handle", because just like the handle on briefcase, it allows you to grab the *plot* you just made and adjust it. The command **gca** on line 6 stands for "get current axes". It gets the handle for the *axis* and sets the variable **g** to it. Once you have the handle to a plot feature, you can get a list of all the adjustable properties, using **get()**. This is illustrated on line 9. The **get()** command generated about forty lines of output, only twenty of which are reproduced here. Note that the general form is "key" and "value". ('Key-value' lists are another common feature of modern

programming languages.) For example, "Color" takes the values in a 1×3 array. The first entry is the amount of red, second the amount of green, and third the amount of blue in the color. If you have defined custom colors in *HTML*, when doing a web page, you have seen something similar to this approach to color.

The meanings of the keys "FontName", and "FontSize" on lines 12–13 should be fairly self explanatory. Now is the time to explain how line 14 in the previous script adjusted the axis labels on the ballooning plot of figure 4.2. The `a=gca;` picked up the *handle*for the axes just plotted. Using the *handle*, we adjusted the fonts via the **set()** command.

Note on line 26 is a key called *Children*. In computing vocabulary (as in life) *Parents* create *Children*. Note the value of the child, 159.041, is the same number that appears on line 3 for **p**. Thus the "plot object" is the "child" of the "axis object". This is consistent with my previous explanation that there is a hierarchy in a graph made with *Matlab*. Top to bottom (or *parent* to *child*) the hierarchy is, *figure*, *axis*, *plot*.

As a final example, I will show how to change the direction of the tick-marks on each axis. This is not something you are likely to need to do, but it demonstrates the fine level of control you can have over plots as well as pointing out some common errors when using **set()**. For this reason, I recommend you experiment with **set()** at the command line before incorporating it into a script.

```
1  %======= handles2.txt (demonstrates set command) ======
2  >>set(g,'TickDir','out')
3
4  >>set(g,'FontSize','12','FontName','Times','TockDir','mid')
5  ??? Error using ==> set
6  Invalid axes property: 'TockDir'.
7
8  >>set(g,'FontSize','12','FontName','Times','TickDir','mid')
9  ??? Error using ==> set
10 Value must be numeric.
11
12 >>set(g,'FontSize',12,'FontName','Times','TickDir','mid')
13 ??? Error using ==> set
14 Bad value for axes property: 'TickDir'.
15
16 >>set(g,'FontSize',12,'FontName','Times','TickDir','out')
```

Note that we can change one key-value pair at a time, as for *TickDir* on line 1, or several, as on lines 3,7 and 11. Lines 4-5, 8-9 and 11-12 demonstrate several errors made before getting it right on line 15.

To learn still more, you will see that handle graphics has its own section in the web-based *Matlab* help, but it is rather daunting. A shorter description of some of what can be done (which will prepare you for most customizations) can be found simply by typing "help set", "help get", and "help gca" at the *Matlab command line*.

4.3 Numerical Derivatives

You now know how to read and plot scientific data. Once data is in digital form, you can easily do post-processing on it. One obvious thing you might do is to differentiate or integrate the data. For the case of the balloon data we have been studying, one might like to differentiate altitude data with respect to time to yield velocity with respect to time. The result is shown below. Note that the balloon appears to ascend at relatively constant velocity, then descend much faster than it ascended, and to slow down as it gets lower. The balloon is coming down by parachute, but up at 30 km where it begins its descent there is very little air, thus it descends at around 60 miles and hour until the air thickens and the parachute becomes more effective. Note also that the velocity data is "noisy". This is typical of numerical differentiation. The velocity is not *really* jumping around so much. By using different techniques one can somewhat reduce the noise, as you will see in the homework problems.

Numerical differentiation uses the definition of derivative you first learned in introductory calculus.

$$f'(t) = \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{4.1}$$

In calculus, you take the limit in which Δt goes to zero. In numerical differentiation, taking this limit is neither possible, nor necessary. You simply apply the formula above for Δt small (such as the time between data samples). $f(t+\Delta t) - f(t)$ is replaced by $y(k+1) - y(k)$ as in line 13 below. Likewise Δt itself is replaced by the time between data points. Essentially you are calculating average velocity (rather than instantaneous velocity) over fairly small time intervals. That's it! Of course, as your Δt become large compared to the time scale on which your data is changing, your calculated velocity is not very accurate.

Below is the definition of a derivative translated into a *Matlab* script.

```
1  % forwardDeriv: Take forward derivative of y with
2  % respect to t.
3  % Assumes the time (t) points are evenly spaced.
4  % Returns yprime, an array one shorter than y.
5  len=length(y);
6  %The number of intervals in y is one less than number
7  % of points in y.
8  n=len-1;
9  % Initialize the array because it's more efficient then
10 % letting matlab figure it out as it goes along.
11 yprime=ones(1,n);
12 % Take a forward derivative, element-by-element.
13 deltat=t(2)-t(1);
14 for k=1:n
15     deltay=y(k+1)-y(k);
16     yprime(k)=deltay/deltat;
17 end
```

Note that, unless we do something to prevent it, the array yprime will be one shorter than `y`. This is because we have to take the difference of every element of `y` with the next element. There are one fewer differences than there are elements. (Think of 5 pennies in a row, there are 4 spaces between them; the derivatives are the spaces.). Note also that we initialized the array yprime with ones before making it a derivative. While *Matlab* does not strictly require initialization, other languages do, and it is good practice. (It also makes *Matlab* faster than making it guess how long you want your array to be.) Note also that in our script we assume Δt is the same for every array element. Note from the figure above that this is manifestly not true for the data set provided. You will fix this problem in your homework.

The traditional definition of a derivative (equation 4.1) is sometimes referred to as a *forward derivative* in numerical analysis. This is because it uses information about the `k+1`th data point to calculate `fprime(k)`. If we think if `k` as representing time and `y` as representing position, then equation 4.1 uses information from the future to calculate the velocity in the present. By the same argument, equation 4.2 below can be called a *backward derivative*. It uses information from the past to calculate a speed in the present.

$$f'(t) = \frac{f(t) - f(t - \Delta t)}{\Delta t} \tag{4.2}$$

Finally, we mention the *central derivative*. It uses data from the future *and* the past to calculate the derivative in the present according to the formula below:

$$f'(t) = \frac{f(t + \Delta t) - f(t - \Delta t)}{2\Delta t}$$

Why do we have three different derivatives when there is only one in calculus? In calculus, forward, backward, and central derivatives are all essentially equivalent because Δt goes to zero. In numerical analysis, the three derivatives are approximately equal, but not identical. The central derivative has the advantage of being symmetrical. It also may reduce the noisiness that is inherent in looking at differences of measured quantities over small times. In fact, homework problem #9 allows you to see that for the $bflight1$ data set, the central derivative is noticably less noisy than the forward or backward derivatives.

4.4 Review of commands introduced in this chapter

For more information (and to see the related commands not discussed here) on each command type `>>help cmdname` at the *Matlab command line*.

gtext() Allows you to add text to a graph. Lets you click with mouse on where you want text to appear.

length() Gives you the length of an array, or the longest dimension of a matrix.

load() a=load('sample.data') loads the data in text file sample.data into matrix **a**.

print() Can print a figure to paper, more useful to convert a figure to a jpeg or other type of image.

save() save('sample.data',x,y) saves data in arrays x and y to a file called sample data. It can be stored in binary or text form.

size() Gives you the number or rows, then number of columns of a matrix or array. Good to distinguish row from column arrays.

text() Allows you to add text to a graph. Requires coordinates to know where to put text.

title() Allows you to title your plot.

xlabel() Allows you to label your x-axis.

4.5 End of Chapter Problems

(1) Download the file `practice.txt` and reproduce the examples shown in section 4.1.2. Record your experiments in a file called *problem1.txt* using the **diary()** function (See section 3.1.1.2). Experiment all you like. Please clean up your diary file before you submit it so that it contains *only* the requested experiments.

(2) Write a script to do the following: Create array **x** ranging from 0 to 450 in increments of 5. Plot sind(x) vs. x and tand(x) vs. x on the same set of axes. Change the line color for the sine to be red and for the tangent to be black. Also, make the tangent line a dashed line. Change the linewidth to 2 units. Label the axes and change the fontsize of the label to at least 16 point and the font to 'Times'. Title the plot and change the title font to something interesting, maybe even to Chinese or Arabic if you like! The size of the title font should be larger than the label fonts.

Hint: All of these changes can be done directly within the **plot()** and **xlabel()**, **ylabel()**, and **title()** commands. The help for **plot()** is quite useful.

Hint: Remember the **listfonts** command.

(3) Add more customization to your script from problem 2. Use **set()** to do the following:

a. Increase the fontsize labeling the x-axis to 14 point.

b. Move the x-axis to the top of the plot.

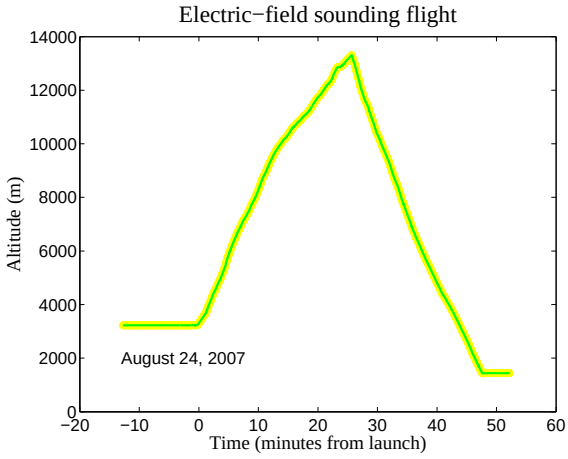
c. Enable minor ticks on the x-axis.

d. Turn on the Y minor grid.

e. Change the tick direction to "out".

Hint: Use **gca** to get a handle to your axes, then use **get()** to list all the properties that are adjustable. You will see all the properties you need to adjust for this exercise in the list.

(4) Download the file `20070824_gps76.flt3` and reproduce the following figure as closely as you can. The data is similar to that discussed in section 4.2.1, but is from a different day of the study and the data column definitions are different. The comments in the file will allow you to understand it.



(5) You will be writing a derivative function and you need data to test it on. Write three short scripts (or one longer one) that generate a two-column by 100-row matrix with the specified data, then write that matrix to an ASCII data file using the **save()** command. The matrices/files should be as follows:

a. A file called `linear.txt` for which the first column (**x**) is the numbers 0:9.9 in 0.1 increments, and the 2nd column (**y**) is generated by the equation `y=-3x`

b. A file called `square.txt` for which the first column (**x**) is the numbers 0:9.9 in 0.1 increments, and the 2nd column (**y**) is generated by the equation `y=(1/2)x^2`

c. A file called `sine.txt` for which the first column (**x**) is 100 numbers between 0 and 2*pi, and the 2nd column (**y**) is generated by the equation `y=sin(2*x)`

(6) Write the function corresponding to this function header:

```
function yprime=forwardDeriv(y,x)
% Takes the forward derivative of y with respect to x.
% x and y are arrays of equal length
% x is not required to be evenly spaced
```

```
% yprime may be one element shorter than y or x.
```
Write a script to test the function which reads each of the files you generated in the last problem, takes the derivative and plots yprime vs. x. The plot does not have to be fancy, but it should demonstrate that your derivative function works as expected.

Hint: Feel free to borrow the sample code from section 4.3. You have to write a function rather than a script. More importantly, you cannot assume that the times are evenly spaced.

(7) Write the function corresponding to this function header:
```
function yprime=centralDeriv(y,x)
% Takes the central derivative of y with respect to x.
% x and y are arrays of equal length
% x is not required to be evenly spaced
% yprime should be the same length as y or x.
```
Write a script to test the function which reads each of the files you generated in the last problem, takes the derivative and plots yprime vs. x. The plot does not have to be fancy, but it should demonstrate that your derivative function works as expected.

Hint: Writing a central derivative is no harder than writing a forward derivative. The only tricky points are the first and last point. I recommend that you treat those points separately and use a forward derivative for the first point only, and a backward derivative for the last point. All the other points can have central derivatives.

(8) Write a script that uses **subplot()** to plot position vs. time on the top panel and velocity vs. time on the bottom panel for the data set `20070824_gps76.flt3`. To do the velocity calculation, the script should call a function. It may be either the `forwardDeriv` or the `centralDeriv` function that you previously wrote.

(9) Write a script that uses **subplot()** to plot velocity vs. time on the top and bottom panel. Use the data set `bflight1_20030725_mod.pos` On the top panel, the script should use `forwardDeriv`. On the bottom panel it should use `centralDeriv`. There is a noticeable difference between these two approaches.

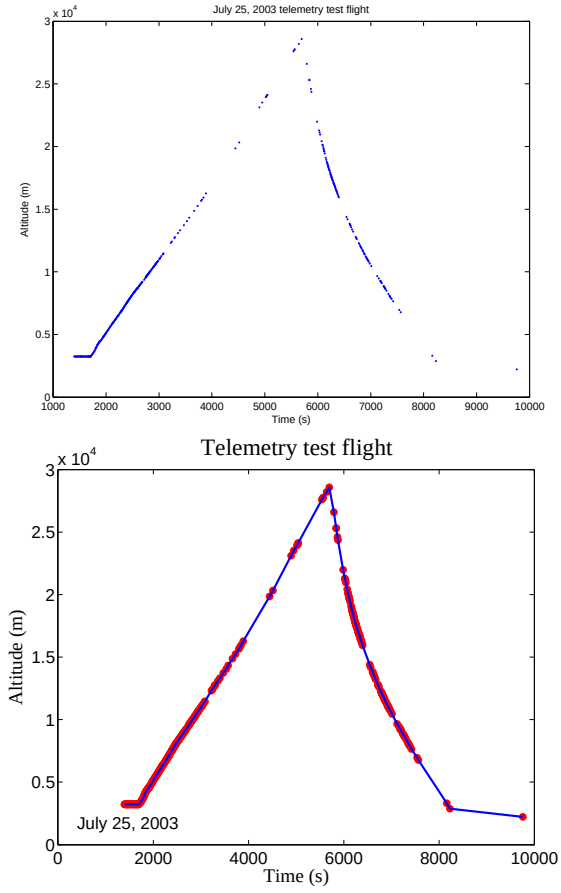


Figure 4.2: Both panels of figure 4.2 show the same data. The right panel is far more legible, however. It demonstrates enhancements possible using built-in functions to adjust fonts and font-sizes. The right panel is closer to publication quality.

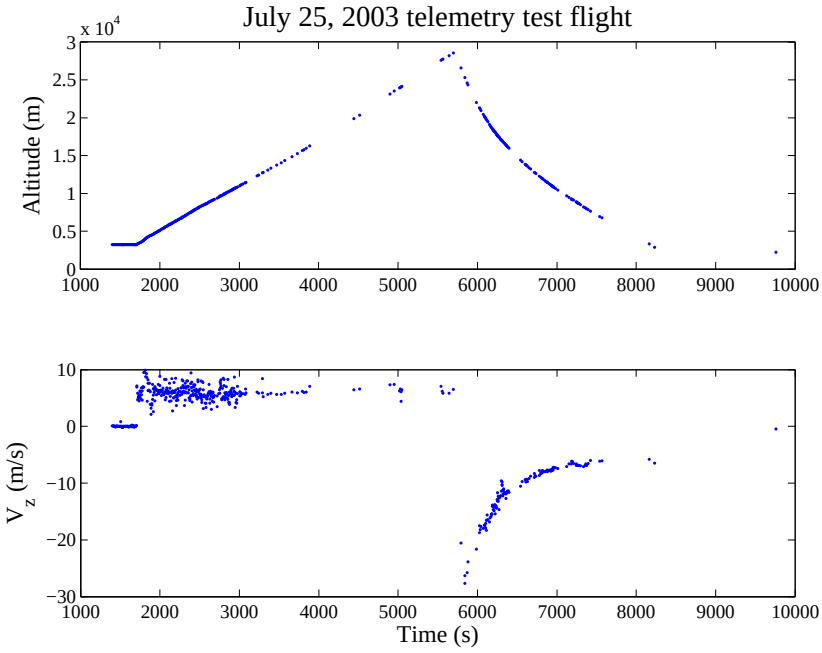


Figure 4.3: We replot the balloon time vs. altitude data along with velocity vs. time calculated by numerical differentiation.

4.6 Code examples for this chapter

```
%GPS Data: 2003-07-25 balloon flight from Langmuir Lab
%  LONG        LATITUDE    ALT    TIME
%  (deg)         (deg)      (m)     (s)
 -107.188606  33.982315  03249.5  1708
 -107.188408  33.982296  03302.0  1717
 -107.188431  33.982269  03306.7  1718
 -107.188477  33.982182  03351.9  1727
 -107.188416  33.982212  03357.1  1728
 -107.188194  33.981983  03407.1  1737
 -107.188217  33.981918  03411.6  1738
```

```
%selectData.m (selecting and saving two columns of data)
a=load('bflight1_20030725_mod.pos');
time=a(:,4);
alt=a(:,3);

clean_data=[time alt];
save('bflight1_20030725_cln.txt','clean_data','-ascii');
save('bflight1_20030725_cln','clean_data');
```

```
%============ls_datafiles.txt=============
>> ls -lh bflight*
-rw-r--r--  18K 2011-09-09 3:25 bflight1_20030725_mod.pos
-rw-r--r--  17K 2011-09-09 9:57 bflight1_20030725_cln.txt
-rw-r--r-- 2.7K 2011-09-09 9:57 bflight1_20030725_cln.mat
```

```
%== bflight1_20030725_clean_excerpt.txt    ==
    1.3970000e+03    3.2338000e+03
    1.3980000e+03    3.2339000e+03
    1.4070000e+03    3.2344000e+03
    1.4080000e+03    3.2345000e+03
    1.4170000e+03    3.2351000e+03
```

```
%=====bflight1_20030725_clean.mat========
MATLAB 5.0 MAT-file, Platform: GLNX86, Created on: Fri
Sep  9 10:40:50 2011  ^@^AIM^O^@^@^ @  ^C^@^@x<9c>-<94>{
x<8e>u^\  _                                V    B  <85><9c><8a><8c>
4  <9c>I^X^]<8c> X   ^ Z  6  <9c><86>(Z 3  (  0 m   h  ^D<9c><83>
p^M^<87>    7 (^D<8c>   0 <84>^ N   F  ^ P 6    ^ B  a g  <82>    7
^X<93>p$<8e>     ^ N N ^Op^ Z   l <9c><83>^ Y  ^P<97>  2   u  5
X<88>  q ^ O   C x ^ X   4 ^  K  ^ G V ] d  ^ T   ^TM5(<86>  P ] jL M %
^TG^]   ^[ u  ^ D C <83>(<91><92>i(^ M 14 <96>  Q
```

```
%=== practice.txt (sample ballooning data) =====
```

```
 -107.188606  33.982315  03249.5  1708
 -107.188408  33.982296  03302.0  1717
 -107.188431  33.982269  03306.7  1718
 -107.188477  33.982182  03351.9  1727
 -107.188416  33.982212  03357.1  1728
 -107.188194  33.981983  03407.1  1737
```

```
% ======matrix1.txt (selecting matrix elements)=========
>> a=load('practice.txt');
>> a(2,3)
ans = 3302

>> a(1,1:3)
ans =    1.0e+03 *    -0.1072     0.0340     3.2495

>> a(1,[2,4])
ans =    1.0e+03 *     0.0340     1.7080

>> a(1,[2:4])
ans =    1.0e+03 *     0.0340     3.2495     1.7080

>> a(1,1:5)
{??? Index exceeds matrix dimensions.}

>> a(2,:)
ans =   1.0e+03 *   -0.1072    0.0340    3.3020    1.7170
```

```
% ==================matrix2.txt ========================
% Demonstrate selecting/reassembling matrix rows/columns
>> long = a(:,1)
long = -107.1886
       -107.1884
       -107.1884
       -107.1885
       -107.1884
       -107.1882

>> lat = a(:,2);

>> latandlong=[lat' ; long']
latandlong =    33.9823    33.9823    33.9823    33.9822
                33.9822    33.9820
              -107.1886  -107.1884  -107.1884  -107.1885
              -107.1884  -107.1882

>> size(latandlong)
ans =      2      6
```

```
>> length(latandlong)
ans =     6

>> latandlong2=[lat  long]
latandlong2 =
   33.9823 -107.1886
   33.9823 -107.1884
   33.9823 -107.1884
   33.9822 -107.1885
   33.9822 -107.1884
   33.9820 -107.1882

>> size(latandlong2)
ans =     6     2

>> length(latandlong2)
ans =     6

>> latandlong3=[lat ; long]
latandlong3 =
   33.9823
   33.9823
   33.9823
   33.9822
   33.9822
   33.9820
 -107.1886
 -107.1884
 -107.1884
 -107.1885
 -107.1884
 -107.1882

>> length(latandlong3)
ans =    12
```

```
%======== plotting.m (basic plotting demo) ============
d=linspace(1,10,25);
f=d.^(0.5);
subplot(2,1,1)
   %Set up 2 plots stacked vertically.  This is first.
plot(f,'+')
   %Plot sqrt of numbers from 1-10 vs. index 1-25
subplot(2,1,2)
   %This is 2nd of two vertically stacked plots.
```

```
plot(d,f,'s')
   %Plot sqrt of 1-10 vs. the numbers from 1-10
```

```
%================ a20030725_ZvsT.m ====================
% Loads & plots a weather balloon trajectory.

a=load('bflight1_20030725_mod.pos');
time=a(:,4);
alt=a(:,3);
plot(time,alt,'.');
xlabel('Time (s)');
ylabel('Altitude (m)');
title('July 25, 2003 telemetry test flight');
```

```
%============ a20030725_ZvsTfancy.m ====================
% Loads and plots a balloon trajectory.
% Demonstrates changing fonts.
% ("listfonts" command gives you font options)
a=load('bflight1_20030725_mod.pos');
time=a(:,4); alt=a(:,3);
plot(time,alt,'r.',time,alt,'b-','MarkerSize',22,...
      'LineWidth',2);
xlabel('Time (s)','Fontname','Times','FontSize',18);
ylabel('Altitude (m)','FontName','Times','FontSize',18);
title('Telemetry test flight','FontName','Times', ...
      'FontSize',22);
text(400,1700,'July 25, 2003','FontSize',16)
a=gca; set(a,'FontName','Arial','FontSize',16)
```

```
>> print(1,'-djpeg','homework2_2.jpeg')
>> print(1,'-djpeg','-r300','homework2_2.jpeg')
>> print(1,'-deps','homework2_2.eps')
%======end of Printing Examples==========
```

```
%===== handles1.txt (demonstrates get command) ========
>> x=1:5:360;
>> p=plot(x,sind(x))
p = 159.0414e+000

>> g=gca
g = 158.0409e+000

>> get(g)
          Color = [1 1 1]
          FontAngle = normal
          FontName = Helvetica
          FontSize = [10]
```

```
          FontUnits = points
          FontWeight = normal
          LineStyleOrder = -
          LineWidth = [0.5]
          TickDir = in
        ...
          XLim = [0 400]
          YLabel = [162.041]
          YAxisLocation = left
          YLim = [-1 1]
          YScale = linear

          Children = [159.041]
```

```
%======= handles2.txt (demonstrates set command) ======
>>set(g,'TickDir','out')

>>set(g,'FontSize','12','FontName','Times','TockDir','mid')
??? Error using ==> set
Invalid axes property: 'TockDir'.

>>set(g,'FontSize','12','FontName','Times','TickDir','mid')
??? Error using ==> set
Value must be numeric.

>>set(g,'FontSize',12,'FontName','Times','TickDir','mid')
??? Error using ==> set
Bad value for axes property: 'TickDir'.

>>set(g,'FontSize',12,'FontName','Times','TickDir','out')
```

```
% forwardDeriv: Take forward derivative of y with
% respect to t.
% Assumes the time (t) points are evenly spaced.
% Returns yprime, an array one shorter than y.
len=length(y);
%The number of intervals in y is one less than number
% of points in y.
n=len-1;
% Initialize the array because it's more efficient then
% letting matlab figure it out as it goes along.
yprime=ones(1,n);
% Take a forward derivative, element-by-element.
deltat=t(2)-t(1);
for k=1:n
    deltay=y(k+1)-y(k);
    yprime(k)=deltay/deltat;
```

```
17 end
```

Chapter 5

Projectile Motion with Drag

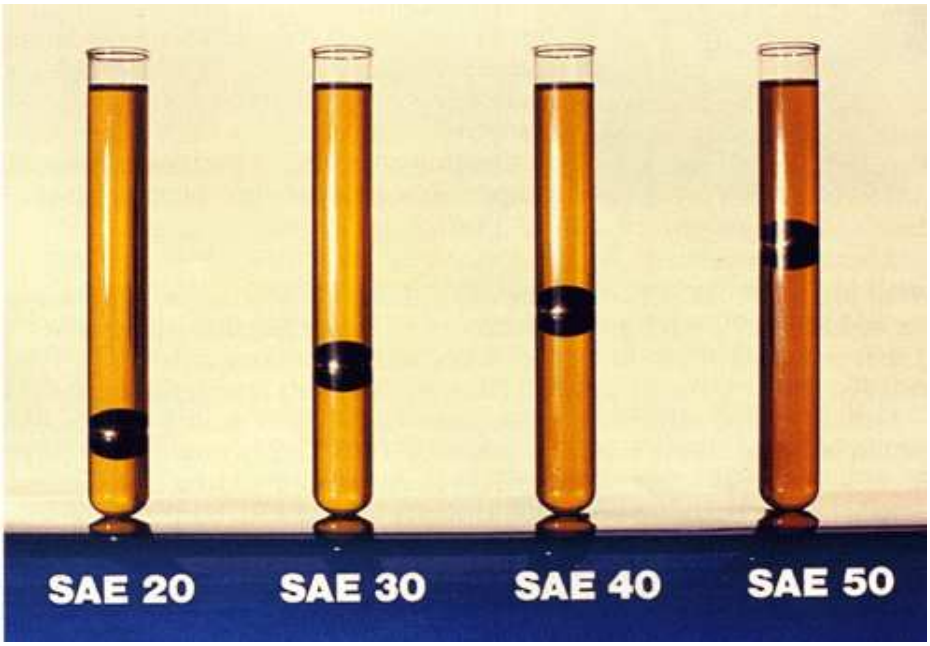


Figure 5.1: Measuring the speed of a ball falling in a fluid is a pretty good way to measure viscosity

5.1 Trajectory and range of a heavy projectile

In section 2.3 we developed the basic equations for 2-dimensional motion in the presence of a constant gravitational force. We called the projectile *heavy* in order to justify neglecting air drag. For the heavy projectile, equations 2.5 tell you all you need to know, but are both written in terms of the time from launch. Obviously one can eliminate time completely from these equations, to give y of x directly. This gives what is referred to as the *trajectory* of the projectile. Let us eliminate t.

Solving 2.5 for t in terms of x gives

$$x = x_0 + v_{0x}t \quad t = \frac{x - x_0}{v_{0x}} \tag{5.1}$$

Plugging 5.1 into the y vs. t equation immediately yields

$$y = y_0 + v_{0y}\frac{x - x_0}{v_{0x}} - \frac{1}{2}g(\frac{x - x_0}{v_{0x}})^2 \tag{5.2}$$

Since $v_{0y} = v_0\ sin(\theta)$ and $v_{0x} = v_0\ cos(\theta)$, this simplifies to

$$y = y_0 + tan(\theta)(x - x_0) - g\frac{(x - x_0)^2}{2(v_0\ cos(\theta))^2} \tag{5.3}$$

Which gives altitude in terms of distance along the ground, launch angle, and launch speed.

By solving for x in equation 5.3 when $y = 0$, we can arrive at a formula for the range R of a projectile, the distance it travels before landing. For simplicity, we first set $y_0 = 0$. This saves the trouble of having to use the quadratic formula on a slightly messy set of coefficients because it allows one of the factors of $x - x_0$ to factor out, leaving:

$$0 = tan(\theta) - g\frac{R}{2(v_0\ cos(\theta))^2} \tag{5.4}$$

where we have replaced the remaining $x - x_0$ by R for *range*. Multiplying both sides by $2{v_0}^2 cos^2(\theta)$ and using a double-angle formula leads to:

$$R = \frac{{v_0}^2 sin(2\theta)}{g} \tag{5.5}$$

By inspection, equation 5.5 gives a maximum range $R_{max} = {v_0}^2/g$ when $\theta = 45°$. Note also that $R(10°) = R(80°)$ and in general $R(\theta) = R(90 - \theta)$.

5.2 More Plotting data with *Matlab*

Since we have something moderately interesting to plot, a trajectory, let us plot it.

Below is a script that generates a plot of the trajectory of a projectile.

```
1  %DOCUMENT
2  %Calculates and plots trajectory of a heavy projectile
3  %given arbitrary initial conditions
4  %Usage:  >> projectile1
5  %DEFINE
6  g= 9.81;      %g, m/s^2
7  v0=30;        %initial speed, m/s
8  y0=10;       %launch altitude, m
9  x0=-20;       %launch position, m
10 theta=25;     %launch angle, degrees above horizontal
11 x=linspace(-20,100,150); %generates 150 evenly spaced
12                          %points between 0 and 100 m
13 %DERIVE
14 y=y0+tand(theta)*(x-x0) ...
15   -g*((x-x0).^2)/(2*(v0*cosd(theta))^2);
16 %DISPLAY
17 plot(x,y,'k')  %The 'k' makes the plotted line black
18                %without it, line would be blue.
19 title('Trajectory of a heavy projectile')
20 xlabel('Position downrange (m)')
21 ylabel('Altitude (m)')
22 % ============  end of projectile1.m  ================
```

Lines 5-9 define initial conditions for the problem. The meanings of the variables on these lines follow typical physics conventions, and comments (%) were used to the right of each definition to further clarify the meaning and units of the constant. Line 10 defines a range of times appropriate to the problem at hand using **linspace**. The physics action all happens on line 13, which translates equation 5.3 into computer code. The resulting plot appears in the left panel of figure 5.2.

We made the effort to develop equation 5.3 because it gives altitude **y** of the projectile directly as a function **x**. However, *Matlab* can plot the trajectory directly from the expressions for altitude vs. time and horizontal position x vs. time listed in equations 2.5 without the need for equation 5.3. All we need to do is replace lines 10-13 above with 1-4 below. Note in this case that we used **linspace** to generate an array of times rather than an array of x-coordinates. We then used the formulae on lines 3 and 4 below to create **x**(t) and **y**(t). In math and physics, we would say that we "reparameterized" the problem in terms of **t**. *Matlab* does not care however. The arrays **x** and **y** are just lists of numbers, regardless of how they were generated, and the **plot()** command will plot them against eachother.

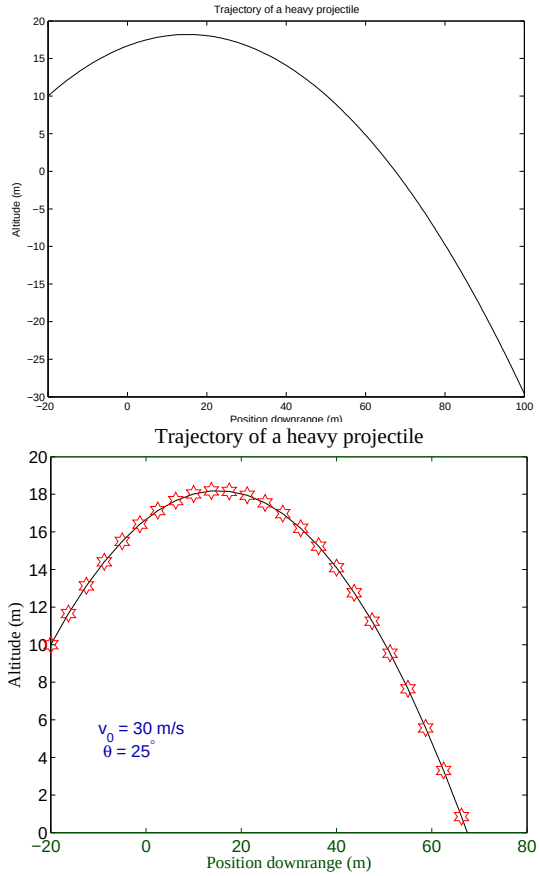


Figure 5.2: At left is our first plot. Lines 16-18 above generate the titles and axis labels. The right plot demonstrates enhancements using "handle graphics" aimed at making the figure closer to publication quality.

```
1 t=linspace(0,4,150); %generates 150 evenly spaced points between
2 %DERIVE
3 x=x0+v0*cosd(theta)*t;
4 y=y0+v0*sind(theta)*t-(1/2)*g*t.^2;
5 %DISPLAY
6 plot(x,y,'k')
```

5.2.1 Making publication quality figures

In the last chapter we learned how to customize plots. We reinforce some of what was learned previously and introduce some new parameters that can be adjusted. The right panel of figure 5.2 was generated by the script below.

```
1  %DOCUMENT Calculates and plots trajectory of a heavy projectile given
2  %for a certain range of times.
3  %Usage:  >> projectile3
4  %DEFINE
5  g= 9.81;                %g, m/s^2
6  v0=30;                  %initial speed, m/s
7  y0=10; x0=-20;          %launch altitude, position, m
8  theta=25;               %launch angle, degrees above horizontal
9  t=linspace(0,4,30); %generates 30 evenly spaced points between 0 and
10 %DERIVE
11 x=x0+v0*cosd(theta)*t;
12 y=y0+v0*sind(theta)*t-(1/2)*g*t.^2;
13 %DISPLAY
14 p=plot(x,y,'k',x,y,'rh');
15 set(p, 'MarkerSize',16)
16 ax=gca;
17 set(ax,'FontSize',16,'XColor',[0 0.3 0]);
18 t=title('Trajectory of a heavy projectile');
19 set(t,'FontSize',20,'FontName','Times')
20 xl=xlabel('Position downrange (m)');
21 set(xl,'FontSize',16,'FontName','Times')
22 yl=ylabel('Altitude (m)');
23 set(yl,'FontSize',16,'FontName','Times')
24 xlim([-20 80])
25 ylim([0 20])
26 tx=text(-10,5,'v_0 = 30 m/s \newline \theta = 25^{\circ}');
27 set(tx,'FontSize',16,'Color',[0 0 0.7])
28 % ==========  end of projectile3.m  ==============================
```

Notice that the *projectile3* script is the same as *projectile1* script except in the %DISPLAY section beginning on line 13. This shows the value of the D^4 structure for code. One can make a plot prettier by working only on the %DISPLAY section of code; leaving alone the %DOCUMENT, %DEFINE and %DERIVE sections that have already been debugged.

Lines 24 and 25 introduce the **xlim()** and **ylim()** functions. These set the x and y limits in the right panel to -20, 80 and 0, 20. In their absence (the left panel of figure 5.2), *Matlab* autoscales the axes to fit all the data. Autoscaling is not always desirable. For example, for projectile motion, we might want to limit the plot to just those points with altitude greater than zero. In multiple plots of experimental data, it is often recommended to keep limits consistent across multiple plots. This allows a reader to easily tell if your data values have gotten larger or smaller by comparing the apparent sizes of the two graphs on the paper[1].

[1] I recommend when plotting new data for the first time that **xlim()** and **ylim()** *not* be used. Sometimes the data has surprising values, and it may end up plotting off-screen. A blank plot is hard to debug!

In technical publications, figures are often single column, or about the size of figures 5.2. Note that the axis labels and numbers on the left figure are almost unreadable at this size. As in Chapter 4, the panel on the right of figure 5.2 has the more legible labels generated by using handle graphics to increase the font sizes on the axis labels, title, and axes. (See the code on lines 17, 19, 21, 23 and 27.)

The right figure also shows the actual data points overlaid on a line joining the data points. It is often beneficial to show experimental data this way. You want to show the actual data, not merely the fit to the data. This is achieved with line 14 which plots the same data twice, first with a black line using the "k" control character, then again with a red hexagonal marker (the "rh" control character).

The x-axis of the right figure is green. I did this mostly to show you how to do it, however color can be a useful tool to aid in descriptions or draw the readers eye to some important feature of your graph.

Note that the right panel also shows the initial conditions for the problem, e.g. $v_0 = 30m/s$, $\theta = 25°$. This is good practice. It allows the reader to see directly on the figure all the conditions that are relevant to defining the data being plotted. Line 26 of the script shows how this is done with the *Matlab* **text** function. The -10, 5 are the coordinates on the graph at which the text starts. (You can give coordinates that are not on the graph, in which case *Matlab* will gladly print your explanatory text in some invisible place off the screen.)

The rest of line 26 is the message to be displayed, using LaTeX. LaTeX is a powerful scientific publications language with excellent mathematical formula capabilities; and it is built into *Matlab*. It allows *Matlab* to show `v_0` with a proper subscript (v_0), and `\theta` as θ. (This book is written with LaTeX.)

5.2.2 Adding authorship information

If you want to take credit for your work, in print, at a talk, or on the web, it is good to get your name on the plot. It is not a bad idea to add the date as well. The script below can be run after your plot is finished. It will add your name and the current date.

```
1  %This puts name and date at bottom of current figure.
2  name='R. Sonnenfeld, NMM Tech';
3  b=gca;  % Get current axis
4  x=get(b,'XTick'); %Get list of all tick marks on X-axis
5  xmin=x(1);xmax=x(length(x)); %First/last tick marks
6    %give you range of plot
7  width=xmax-xmin;
8  y=get(b,'YTick'); %Get list of all tick marks on Y-axis
9  ymin=y(1);ymax=y(length(y));
10   %y(length(y)) gives the last element of y.
11 height=ymax-ymin;
12 %Puts your name 10% below and
```

```
13 %    to the left of the origin of your plot.
14  text(xmin-0.1*width,ymin-0.1*height,name,'Fontsize',10)
15 %Puts date.  Note "date" is a function, not a variable.
16  text(xmax-0.1*width,ymin-0.1*height,date,'Fontsize',10)
17 % ==========  end of AddNameDate.m  ==================
```

The interesting parts of the script occurs on lines 4,5 and again on 7,8. On line 4, the axis handle is used to request what XTick marks *Matlab* has chosen to put on the graph. This returns an array whose first element is the point where the X-axis starts and whose last is where it stops. On line 5, those values are assigned to variables. Note that the function **length()** is used inside the parenthesis for an array to automatically get the last element of the array of Tick-marks without knowing its length in advance. This nesting of one feature of *Matlab* inside another is allowed and can be efficient if it is not allowed to become confusing!

5.3 Document, Define, Derive, Display (D^4)

So far our scripts are not very long. We are headed for increasing complexity, so this is a good point to discuss a bit about program structure. A finished script will tend to have four sections, which I call D^4. You have already seen these four sections illustrated in some of the example scripts we have studied:

Document: At the top of the script. Should describe in plain English what the script does and give an example of usage.

Define: Variables are defined here. Since this is physics, it is important to state what units the variables represent. These variables often represent the "initial conditions" of the physics problem.

Derive: The physics is here. Sometimes this is only one or two lines, but it can get arbitrarily complex. For example, the derive section might contain a global model of atmospheric circulation, a quantum-chemistry calculation, the velocity distribution of stars in a galaxy, or numerous other complex physical systems.

Display: The physics of interest is typically complete by the end of the Derive section. However, much of the point of writing a physical simulation yourself is to increase your insight into a phenomenon, or to provide to a peer or lay audience a clear understanding of your work. Thus the end product of a model or simulation is often a display of some sort.

The usage section is particularly important for user-defined functions. It is important to show sample inputs and sample output. Variable definitions may be obvious, (e.g. "x0", "vx0") but anything the least bit non-standard should be clearly explained. Without proper documentation, even the author of a script

(you) is likely to forget what it does or how to use it within a few weeks of writing it.

The derive section can be complicated, but over time I will demonstrate how to break that section into small functions, which are themselves clearly documented, so that the derive section may ultimately just become a list of a series of special functions to implement the physics.

One reason to break out the display section is that there are often several different ways to look at the data. You have already noticed that the difference between *projectile1.m* and *projectile3.m* was all in the %DISPLAY section. We will soon see that we can write the %DOCUMENT, %DEFINE, and %DERIVE sections of a program, then add a %DISPLAY section with either a plot, an animation, or both.

5.4 Simple animation with *Matlab*

Matlab affords multiple methods of animating data. One is the set of **movie** functions. You can do a succession of plots, which then become the frames of a movie. The **movie** function will play these frames. This method is versatile, but somewhat tedious. For simple simulations such as projectile or orbital motion, there are ways to directly animate plots without first creating frames. This method is discussed next.

To animate the trajectory of a ball, one begins with a normal plot to plot the entire trajectory and set up the axes and labels. An additional plot is placed on top of the first to put the ball at its initial position. The additional plot gives a handle to the ball's position. The handle can be used to directly change the X and Y values of the ball without replotting the whole figure. This method is far faster than using the **plot()** command many times, and has the desired effect of showing a marker moving along the trajectory. Our script below illustrates this. It is almost identical to the *projectile3.m* script, but has additional features in the %DISPLAY section.

```
1  %DOCUMENT Calculates and animates trajectory of a heavy projecti
2  %for a certain range of times.
3  %Usage:  >> projectile4
4  %DEFINE
5  g= 9.81;                %g, m/s^2
6  v0=30;                  %initial speed, m/s
7  y0=10; x0=-20;          %launch altitude, position, m
8  theta=25;               %launch angle, degrees above horizontal
9  len=100;
10 t=linspace(0,4,len); %generates len evenly spaced points between
11 %DERIVE
12 x=x0+v0*cosd(theta)*t;
13 y=y0+v0*sind(theta)*t-(1/2)*g*t.^2;
14 %DISPLAY
```

```
15 close all
16 plot(x,y,'k');
17 ax=gca; set(ax,'FontSize',16);
18 title('Trajectory of a heavy projectile','FontSize',20,'FontName','Ti
19 xlabel('Position downrange (m)','FontSize',16);
20 ylabel('Altitude (m)','FontSize',16);
21 xlim([-20 80])
22 ylim([0 20])
23 tx=text(-10,5,'v_0 = 30 m/s \newline \theta = 25^{\circ}');
24 set(tx,'FontSize',16,'Color',[0 0 0.7])
25 %ANIMATE
26 hold on
27 q=plot(x(1),y(1),'ro');
28 set(q, 'MarkerSize',16)
29 for i=1:len
30     if y(i) < 0
31         break;
32     end
33     set(q,'XData',x(i),'YData',y(i))
34     pause(0.03)
35 end
36 % ==========  end of projectile4.m  ==============================
```

Type this script in and try it. Line 27 plots a single data point representing the ball's initial position on top of the existing plot of the trajectory. This creates a handle for subsequent use. In line 28, the handle is used to make the ball a bit bigger; this line could be eliminated with no loss of function. The real action happens on line 33. 'XData' and 'YData' are both properties of the plot with handle **q**. 'XData' and 'YData' can be loaded with arrays of data, but in this case we are just sticking single data values in them, and updating the position every time through the **for** loop.

Once you have the script *projectile4* working, you can learn more by commenting out various single lines and seeing how the behavior of the script changes.

First comment out line 15. The script might work the first time executed, but it is likely that for subsequent runs the figure will end up in the background on your computer's desktop. The animation will be running but you will not see it. The **close** command closes all open figures. Then when the **plot()** command on line 16 creates a new figure, it ends up on the top layer of your screen – so you see it.

Next try commenting out line 26. You should see a ball jiggling around on the screen, but not following a trajectory. The **hold** told *Matlab* not to replot the figure on subsequent plot commands but just to plot new data on top of the old. Once you comment it out, the **for** loop which does the animation just plots single data points, and then the *Matlab* axis auto-scaling continually rescales the axes to keep the point centered.

Next try commenting out line 34. The animation worked, but without the **pause** command, it is likely the ball moved so fast you saw it only at its end point.

5.5 Viscosity and Stoke's Law

It is time to extend the discussion of projectile motion to include drag. There are two common types of drag, *viscous* drag and *inertial* drag. The viscous drag force is proportional to particle velocity, whereas inertial drag is proportional to velocity squared. Because viscous drag is linear in velocity, it is mathematically tractable. For typical objects in air like baseballs and cannonballs and skydivers, the viscous drag term is not the important term. Nonetheless, we start with viscous drag because of its relative simplicity. When we take up inertial drag next chapter we will discover that we really need *Matlab*'s numerical capabilities, as there are no simple analytical solutions for some fairly simple problems (like projectile motion).

5.5.1 Defining Viscosity

You probably have an intuitive idea of the meaning of viscosity; but a rigorous definition is somewhat slippery. Viscosity is a measure of the resistance of a fluid to shear stress. Shear stress is defined as the force per unit area in the direction of motion. Figure 5.3 illustrates this definition.

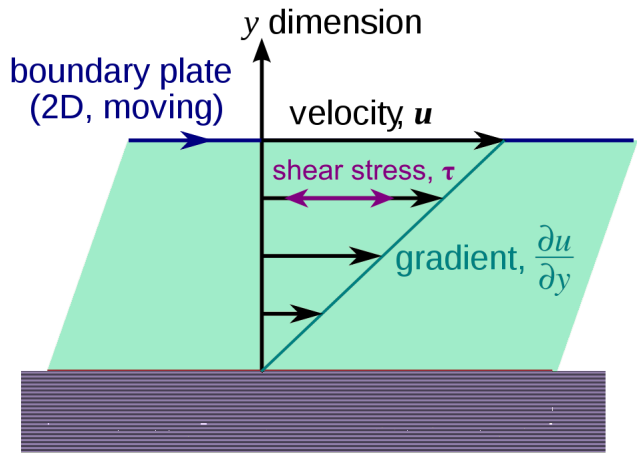


Figure 5.3: For any given velocity gradient $\partial u/\partial y$, the necessary force to drive the top panel to the right is proportional to the viscosity η of the intervening fluid.

The rectangular block at the bottom of figure 5.3 is a fixed flat surface. The light green parallelogram above it represents the fluid of interest, and the thin black

line on top represents another flat plate being driven to the right. It is a good approximation to assume that the fluid touching the bottom plate is not moving at all, that the fluid touching the top plate is moving to the right at the same velocity as the plate, and the fluid in-between moves at a speed proportional to its distance from the bottom plate. (The lengthening black arrows in the figure indicate this.) In the situation where the velocity profile is a simple linear function like that illustrated, $\partial u/\partial y$ can be simply interpreted. It is just the speed **u** of the top plate divided by the fluid thickness.

The following three equations describe the situation of figure 5.3, and can be used to define viscosity.

$$F = \eta A \frac{\partial u}{\partial y} \tag{5.6}$$

$$\tau = \eta \frac{\partial u}{\partial y} \tag{5.7}$$

Equation 5.6 gives the horizontal force you must exert on the top plate in order to maintain its motion. Not surprisingly, this force is proportional to the area of the plate. Equation 5.7 is a restatement of 5.6 with different terminology. Here τ represents "shear stress", which is shear (horizontal) force per unit plate area[2].

Both 5.6 and 5.7 involve velocity gradients. If we assume a fluid velocity profile varying linearly with **y** as indicated in figure 5.3, equation 5.6 simplifies to:

$$F = \eta A \frac{u}{y} \tag{5.8}$$

Here **u** is just the plate velocity and **y** the fluid thickness. The SI units for η, $N \cdot s/m^2$, can be readily determined from dimensional analysis of equation 5.8.

5.5.2 Examples of Viscosity

Below is a table of viscosities of some common substances. Note that the viscosity of honey is much greater than that of oil, which is greater than that of water. This should fit with your intuitive concept of viscosity.

The SI units for viscosity are $N \cdot s/m^2$ or, equivalently $Pa \cdot s$. (`Pa = Pascal`, usually the SI unit of pressure). There is no named unit for viscosity in SI. However in the CGS system, the unit of viscosity is the *Poise* (pronounced "pwah"), after the physicist Poiseuille. For this reason you will often find viscosities quoted in Poise, and particularly in cP, or centiPoise, because water, a very important fluid, has a viscosity of roughly one cP. The table above quotes viscosities of some common fluids in cP and also in $milliNewton \cdot s/m^2$, which happens to be the same value. To convert to straight SI units, divide all figures in the table by 1000.

[2] "Stress" is everywhere in American culture, but the technical definition of stress is always and only a Force per unit area. Thus what we call pressure in physics can also be called compressive stress.

Substance	Viscosity $mN \cdot s/m^2$ centiPoise
Air (STP)	0.02
Water	1.0
Water (99 C)	0.3
Antifreeze	20
Olive Oil	80
SAE 30 motor oil	200
Honey	10000
Peanut Butter	250,000

Table 5.1: All quantities are measured at 20 C unless otherwise specified. Viscosities are all in round numbers (1 significant figure). The SI units for viscosity are $N \cdot s/m^2$.

5.5.3 Two types of viscosity

There is a second unit for viscosity, the *Stoke*, for George Stokes, the physicist whose law we discuss in section 5.5.5. The relation between viscosity in Stokes (typically designated ν) and viscosity in Poise (typically designated η) is as follows:

$$\nu = \frac{\eta}{\rho} \tag{5.9}$$

The quantity η is typically just called viscosity, whereas ν is more properly called *kinematic viscosity*. Kinematic viscosity is ordinary viscosity divided by the density (ρ) of the fluid or gas under study. Not all references make clear which type of viscosity is being quoted, however the unit (Stokes vs. Poise) should allow you to figure it out.

5.5.4 What causes Viscosity?

There seem to be two causes for viscosity. In the case of highly viscous fluids like motor oil, peanut butter or honey, long-chain molecules, be they hydrocarbons or proteins, get tangled and make intermittent connections as they slide past eachother. This tangling helps a molecule that is in contact with a moving surface to drag its neighbors along with it. As temperatures decrease the intermittent bonds become more effective; thus most highly viscous materials become still more viscous as they are cooled.

For low viscosity fluids, like air, or galaxies, viscosity is defined in terms of momentum transport across velocity gradients which occurs because of diffusion of

fast moving air molecules (interstellar gas / stars) into slow moving regions. Since diffusion is faster at higher temperatures, one would expect the viscosity of air to increase at higher temperatures, and it does.

In summary, the two mechanisms of viscosity have different origins and different temperature dependencies.

Lewis Richardson, a mathematical physicist who pioneered numerical techniques for weather forecasting, is perhaps best known for this bit of doggerel about the second mechanism (from his 1920 paper on atmospheric eddies): *Big whorls have little whorls that feed on their velocity, and little whorls have lesser whorls, and so on to viscosity.*

5.5.5 Stokes' Law

Look again at equation 5.8. The drag force on the top plate is proportional to η, $\mathbf{u}$, and A/y. Thus if we tried to calculate the drag force in some other configuration, for example, a sphere in a fluid, we might guess it would be of the following form,

$$F \simeq \eta L u \tag{5.10}$$

where L is a characteristic dimension of the object. For a sphere, the characteristic dimension is the diameter or radius. In 1851, George Gabriel Stokes derived the expression in equation 5.11 for the drag force on spherical objects in a viscous fluid. It has exactly the form of 5.10, the characteristic dimension of the sphere is its diameter, and there is a constant factor of 3π out front.

$$\vec{F}_d = -3\pi\eta D\vec{v} \tag{5.11}$$

The minus sign in equation 5.11 shows the viscous force always acts to oppose the current velocity of a particle, which fits our intuitive concept of drag.

5.6 Linear Drag in One-Dimension, Terminal Velocity

Armed with Stokes's formula for drag force, we can apply Newton's laws of motion to the study of objects subject to gravity and drag. To keep life simple, and true to the great history of the practice of physics, all objects of study will be spherical!

Consider the case of a small metal sphere at rest in water and released at time zero. Figure 5.4 shows a free-body diagram of this situation. Newton's 2nd law is only needed in the y-direction.

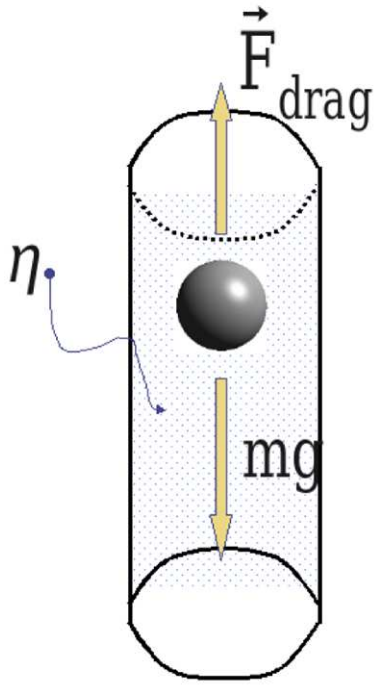


Figure 5.4: A sphere in a fluid will also experience a buoyant force that reduces its apparent weight. For simplicity, we merely consider the weight of the sphere acting downward and the drag force acting upward.

$$\vec{F}_{net} = m\vec{a} \tag{5.12}$$

$$\vec{F}_{net} = m\frac{d\vec{v}}{dt} \tag{5.13}$$

$$F_{net_y} = m\frac{dv_y}{dt} \tag{5.14}$$

$$mg - 3\pi\eta D v_y = m\frac{dv_y}{dt} \tag{5.15}$$

$$mg - bv = m\frac{dv}{dt} \tag{5.16}$$

Equation 5.12 is the full vector form of Newton's 2nd law. In 5.13 acceleration is rewritten as the derivative of velocity. In 5.14 only the y-component is considered. In 5.15, the the full form of Stokes' law is substituted and we define the positive direction for velocity as downward. Equation 5.16 groups all constants into **b**, thus: $b = 3\pi\eta D$. The y-subscript is dropped since we know velocity is only in the y-direction.

Equation 5.16 is called an ordinary differential equation because it is an equation with a derivative in it. You will learn general techniques for solving these in math class, if you have not already. As physicists though, we can make a lot of progress by guessing a solution. If our guess works, it is the solution. We guess a solution of the form 5.17. C_1 and C_2 are free constants that will be adjusted to fit the initial conditions of the problem.

$$v(t) = C_1 + C_2 e^{kt} \tag{5.17}$$

We plug 5.17 into 5.16 and see if a self-consistent solution arises. The following occurs.

$$\begin{aligned} mg - b(C_1 + C_2 e^{kt}) &= mC_2 k e^{kt} \\ mg - bC_1 &= mC_2 k e^{kt} + bC_2 e^{kt} \\ mg - bC_1 &= (mk + b)C_2 e^{kt} \end{aligned} \tag{5.18}$$

Equation 5.18 can only be true for all times if the left and right sides are separately zero.

$$\begin{aligned} mg - bC_1 &= 0 \\ \therefore C_1 &= \frac{mg}{b} \\ (mk + b)C_2 e^{kt} &= 0 \\ \therefore (mk + b) &= 0 \\ k &= -\frac{b}{m} \end{aligned} \tag{5.19}$$

Having found possible solutions for k and C_1, we still need C_2. Our initial condition is that $v(0) = 0$. This means

$$v(0) = C_1 + C_2 e^0 = C_1 + C_2 \therefore C_2 = -C_1 \tag{5.20}$$

$$v(t) = \frac{mg}{b}(1 - e^{-\frac{b}{m}t}) \tag{5.21}$$

$$v(t) = \frac{mg}{3\pi\eta D}(1 - e^{-\frac{3\pi\eta D}{m}t}) \tag{5.22}$$

Examine equation 5.22 for large times. The exponential term goes to zero, leaving

$$v(\infty) = \frac{mg}{3\pi\eta D}. \tag{5.23}$$

The quantity on right-hand side of equation 5.23 is referred to as *terminal velocity.*

5.7 Linear Drag in Two Dimensions

The case of linear drag in two dimensions is a straightforward extension of the one-dimensional problem and is a very worthwhile homework problem. One can even guess a solution of the form 5.17 for each component, though the constants will differ. Below we set it up. As in the last section, we begin with Newton's second law (5.24), and explicitly list the forces acting (5.25). As before, we define **b** (5.26). Because equation 5.27 is completely linear, the x-component and y-component may be treated independently. Equations 5.28 and 5.29 may be solved separately and the results combined to give the vector velocity versus time. Equation 5.28 is identical to equation 5.16, which was solved previously. The solution is not changed by the presence of an x-component of velocity. (It may be changed by different initial conditions, but these merely change the values of C_1 and C_2 from equation 5.17).

$$\vec{F}_{net} = m\vec{a} = m\frac{d\vec{v}}{dt} \tag{5.24}$$

$$m\vec{g} - 3\pi\eta D\vec{v} = m\frac{d\vec{v}}{dt} \tag{5.25}$$

$$3\pi\eta D \to b \tag{5.26}$$

$$m\vec{g} - b\vec{v} = m\frac{d\vec{v}}{dt} \tag{5.27}$$

$$mg - bv_y = m\frac{dv_y}{dt} \tag{5.28}$$

$$-bv_x = m\frac{dv_x}{dt} \tag{5.29}$$

5.8 Review of commands introduced in this chapter

For more information (and to see the related commands not discussed here) on each command type `>>help cmdname` at the *Matlab command line.*

hold() After plotting something, `hold on` makes subsequent plots overlay the existing plot instead of replacing it. The axes do not change. Goes with `hold off`.

close() `close(2)` closes figure 2. `close all` closes all figures. Because newly created figures pop up to the foreground of all open windows, but existing figures may remain hidden in the background until clicked, it can be convenient to close all figures before running a script.

logspace() Gives a logarithmically spaced array. `logspace(-1,3,50)` would give 50 numbers logarithmically spaced between 0.1 and 1000

movie() Plays a set of frames made with *Matlab* as a movie. This is the most general way to do animation. It was not discussed in any detail in this chapter.

pause() `pause(0.1)` would wait 0.1 seconds before continuing. Useful in animations to slow things down.

set(h,'XData', ...) Command introduced in previous chapter. Here we used it to directly set XData and YData for faster animation

xlim() Manually sets x-axis limits (disables autoscaling). `xlim([-1 7])` Would set x-axis limits at -1 and 7.

ylim() Manually sets y-axis limits.

5.9 End of Chapter Problems

(1) [**On paper**] Solve equation 5.3 for the range of the projectile when $y_0 \neq 0$. Show that it reduces to 5.5 for $y_0 = 0$.

(2) Plot sind(x) for x equal 1 to 450 with 5 degree increments. Make a title for your plot and set the fontsize to 16 point and change the font-type to "Times". Label x, and y axes and increase the fontsize to 14 points. Use the `p=plot ...` construction to get the handle to your plot. Use `get(p)` and `set(p)` to customize four different things about your plot. Explain what you changed.

(3) Write a script that plots the trajectory for two projectiles in the absence of air resistance. One is launched at 15° above horizontal and the other at 75° above horizontal. In this script use equation 5.3 to generate trajectories of y vs. x. Write a second script that generates the functions y(t) and x(t). Plot one vs. the other. Show the result of the two scripts is the same.

(4) Beginning with the script for exercise 3 above, add handle graphics to enlarge the text on the title and axis labels. Also include two separate text statements to document the initial conditions of the two projectiles. Plot data markers on top of the lines, and use different markers for each of the two plots.

(5) Beginning with the script for exercise 3 above, write a script that animates the trajectories for two projectiles simultaneously. Choose launch angles of $\theta = 25°$ and $\theta = 65°$, respectively.

Hint: A straightforward way to do this is to define two arrays x1, y1 for the first and x2, y2 for the second projectile. When you get the handle to do the animation, you probably want to get two separate handles, perhaps call them q1 and q2. You should make a pretty plot, as you have in prior exercises, but you need not bother with the fancy labels for initial angle and speed.

(6) The piston in a cold automobile engine has an 0.1 mm thick layer of SAE 30 motor oil between it and the cylinder wall. Assume the piston ring is a cylinder 1 cm high and 10 cm in diameter that is moving at 20 m/s relative to the wall. Write a script that prints out the test conditions above and then calculates and prints the force that the engine exerts overcoming the viscous drag of the oil.

(7a) Write a function that calculates the terminal velocity for spheres of arbitrary density and diameter falling in fluids of arbitrary viscosity.

(7b) Write a script that calls the function and generates a table with columns diameter, weight and terminal velocity for a steel sphere falling in water (temperature 20 C). The diameter of the sphere should range from one micron up to one meter and include 18 points logarithmically spaced between these radii.

Hint: Check out the **logspace** function.

(7c) Write a second script (or extend the first) to create the same table for steel falling in air at 20 C.

(8) [**On paper**] Calculate the terminal velocity for a 1 mm diameter steel sphere falling in water at temperature 20 C, ignoring buoyancy. (The density of steel is 7.8 g/cm^3.) Now repeat the calculation including buoyant force.

(9) Use your function from problem 7 to calculate the terminal velocity (in miles per hour) for a human falling in air. Document your assumptions in your script. The number you will come up with based on linear drag is going to sound very high compared to what you may have heard in stories about sky-diving. The reason for this is that the dominant drag force for a falling human is not linear drag. We will discuss this in the next chapter.

(10) [**On paper**] Examine equation 5.22. We have shown that the velocity goes to a constant velocity at very large times. Equation 5.22 also simplifies at very small times. To what does it simplify? Now integrate 5.22 to solve for y(t). What is the behavior of y(t) for very large and very small times. Explain why this is a reasonable result.

(11) [**On paper**] Work out the analytical solution for $v_y(t)$ and y(t) for a 0.1 mm raindrop starting from rest and falling down vertically.

(12) [**On paper**] Using the procedures of sections 5.6 and 5.7 as your guide, solve equations 5.29 and 5.28 for both components of velocity vs. time for a 1 cm steel ball which is initially travelling horizontally at speed 10 m/s in water at 20 C.

(13) [**On paper**] Calculate x and y vs. time for a mass subject to linear drag with initial velocity v_{0x}, and v_{0y}.

(14) Using the results of problem 13, write a script to plot the trajectory of a 1 mm steel ball fired at 30° above horizontal and 100 m/s. On the same plot include the trajectory of the ball if fired in vacuum. In your script, label the %DOCUMENT, %DEFINE, %DERIVE and %DISPLAY sections.

5.10 Code examples for this chapter

```
%DOCUMENT
%Calculates and plots trajectory of a heavy projectile
%given arbitrary initial conditions
%Usage:  >> projectile1
%DEFINE
g= 9.81;      %g, m/s^2
v0=30;        %initial speed, m/s
y0=10;       %launch altitude, m
x0=-20;       %launch position, m
theta=25;     %launch angle, degrees above horizontal
x=linspace(-20,100,150); %generates 150 evenly spaced
                         %points between 0 and 100 m
%DERIVE
y=y0+tand(theta)*(x-x0) ...
  -g*((x-x0).^2)/(2*(v0*cosd(theta))^2);
%DISPLAY
plot(x,y,'k')  %The 'k' makes the plotted line black
               %without it, line would be blue.
title('Trajectory of a heavy projectile')
xlabel('Position downrange (m)')
ylabel('Altitude (m)')
% ============  end of projectile1.m  =================
```

```
%DOCUMENT Calculates and plots trajectory of a heavy projectile
%for a certain range of times.
%Usage:  >> projectile3
%DEFINE
g= 9.81;              %g, m/s^2
v0=30;                %initial speed, m/s
y0=10; x0=-20;        %launch altitude, position, m
theta=25;             %launch angle, degrees above horizontal
t=linspace(0,4,30); %generates 30 evenly spaced points between 0
%DERIVE
x=x0+v0*cosd(theta)*t;
y=y0+v0*sind(theta)*t-(1/2)*g*t.^2;
%DISPLAY
p=plot(x,y,'k',x,y,'rh');
set(p, 'MarkerSize',16)
ax=gca;
set(ax,'FontSize',16,'XColor',[0 0.3 0]);
t=title('Trajectory of a heavy projectile');
set(t,'FontSize',20,'FontName','Times')
xl=xlabel('Position downrange (m)');
set(xl,'FontSize',16,'FontName','Times')
```

```
yl=ylabel('Altitude (m)');
set(yl,'FontSize',16,'FontName','Times')
xlim([-20 80])
ylim([0 20])
tx=text(-10,5,'v_0 = 30 m/s \newline \theta = 25^{\circ}');
set(tx,'FontSize',16,'Color',[0 0 0.7])
% ==========  end of projectile3.m  ==============================
```

```
%This puts name and date at bottom of current figure.
name='R. Sonnenfeld, NM Tech';
b=gca;  % Get current axis
x=get(b,'XTick'); %Get list of all tick marks on X-axis
xmin=x(1);xmax=x(length(x)); %First/last tick marks
  %give you range of plot
width=xmax-xmin;
y=get(b,'YTick'); %Get list of all tick marks on Y-axis
ymin=y(1);ymax=y(length(y));
  %y(length(y)) gives the last element of y.
height=ymax-ymin;
%Puts your name 10% below and
%   to the left of the origin of your plot.
text(xmin-0.1*width,ymin-0.1*height,name,'Fontsize',10)
%Puts date.  Note "date" is a function, not a variable.
text(xmax-0.1*width,ymin-0.1*height,date,'Fontsize',10)
% ==========  end of AddNameDate.m  ==================
```

```
%DOCUMENT Calculates and animates trajectory of a heavy projectile gi
%for a certain range of times.
%Usage:  >> projectile4
%DEFINE
g= 9.81;                %g, m/s^2
v0=30;                  %initial speed, m/s
y0=10; x0=-20;          %launch altitude, position, m
theta=25;               %launch angle, degrees above horizontal
len=100;
t=linspace(0,4,len); %generates len evenly spaced points between 0 an
%DERIVE
x=x0+v0*cosd(theta)*t;
y=y0+v0*sind(theta)*t-(1/2)*g*t.^2;
%DISPLAY
close all
plot(x,y,'k');
ax=gca; set(ax,'FontSize',16);
title('Trajectory of a heavy projectile','FontSize',20,'FontName','Ti
xlabel('Position downrange (m)','FontSize',16);
ylabel('Altitude (m)','FontSize',16);
xlim([-20 80])
```

```
22 ylim([0 20])
23 tx=text(-10,5,'v_0 = 30 m/s \newline \theta = 25^{\circ}');
24 set(tx,'FontSize',16,'Color',[0 0 0.7])
25 %ANIMATE
26 hold on
27 q=plot(x(1),y(1),'ro');
28 set(q, 'MarkerSize',16)
29 for i=1:len
30     if y(i) < 0
31         break;
32     end
33     set(q,'XData',x(i),'YData',y(i))
34     pause(0.03)
35 end
36 % ==========  end of projectile4.m  ==============================
```

Chapter 6

Quadratic Drag and the Euler Method

Figure 6.1: Sky-divers in wing suits enjoying quadratic drag.

6.1 Inertial Drag

6.1.1 What causes inertial drag?

The key to understanding inertial drag is "inertia", which means

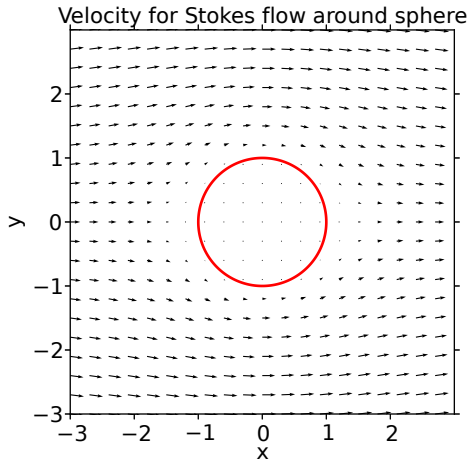


Figure 6.2: The sphere in this picture is moving to the right, which is equivalent to a stationary sphere and the fluid moving to the left, as shown. In the low speed (low Reynold's number) regime, flow looks similar to this. Note how the fluid speed goes to zero in the immediate vicinity of the sphere and only a volume about 3X the diameter of the sphere is disturbed by its presence.

resistance to change in momentum. If a body moves slowly enough through a fluid (or is small enough) the fluid can move out of its way by flowing around it as illustrated in figure 6.2. In this case the body only affects the layer of the fluid that is closest to it. Note in the figure above that the flow vectors only a bit more than two radii from the sphere center are almost undisturbed. This is the situation for viscous drag.

However, when the body moves more quickly (or is larger), the fluid does not have time to move around the body, some fraction of it is made to move forward; it is pushed ahead of the body. This situation is shown in figure 6.3. A plate of area $\mathbf{A}$ is moving to the right through a fluid of density ρ. After a time Δt the plate has moved to the right by a distance $v\Delta t$ and now occupies the position indicated by the dotted plane at the right of the figure. If we assume that *none* of the fluid leaks around the edges of the plate, then a volume of fluid given by $V = Av\Delta t$ is displaced by the plate. In addition, the volume of fluid affected by

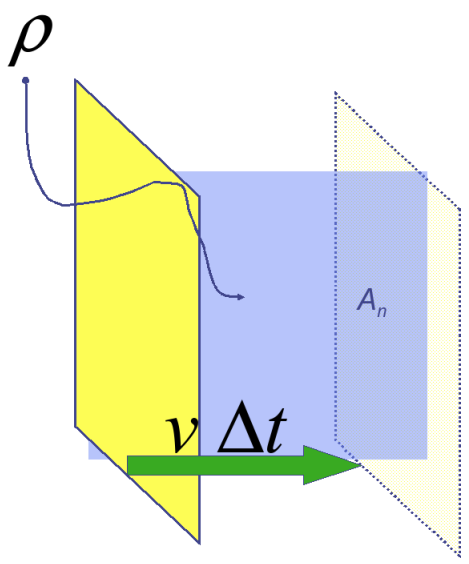


Figure 6.3: A flat plate of area **A** moves to the right at speed **v** in a fluid of density ρ.

the plate has to develop a velocity **v** to the right. It is useful to consider this in terms of momentum change. Mass **M** of fluid has acquired a velocity to the right. The momentum change of the fluid is $\Delta p = Mv$. The mass **M** is given by ρV, where ρ is the fluid density and **V** is the volume displaced as calculated above. Putting this all together gives the following relations.

$$\Delta p = Mv = \rho V v = \rho A v \Delta t v = \rho A v^2 \Delta t \tag{6.1}$$

$$\frac{\Delta p}{\Delta t} = \rho A v^2 = F_{drag} \tag{6.2}$$

Equation 6.2 was derived from 6.1 by dividing through by Δt and assuming that no fluid leaks around the edges of the plate. Clearly this is not true for a finite plate. However the following basic arguments hold in general for finite plates and other bodies:

1. Some fluid ahead of the body must acquire its speed **v**.
2. The mass of fluid per unit time that must accelerate is proportional to the speed of the body, the body cross-section, and the density of the fluid.

These two arguments lead us to expect a drag force proportional to v^2 that is also proportional to the plate area and the density of the fluid. More generally,

$$F_{inertial} = -\frac{1}{2} C_D \rho A_n v^2 \tag{6.3}$$

Where C_D is a constant called the "drag coefficient", and A_n represents the "normal area" or "cross-sectional area" of the body. Drag coefficients are the things that car and aircraft designers try to reduce, but they cannot change the fundamental v^2 dependence, which inevitably reduces the mileage of a car on a highway compared to one driving slowly down a country road.

For a sphere, C_D is found empirically to be roughly 1/2 for Reynolds numbers $< 100{,}000$. (Reynolds number is discussed in the next section). This gives us a useful drag law for a sphere of diameter D (our favorite physics shape!):

$$F_{sphere} = -C_D \frac{1}{2} \rho A_n v^2 = -\frac{1}{4} \rho \pi \frac{D^2}{4} v^2 = -\frac{1}{16} \rho \pi D^2 v^2 \tag{6.4}$$

6.2 Reynolds Number

6.2.1 Fluid mechanics and dimensionless parameters

Fluid mechanics problems (e.g. weather and climate models, ocean models, supersonic flight models) stress the limits of the latest supercomputers. The reason should be clear even at this stage of your scientific computing career. So far we have been studying the motion of a single ball (or perhaps two). If one wanted to model a single cubic centimeter of water at the atomic level, which is ultimately necessary since every atom has both vector position and velocity at every time, one would need to model $\simeq 10^{22}$ particles. Despite rapid annual advances in computing power, this approach to fluid mechanics is still impossible. Instead, macroscopic "parcels" of fluid are modeled, the number depending on the power of the computer and the needs of the simulation.

Also, in a still useful approach developed in pre-computer days, one can take combinations of macroscopic properties (such as density, viscosity, buoyancy, thermal conductivity, surface tension, average velocity) and derive empirical relations through experiment. Typically, fluid properties are combined in such a way that their units cancel out, leading to a "dimensionless number". There are many tens of dimensionless numbers that appear in fluid studies, bearing the names of many of the great engineers and scientists that have worked in this area. For example, the Ekman number relates viscous forces to fictitious forces caused by the Earth's rotation, and is useful in ocean studies. The Prandtl number gives the ratio of heat transport by direct conduction to that caused by fluid mixing. The Strouhal number is relevant if one studies the oscillation of wires caused by wind blowing across them. The Mach number gives the ratio of object velocity to speed of sound, and generally is a subject of much excitement among speed enthusiasts. These numbers get pretty specialized, but it is worth knowing that they exist should you ever end up with an exotic fluids problem.

6.2.2 Reynolds number and drag regimes

Of all the dimensionless numbers in classical fluid mechanics, perhaps the best known and most easily understood (next to Mach number) is the Reynolds number ($\Re$) – and it is particularly helpful in our study of drag. Reynolds number is defined as follows:

$$\Re = \frac{\rho v L}{\eta} \tag{6.5}$$

The above definition can be seen to be helpful if one considers the two types of drag (quadratic and linear) on a sphere. These are:

$$F_q = \frac{1}{16}\rho\pi D^2 v^2 \tag{6.6}$$

$$F_l = 3\pi\eta D v \tag{6.7}$$

Dividing 6.6 by 6.7 gives the relative importance of quadratic drag force to viscous drag force. We obtain:

$$\frac{F_q}{F_l} = \frac{\frac{1}{16}\rho\pi D^2 v^2}{3\pi\eta D v}$$

$$\frac{F_q}{F_l} = \frac{\frac{1}{48}\rho D v}{\eta} = \frac{\Re}{48} \tag{6.8}$$

Note that $\Re$ is precisely proportional to the ratio of the quadratic to linear drag, other than a constant factor (48). In general, the length L that appears in equation 6.5 is a "characteristic length of the object under study", a quantity subject to interpretation. For the case of a sphere, we have set $L = D$.

6.5 can tell us whether viscous or inertial forces are more important for any given problem. Based on 6.8, we would say that for values $\Re < 1$, quadratic drag is unimportant, for $\Re > 300$, linear drag is unimportant, and for values in between, both should be considered. It is worth a quick calculation of Reynolds number to see what form of drag (viscous or inertial) you should use in solving a drag problem[1].

[1]Reynolds number has other applications. Although the criterion is quite sensitive to the system under study, there is generally a threshold value of Reynolds number above which the flow around the object becomes turbulent and neither of the two drag equations listed above is correct.

6.3 One-dimensional (1-D) analytic solution for quadratic drag

The problem of linear drag (discussed in the last chapter) can be solved analytically for motion in one dimension, two dimensions, or three dimensions. The solution is straightforward because the drag force in each direction depends only on the velocity in that direction. The differential equation of motion is said to be "separable". The case for motion of a body under both gravity and square-law (inertial) drag is more difficult. The drag force in the x-direction depends on the velocity in x, y, and z directions. Only the 1-D case, where velocity is confined to the vertical direction (the same direction as gravity) can be solved analytically. All other cases (such as projectile motion in two dimensions) can only be solved numerically. We will proceed to numerical solutions, but it is worth first solving the one case that is amenable to an analytical solution.

In equation 6.10, we drop the "y" subscripts and define the downward direction as positive.

$$F_{net_y} = ma_y = m\frac{dv_y}{dt} \tag{6.9}$$

$$mg - \frac{1}{16}\pi\rho D^2 v^2 = m\frac{dv}{dt} \tag{6.10}$$

In this case, it is convenient to find the terminal velocity v_T before going further. We can do this by setting the right hand side of equation 6.10 to zero, since by definition the velocity is constant (and the derivative zero) at terminal velocity. This gives:

$$mg - \frac{1}{16}\pi\rho D^2 {v_T}^2 = 0 \quad \rightarrow \quad {v_T}^2 = \frac{16mg}{\pi\rho D^2} \tag{6.11}$$

Plugging 6.11 into 6.10 and rearranging, yields:

$$\frac{dv}{dt} = g(1 - \frac{v^2}{{v_T}^2}) \tag{6.12}$$

Equation 6.12 can be solved by direct integration if the velocity terms are grouped together on the left hand side and the time term on the right, thus:

$$\int_0^v \frac{dv}{(1 - \frac{v^2}{{v_T}^2})} = g\int_0^t dt \tag{6.13}$$

Note that the initial condition $v(0) = 0$ was used to set the lower limits of integration in 6.13. Next, a change of variable on the left side is made, and the trivial integral with respect to time on the far right of 6.13 is evaluated:

$$u = \frac{v}{v_T} \quad \rightarrow \quad dv = v_T\, du \qquad \rightarrow \qquad \int_0^u \frac{v_T\, du}{(1 - u^2)} = gt \tag{6.14}$$

Finally an appropriate trigonometric substitution is found, as follows:

$$\int \frac{dx}{(1-x^2)} = arctanh(x) \tag{6.15}$$

Solving the definite integrals in 6.15 using the indefinite integral from 6.14 gives:

$$v_T \; arctanh(v/v_T) = gt \quad \rightarrow \quad v(t) = v_T \; tanh(gt/v_T) \tag{6.16}$$

In the limit of large values of x, tanh(x) = 1. Thus, as $t \rightarrow \infty, \quad v \rightarrow v_T$. If our solution is reasonable, one would have expected the limiting velocity to be v_T. Since hyperbolic functions are not typically taught in a basic trigonometry course, they are introduced below in order that you may understand how to calculate with them.

6.3.1 Hyperbolic functions

The ordinary trigonometric functions can be defined in terms of unit circles, while the hyperbolic functions can be defined in terms of unit hyperbolae. While this fact explains why they are called hyperbolic functions, it is more useful to understand how they relate to the more familiar trig functions and to exponential functions.

You have doubtless seen Euler's identity:

$$e^{ix} = cos(x) + i \; sin(x) \tag{6.17}$$

Euler's identity can be inverted to express ordinary trig functions in terms of exponentials, thus:

$$cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \qquad sin(x) = \frac{e^{ix} - e^{-ix}}{2}, \qquad tan(x) = \frac{sin(x)}{cos(x)} = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \tag{6.18}$$

Removing the factor of $\sqrt{-1}$ from the exponentials in equations 6.18 yields useful definitions for the hyperbolic functions:

$$cosh(x) = \frac{e^{x} + e^{-x}}{2}, \qquad sinh(x) = \frac{e^{x} - e^{-x}}{2}, \qquad tanh(x) = \frac{sinh(x)}{cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \tag{6.19}$$

Given the good intuition physicists have for the behavior of exponentials, equations 6.19 should provide insight into hyperbolic functions. An additional interesting note is that, for angles that are imaginary numbers, the ordinary trig functions behave like hyperbolic functions, while the hyperbolic functions act like ordinary sines and cosines in imaginary space.

6.4 Euler Method

It was previously stated that the quadratic drag problem requires numerical methods for its general solution. The Euler method is the simplest of all the numerical

methods for solving differential equations[2]. It has adequate accuracy for first-order differential equations such as those involving drag. Its limitations appear if it is used for second-order equations like those appearing in harmonic oscillators and orbital motion. The greatest benefit of the Euler method is that proceeds directly from the definition of a derivative.

Consider the definition of instantaneous acceleration:

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \tag{6.20}$$

In calculus, we require Δt to become arbitrarily small. However, if we let Δt be small, but not infinitesimal, then equation 6.20 can still be a good approximation to a derivative. Rather than being a derivative, equation 6.20 is now called a "difference" equation, because it merely shows the difference between $v(t + \Delta t)$ and $v(t)$. If we are given the initial velocity v_0 and acceleration at time t=0, we can use 6.20 with a finite Δt to find the velocity (which we call v_1) at time $t = \Delta t$. Knowing v_1 allows one to use equation 6.20 again to calculate v_2, the velocity at time $t = 2\Delta t$. This procedure is continued until v has been approximated for all times of interest.

Let us apply the procedure to quadratic drag to make this all clear.

6.5 Applying Euler Method to Quadratic drag

6.5.1 One dimensional case

We begin with the one dimensional case already solved. Defining $c = \frac{\pi}{16}\rho D^2$ and dividing equation 6.10 by m yields:

$$\frac{dv}{dt} = g - \frac{c}{m}v^2 \tag{6.21}$$

Given $v_0 = 0$, we calculate v_1, v_2, v_3, etc. as follows:

$$\frac{dv}{dt} \simeq \frac{v_1 - v_0}{\Delta t} = g - \frac{c}{m}{v^2}_0 \tag{6.22}$$

$$v_1 = v_0 + (g - \frac{c}{m}{v^2}_0)\,\Delta t \tag{6.23}$$

$$v_1 = 0 + (g - 0)\,\Delta t \tag{6.24}$$

$$v_2 = v_1 + (g - \frac{c}{m}v_1^2)\,\Delta t \tag{6.25}$$

$$v_3 = v_2 + (g - \frac{c}{m}v_2^2)\,\Delta t \tag{6.26}$$

[2]Euler was such a great mathematician that his name appears in very many places. Please note that Euler's method in this section has nothing to do with Euler's identity in the previous section

Lest this still seem vague, recall that g, c, and m are all constants that are given with the problem. They are numbers. Also Δt is not just a symbol, it is a number. It may be 1 second, or 0.1 second, or 0.001 second, but it is well-defined. Thus the initial condition v_0 is used to calculate a numerical value for v_1. That newly calculated v_1 is used to calculate v_2, and so on for arbitrarily many time steps. Following the hitherto established pattern, we write an expression for v_n (the speed at time $t = n\Delta t$).

$$v_n = v_{n-1} + (g - \frac{c}{m} v_{n-1}^2)\,\Delta t \tag{6.27}$$

6.5.2 Matlab code for 1-D Euler Method

Equation 6.27 can be rewritten in *Matlab* syntax and inserted into a **for** loop as below:

```
function QuadEuler
%Simple numerical solution of 1-D drag
%================
%DEFINE
%Set Initial Conditions
v0=1;              %m/s A positive number here means ball is falling init
%----------------
%Set Parameters
% Fd=(1/16)*pi*D^2*rho*v^2=c*v^2
D=0.24;  % [m]       ball diameter
rho=1.3; % [kg/m^3]  air-density
m=0.624; % [kg]      ball mass
g=9.8;   % [m/s^2]
c=(1/16)*pi*D^2*rho;  %
%----------------
%Set Time Range
t0=0;              % [s] initial time
tf=6.333;          %     final time
deltaT=0.333;      %     time step
t=t0:deltaT:tf;
len=length(t);
v=zeros(1,len); %[m/s]  Set up array for velocity
v(1)=v0;           % Set initial condition
%================
%DERIVE
for i = 2:len
    v(i) = v(i-1) + (g- (c/m)*v(i-1)^2)*deltaT;
end  %The Euler method can be accomplished in one line of code
%================
%DISPLAY
plot(t,v,'r+',t,v,'r-')
```

```
32 xlabel('Time (s)','Fontsize',14);
33 ylabel('Y-speed (m/s)','Fontsize',14);
34 ttx=4; ytx=9; %Location of text on t, y axes
35 note1=sprintf('m=%6.3f kg',m); note2=sprintf('D=%5.2f m',D);
36 text(ttx,ytx,note1,'Fontsize', [16]) %top line of explanatory te
37 text(ttx,ytx-3,note2,'Fontsize', [16]) %2nd line of text
38 text(ttx,ytx-6,'\rho_{air} = 1.3 kg/m^3', 'Fontsize', [16]) %3rd
39 % LaTeX recognizes greek letters (with a slash) and ^ for supers
40 end
41 % ========== end of QuadEuler.m ==============================
```

Note that the Euler method took only three lines of code (26-28). Most of the script is concerned with setup and display of the problem.

6.5.3 Using sprintf() to annotate plots

A scientific plot should be as self-documenting as possible. In other words, the content should be understandable without reference to other data. In this regard, there are features of lines 34-38 above worth noting. Using `ttx` and `ytx` as variables on line 34 allows the easy positioning of explanatory text by changing only two numbers, rather than having to adjust lines 36-38 to move the annotations around on the plot. Line 38 reminds you that LaTeX can print any greek letter on a plot, provided you preceed it with a backslash. Finally, **sprintf()** can be used just like **fprintf()**. The difference is that **fprintf()** outputs to the screen or a file. On the other hand **sprintf()** outputs to a variable, as shown on line 35. Plugging the strings created on line 35 into the **text** commands on 36, 37 allows us to annotate our plot with the value of the parameters used in the simulation, rather than having to update by hand the explanatory text.

6.5.4 Quadratic drag in 2-Dimensions

Having solved numerically the 1-D problem which is also solvable analytically, let us extend our methods to the intractable 2-D problem. Adding a second dimension requires us to tackle two new challenges:

1. How to create a velocity unit vector, $\hat{v}$
2. How to solve a coupled differential equation.

In full vector form, equation 6.21 becomes:

$$\frac{d\vec{v}}{dt} = \vec{g} - \frac{c}{m} v^2 \, \hat{v} \tag{6.28}$$

Remembering that $\hat{u}$ for arbitrary vector $\vec{u}$ is by definition $\vec{u}/|u|$, we can write:

$$v^2 \, \hat{v} = (v_x^2 + v_y^2) \frac{(v_x \, \hat{i} + v_y \, \hat{j})}{\sqrt{v_x^2 + v_y^2}} \tag{6.29}$$

or by components:

$$v^2\,\hat{v} = \sqrt{v_x^2 + v_y^2}\,v_x\hat{i} + \sqrt{v_x^2 + v_y^2}\,v_y\hat{j} \tag{6.30}$$

By components, 6.28 now becomes

$$\frac{dv_x}{dt} = -\frac{c}{m}\,v_x\sqrt{v_x^2 + v_y^2} \tag{6.31}$$

$$\frac{dv_y}{dt} = g - \frac{c}{m}\,v_y\sqrt{v_x^2 + v_y^2} \tag{6.32}$$

This is called a coupled differential equation because $\frac{dv_x}{dt}$ is affected by v_y as well as v_x, while $\frac{dv_y}{dt}$ depends on both v_x and v_y. There are tricks for dealing with coupled systems. A change of variable may be able to decouple them. However, this system cannot be decoupled and cannot be solved analytically. Numerical methods are the simplest way to proceed.

Preparing to replace our differential equation by difference equations, we first replace $\sqrt{v_x^2 + v_y^2}$ with the less clunky notation v, thus:

$$\frac{dv_x}{dt} = -\frac{c}{m}\,v_x\,v \tag{6.33}$$

$$\frac{dv_y}{dt} = g - \frac{c}{m}\,v_y\,v \tag{6.34}$$

The first three difference equations for v_y become:

$$v_{1y} = v_{0y} + (g - \frac{c}{m}v_0\,v_{0y})\,\Delta t \tag{6.35}$$

$$v_{2y} = v_{1y} + (g - \frac{c}{m}v_1\,v_{1y})\,\Delta t \tag{6.36}$$

$$v_{3y} = v_{2y} + (g - \frac{c}{m}v_2\,v_{2y})\,\Delta t \tag{6.37}$$

and the general expression:

$$v_{ny} = v_{n-1\,y} + (g - \frac{c}{m}v_{n-1}\,v_{n-1\,y})\,\Delta t \tag{6.38}$$

We leave to the homework problems the translation of the difference equations to *Matlab*, but the difficult work is already done. Likewise, the student is expected to be able to fill in the steps to generate the x-axis difference equations.

6.5.5 Getting position from velocity

We have learned how to solve the difference equations to get velocity vs. time. In order to calculate trajectories, however, position vs. time is needed. The Euler method can be readily used to integrate velocity $\vec{v}$ to yield position $\vec{r}$.

$$\frac{d\vec{r}}{dt} = \vec{v} \tag{6.39}$$

The vector equation 6.39 is generally separable into independent component equations:

$$\frac{dy}{dt} = v_y, \qquad \frac{dx}{dt} = v_x, \qquad \frac{dz}{dt} = v_z \tag{6.40}$$

While equations 6.40 look merely like definitions of velocity, they can in fact be converted into difference equations for the calculation of position. For the y-component this yields:

$$\frac{dy}{dt} \simeq \frac{y_1 - y_0}{\Delta t} \tag{6.41}$$

$$y_1 = y_0 + (v_{y_0})\,\Delta t \tag{6.42}$$

$$y_2 = y_1 + (v_{y_1})\,\Delta t \tag{6.43}$$

$$y_3 = y_2 + (v_{y_2})\,\Delta t \tag{6.44}$$

$$\vdots$$

$$y_n = y_{n-1} + (v_{y_{n-1}})\,\Delta t \tag{6.45}$$

The approach for the x and z-components is identical.

6.6 End of Chapter Problems

(1) [**On paper**] What is the terminal velocity for a human falling in air at sea-level and 20 C assuming quadratic drag? Calculate the Reynolds number for the human traveling at this velocity. Discuss its meaning in this problem and explain whether we were justified to neglect linear drag.

(2a) Type the function *QuadEuler* given in 6.5.2 into an *m-file* and run it. Add your name to the lower left corner of the figure it produces. Print a .jpeg of the figure and submit it, along with your script.

(Hint1) You can add your name using **gtext** or the *AddNameDate* script from section 5.2.2.

(Hint2) See section 4.2.3 for advice regarding the printing of figures to `jpegs`. Please submit jpegs, NOT `.fig` files.

(2b) Run the script from part (a) twice more with the following changes. Set `deltaT=1` and plot **v** vs. **t** using black lines and + signs. Set `deltaT=0.1` and plot using green lines and pentagrams. Add your name to lower left corner, print a `.jpeg` of the figure and submit it. Add a comment to your script describing how the difference between the three graphs might help you know when you have chosen a correct timestep `deltaT`.

(Hint) All three plots should appear on the same graph. If you type **hold on** after first plot, this will happen.

(3a) Write the function corresponding to this function header:

```
function [v,t]=squareLawDrag1D(v0,time,D,rhoBall,rhoFluid)
% Uses the Euler method to calculate velocity vs time
% for a sphere in an arbitrary fluid.
% v0    [m/s]    Initial velocity (negative for down)
% time    [s]    Total time to run simulation
% D     [m]   Sphere diameter
% rhoBall  [kg/m^3] Density of sphere
% rhoFluid [kg/m^3] Density of fluid
```

You will test it in 3b.

(3b) Write a script that calls the function and plots the results for the following parameters. `v0=[0,5,10,20,25], time=7 s, D=0.25 m, rhoBall=7.81 kg/m^3, rhoFlui` All five plots should appear on the same axes. Add your name, make a jpeg of the plot and upload it along with your script.

(4) Plot velocity vs. time for a 1 cm diameter steel ball dropped from rest in water. Use **subplot()**. On the top plot, show velocity vs. time using the analytical 1-D solution. On the bottom plot, give velocity vs. time based on your numerical solution. Annotate the plots to show the important parameters, diameter, density, etc.

(5) In *Dialogues of two New Sciences*, Galileo wrote:

> `But I, Simplicio, who have made the test, can assure you that a cannon ball weighing one or two hundred pounds or even more, will not reach the ground by as much as a span ahead of a musket ball weighing only half a pound, provided both are dropped from a height of 200 cubits ... the larger outstrips the smaller by two finger-breadths.`

It is rumored (but not certain) that Galileo did this demonstation at the leaning tower of Pisa. Regardless, you can do this experiment in your computer. How close is Galileo? Document your assumptions by comments in your scripts. Approach the problem any way you choose.

(6a) Write the functions corresponding to these function headers:

```
function vterm=vtermQuad(D,rhoBall,rhoFluid)
function vterm=vtermLin(D,rhoBall,etaFluid)
% D     [m]   Sphere diameter
% rhoBall  [kg/m^3] Density of sphere
% rhoFluid [kg/m^3] Density of fluid
% etaFluid [Pa s] Viscosity of fluid
```

The functions should be able to handle **D** as a vector as well as a scalar, and should return `vterm` of the same dimensions as **D**.

(6b) Create a log-spaced array of 50 diameters from 10^{-6} m up to 1 m. Use the functions just written in part (a). Plot terminal velocity vs. diameter for spheres of water falling through air at 20 C for both types of drag. Add a comment explaining what the graphs demonstrate about which type of drag is more important for the case of falling people and for the case of ten-micron cloud droplets.

(7) Plot the vertical and horizontal components of velocity vs. time for a 1 cm steel projectile launched at 300 m/s at 30° above horizontal and subject to quadratic drag. Use the difference equations developed in section 6.5.4. On the same plot, include the two components of velocity for the projectile launched in vacuum.

(8) Plot the trajectory (y(t) vs. x(t)) for a 1 cm steel projectile launched at 300 m/s at 30° above horizontal. On the same plot, include the trajectory of the projectile launched in vacuum.

(Hint) Begin with your velocity vs. time results from the previous problem and integrate to get position vs. time.

(9) Plot the trajectory for a 1 cm steel projectile launched at 300 m/s at 30° subject to only linear drag. Include the trajectory in vacuum on your plot. In your script, label the %DOCUMENT, %DEFINE, %DERIVE and %DISPLAY sections.

(10) Write a script that animates the trajectory you plotted in the previous problem. In your script, label the %DOCUMENT, %DEFINE, %DERIVE and %DISPLAY sections. You should notice that only the %DISPLAY section needs to change.

6.7 Code examples for this chapter

```
function QuadEuler
%Simple numerical solution of 1-D drag
%================
%DEFINE
%Set Initial Conditions
v0=1;           %m/s A positive number here means ball is falling init
%----------------
%Set Parameters
% Fd=(1/16)*pi*D^2*rho*v^2=c*v^2
D=0.24;  % [m]       ball diameter
rho=1.3; % [kg/m^3] air-density
m=0.624; % [kg]      ball mass
g=9.8;   % [m/s^2]
c=(1/16)*pi*D^2*rho;  %
%----------------
%Set Time Range
t0=0;             % [s] initial time
tf=6.333;         %     final time
deltaT=0.333;     %     time step
t=t0:deltaT:tf;
len=length(t);
v=zeros(1,len); %[m/s]  Set up array for velocity
v(1)=v0;          % Set initial condition
%================
%DERIVE
for i = 2:len
    v(i) = v(i-1) + (g- (c/m)*v(i-1)^2)*deltaT;
end  %The Euler method can be accomplished in one line of code
%================
%DISPLAY
plot(t,v,'r+',t,v,'r-')
xlabel('Time (s)','Fontsize',14);
ylabel('Y-speed (m/s)','Fontsize',14);
ttx=4; ytx=9; %Location of text on t, y axes
note1=sprintf('m=%6.3f kg',m); note2=sprintf('D=%5.2f m',D);
text(ttx,ytx,note1,'Fontsize', [16]) %top line of explanatory text
text(ttx,ytx-3,note2,'Fontsize', [16])  %2nd line of text
text(ttx,ytx-6,'\rho_{air} = 1.3 kg/m^3', 'Fontsize', [16]) %3rd line
% LaTeX recognizes greek letters (with a slash) and ^ for superscript
end
% ==========  end of QuadEuler.m  ============================
```

Chapter 7

Newton's Universal Law of Gravitation

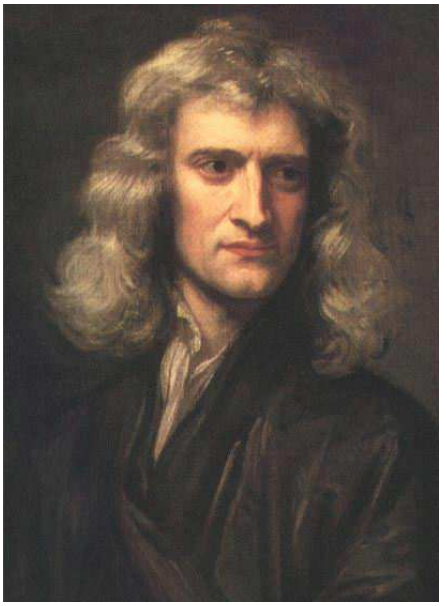


Figure 7.1: Nature and nature's laws lay hid in night; God said "Let Newton be" and all was light. – Alexander Pope

7.1 Introduction

Using Newton's Law of Universal Gravitation, we will begin to discuss orbital motions of planets. Along the way, we will have to review a lot of physics. In particular, we will talk about uniform circular motion, angular momentum, gravitational potential energy, and the division of motions between relative motion and center of mass motion. All of these concepts are broadly applicable in mechanics, but we'll use them to better explain orbits, and use orbits to give concrete examples of each of these important concepts.

7.1.1 Linear and Angular motion

All of the quantities of linear motion have equivalents in rotational motion. An entire chapter of a physics text is usually dedicated to explaining the relationships between the linear quantities and the angular quantities. We will not do that here, but as we will need to refer to these quantities in the rest of this chapter, a table is provided to remind you of the various quantities and clarify their relationships.

Linear Qty	Unit	Linear Name	Angular Qty	Unit	Angular name
x	m	position	θ	radians	angle
$v = dx/dt$	m/s	velocity	$\omega = d\theta/dt$	rad/s	angular velocity
$a = dv/dt$	m/s^2	acceleration	$\alpha = d\omega/dt$	rad/s^2	angular accel.
$\vec{F}$	N	Force	$\vec{\tau} = \vec{r} \times \vec{F}$	N $\cdot$ m	Torque
$\vec{p}$	kg $\cdot$ m/s	momentum	$\vec{L} = \vec{r} \times \vec{p}$	kg $\cdot$ m^2/s	angular momentum

Table 7.1: For every linear quantity defined in physics, there is a rotational equivalent. In this chapter we are most concerned with angular velocity, force, and torque, as well as linear and angular momentum.

If this is really the first time you are seeing these definitions, bring it to the attention of your professor for some suggested references.

7.2 Newton's Law of Universal Gravitation

In 1687 Isaac Newton published the *Principia Mathematica* in which he included his universal law of gravitation. In modern notation, the law allows one to calculate the mutual attraction between any masses m_1 and m_2 knowing only the distance separating them r thus:

$$F = \frac{Gm_1m_2}{r^2} \quad G = 6.67 \times 10^{-11} \quad [SI\ Units] \tag{7.1}$$

The universal gravitational constant G is a difficult constant to measure. G was first measured by Henry Cavendish in 1796. He measured the force of attraction

between two 160–kg lead spheres and 700–g test masses suspended from a torsion balance (a beam suspended by a fine thread). Cavendish did not claim to be measuring G, but rather to be "weighing the Earth". In fact, the idea for the experiment and the initial apparatus design came from geologist John Michell.

We know that $W = mg$ gives the weight of an object on the surface of the Earth. The local acceleration of gravity, g, is easy to measure and was well known in Cavendish's time. Using 7.1 to calculate the force that the Earth exerts on any object, Cavendish realized that:

$$W = m_{object}\, g = \frac{G\, m_{object}\, m_{Earth}}{R_{Earth}^2} \quad \rightarrow g = \frac{G m_{Earth}}{R_{Earth}^2}$$

Since R_{Earth} and g are both known, measuring G allows one to calculate m_{Earth}. Thus Cavendish was correct to say that he was weighing the Earth. He obtained a value of 5.448 times the density of water for the average density of the Earth. That is within about 1% of the value used today.

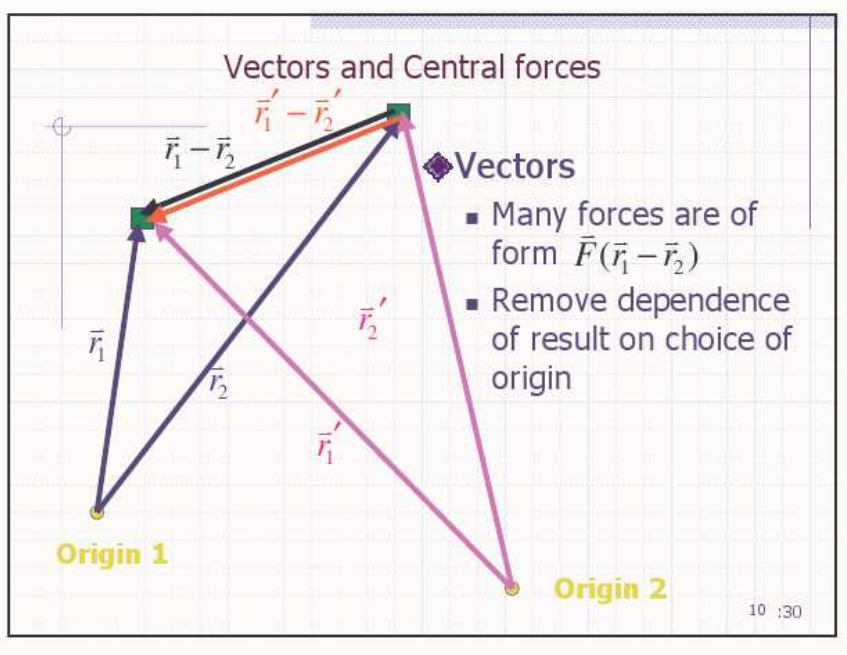


Figure 7.2: Central forces are always directed along the line joining two particles. Vectors $\vec{r}_1$ and $\vec{r}_2$ differ considerably in length and direction from $\vec{r}_1'$ and $\vec{r}_2'$ in this figure. Nonetheless, the difference $\vec{r}_1 - \vec{r}_2$ and $\vec{r}_1\,' - \vec{r}_2\,'$ are the same vector which points from particle 2 toward particle 1 .

7.2.1 Vector form of Law of Gravitation

The fundamental forces of physics (gravitation, electromagnetism, nuclear forces) are directed along lines joining two bodies of interest. Though the force depends on the positions of both particles $\vec{r}_1$ and $\vec{r}_2$, it is only the difference in their position that matters, not their absolute location. We can express this mathematically by saying the force is of the form $\vec{F}(\vec{r}_1 - \vec{r}_2)$, This simple (obvious?) idea can be generalized to say that fundamental forces are the same everywhere in the universe. They depend on the "symmetry" of an isotropic (everywhere the same) universe. Forces which are of the form $\vec{F}(\vec{r}_1 - \vec{r}_2)$ are called "central forces", and expressing them in vector form is particularly advantageous for numerical simulations of motion involving central forces.

Figure 7.2 is intended to clarify that equation 7.2 is independent of the origin chosen in which to solve the problem at hand. Note that vectors $\vec{r}_1$ and $\vec{r}_2$ differ considerably in length and direction from $\vec{r}_1\,'$ and $\vec{r}_2\,'$. However, the difference $\vec{r}_1 - \vec{r}_2$ and $\vec{r}_1\,' - \vec{r}_2\,'$ are the same vector which points from particle 2 toward particle 1. This means that we are free to solve our problem by choosing whatever origin makes the mathematics most convenient. One good choice is to choose the origin at the location of the center of mass of the two-particle system, but ANY choice leads to the same result.

Reformulating equation 7.1 in vector form gives:

$$\vec{F} = Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \tag{7.2}$$

At first glance equation 7.2 may trouble you. You know perfectly well that gravity is an inverse-square law force. Why then is there a cubic denominator in 7.2? Taking the magnitude of the numerator, we breathe a sigh of relief to see it cancel with one term in the denominator to give back the inverse square law dependence. The power of 7.2 is most evident when you do not take the magnitude of the denominator, because this is the correct vector expression of the direction as well as the magnitude of the gravitational force. In fact, in two dimensions we can write

$$F_x = Gm_1m_2\frac{x_1 - x_2}{|\vec{r}_1 - \vec{r}_2|^3} \tag{7.3}$$

$$F_y = Gm_1m_2\frac{y_1 - y_2}{|\vec{r}_1 - \vec{r}_2|^3} \tag{7.4}$$

This is another example of the power of vector notation. The gravitational law written in the form of equations 7.3 is the most directly useful for writing numerical simulations of orbits (the subject of the next chapter). While numerical simulations are not restricted to the simplest cases, it is necessary to understand the exactly solvable problems in order at least to confirm that ones numerical simulation agrees with well-established physics. It is also important to you as a physics student to be able to solve the easy cases with pencil and paper – thus we come to Kepler's laws.

7.3 Kepler's Laws

7.3.1 Introduction

Roughly 80 years before Newton published *Principia*, Johannes Kepler published three laws of planetary motion. Kepler's laws were empirical; they succinctly described important properties of planetary orbits, but they did not explain the properties of orbits in terms of fundamental physics, which we, as descendants of Newton, can now do.

Kepler's observations were as follows:

1. Planets move along ellipses, with the sun at one focus.
2. Planets speed up when closer to the sun, and generally move in such a way as to "sweep out equal areas in equal times".
3. The square of a planet's orbital period is proportional the cube of the semi-major axis of its ellipse.

We will now tackle these one-by-one and discuss the physics behind each law.

7.3.2 Planets move in ellipses with the sun at one focus

Kepler's first law turns out to be a direct mathematical consequence of equation 7.1. There is an analytical solution to the differential equation of motion of planets using polar coordinates, and it explicitly demonstrates that orbits are conic-sections (of which the ellipse is the most general closed form). The form of the solution for orbits depends on the exponent in the denominator. If gravity were anything *other* than an inverse square force, planetary orbits would not be closed. We leave the analytical solution to somewhat more advanced texts, but we will check the form of planetary orbits numerically in the next chapter and verify that only an inverse square law gives closed elliptical orbits. Figure 7.3 shows an elliptical orbit with the major axis marked with an a. "Semi-major axis" just means half of a, the major (long) axis of the ellipse. For circular orbits, a becomes R, the constant orbital radius.

Though we are not solving the differential equation here, we can discuss the closed form solution in more detail. The solution is:

$$r(\theta) = \frac{a(1-e^2)}{1 + e\ cos(\theta)} \tag{7.5}$$

Here "e" stands for eccentricity. θ is the polar angle, and r is the distance from the focus of the ellipse to the orbiting body. Figure 7.3 illustrates this. Please note that if point P represents a planet, then the sun is located at focus F_1,

and *not* at the center of the ellipse.[1] Nothing is located at F_2. Looking at this equation and the figure we observe that the planet P is closest to the sun at $\theta = 0$, when $r_{min} = a(1 - e)$. Likewise, setting $cos(\theta) = -1$ and doing some algebra gives $r_{max} = a(1 + e)$.

Equation 7.5 also allows a general understanding of the shapes of orbits as a function of eccentricity:

e = 0 : In this case, $r = a$ for all θ. Clearly, this describes a circular orbit

0 < e < 1 : In this case the orbit is elliptical. Closest approach occurs at $\theta = 0$, and the planet is most distant when $\theta = \pi$.

e = 1 : In this case, the closest approach at $\theta = 0$ is easy to calculate, but when $\theta = \pi$ the denominator goes to zero. For e=1, orbits become parabolae.

e > 1 : Once $e\ cos(\theta) > 1$, equation 7.5 becomes undefined. Thus there is a range of angles forbidden, and these angles describe the asymptotes of a hyperbolic orbit. The closest approach still occurs for $\theta = 0$.

Equation 7.5 is worth understanding, because published data on planetary orbits tends to quote eccentricities. The brief discussion to this point should enable you to interpret an eccentricity in terms of the general shape of the orbit and the relationship between the closest and most distant approach of an object to its orbital companion.

We will skip to Kepler's third law (relating period to semi-major axis) as it will help us develop concepts needed to prove the second law.

7.3.3 Kepler's Period Law

7.3.3.1 Fictitious forces – What holds the planets up?

We can use equation 7.1 to get the magnitude of the attractive force on a planet of mass m_2 at distance R from a much larger object (such as the sun) of mass m_1. So the problem of gravitational attraction is no problem. Some, however may be disturbed by the other problem. "What force holds the planets up"? That is, since we only know of an attractive force and no repulsive force, why do the planets not plummet into the sun? There are two ways to answer this question, one of which is generally deprecated by physics educators. I will give you the deprecated answer first, because it addresses the common intuition that there must be a counterbalancing force "holding up the planets". I will then explain the more general (and ultimately less confusing) answer to the question.

If you consider the problem of planetary motion in an *accelerated reference frame*, in this case a frame attached to Earth that is traveling in a circular path around

[1] The sun could also be located at F_2, but if we define $\theta = 0$ on the positive x-axis, then it is at F_1.

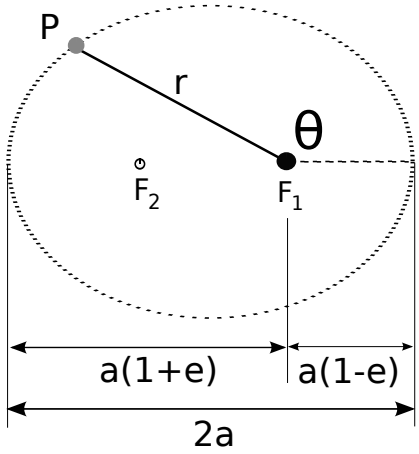


Figure 7.3: Elliptical orbit of planet at point P. Note that **r** is *not* measured from the center of the ellipse, but rather from one focus. **e** represents the eccentricity of the ellipse. **a** is the semi-major axis. The sun would be at focus F_1 such that $a(1-e)$ gives the point of closest approach (perihelion), and $a(1+e)$ is the greatest distance (aphelion). .

the Sun, there is another force, which physicists refer to as a "fictitious force". This force is called "centrifugal" (center-fleeing) force, and its magnitude is given by:

$$F_c = \frac{m_2\, v^2}{R}$$

Here, m_2 is the mass of our planet, v its speed along the circular path it is assumed to follow, and R the radius of the circular orbit. We can readily balance this force against the force from 7.1, and thus calculate the speed at which a planet must orbit the sun to maintain a circular orbit. Knowing the planet's speed and orbital radius, it becomes a simple matter to calculate the travel time for one orbit (called the orbital *period* and represented T). Thus one can verify Kepler's third law for the special case of a circular orbit.

Centrifugal force is something with which we all have direct experience. Everyone has ridden in a car rounding a corner, a carousel, or a roller coaster. Being a passenger in an accelerating vehicle is the very definition of living in an "accelerated reference frame", and in these frames "fictitious forces" are very real. You feel them directly.

The confusion comes rapidly when one tries to handle a body that is not in a circular orbit, or multiple bodies. For a planet not in a circular orbit, the centrifugal force is generally not exactly balancing the force of gravity. Consider the Earth-Sun sytem: At *aphelion*, the gravitational force exceeds the centrifugal force, and thus Earth does indeed get closer to the sun as it continues around

its orbit. Likewise, at *perihelion*, centrifugal force exceeds gravity, and Earth begins to pull away from the Sun again. From the reference frame of Earth, we could calculate the deficit of gravity with respect to centrifugal force, or vice-versa, and thus calculate the radial acceleration of the Earth, but it begins to get confusing. Where the use of an accelerated reference frame gets really out of hand is when trying to understand multiple bodies (e.g. Earth, Sun, Moon). The radial line joining Earth and Moon is not in same direction (generally) as that joining Earth and Sun. One then would need to resolve the centrifugal force on Earth into components along the radial vector joining Earth to Moon, and constantly update it as the three bodies continued to move. Even worse, an object which is moving in an accelerated reference frame feels an additional fictitious force called the Coriolis force. One *can* calculate in accelerated frames, but it is not simpler than working in non-accelerated frames. Having milked the centrifugal force for what we could and seen where it becomes more trouble than it is worth, let us return to the conventional way of describing orbits using *inertial reference frames*.

7.3.3.2 Inertial reference frames

A *reference frame* in physics is roughly the same as a coordinate system in mathematics. The difference is that mathematicians generally do not talk about where their coordinate system is anchored. It is "on the blackboard", or on a piece of paper, or a computer screen. Physicists are a little more grounded. A reference frame for us is a coordinate system that is attached to an object. Thus a "laboratory reference frame" is an imaginary set of axes where the origin is anchored to some location in a room[2]. An inertial reference frame is a reference frame anchored to an object that is considered to be moving at constant velocity with respect to other inertial reference frames. Although it may sound like a recursive definition, an inertial reference frame is one in which Newton's first law is true. If you place a marble on a desk in a class room, or the deck of a ship moving over smooth seas, it will not roll (if the desk/deck are flat). If you place a marble on the floor of a merry-go-round or of an airplane accelerating for takeoff, the marble WILL roll. In an inertial reference frame, an object at rest remains at rest. In a non-inertial (accelerated) frame, it does NOT. Thus the classroom/ship are considered to be inertial reference frames, and the merry-go-round/airplane to be accelerated reference frames in which the rolling of the marble is explained by "fictitious forces".

7.3.3.3 Uniform circular motion

To handle circular motion in an inertial reference frame, we will need to calculate centripetal acceleration. Centripetal (*center seeking*) acceleration is the acceleration needed to keep an object on a circular path at constant speed. The formula

[2]Einstein famously pointed out that our coordinate system had to include a time axis as well as x, y, and z axes, but for the cases treated in this book, we can treat time as fixed and immutable in all coordinate systems or reference frames.

$a_c = v^2/r$ is used in introductory physics courses, but some find it confusing that an acceleration term appears up in a problem wherein the speed of the moving object never changes. Conceptually, acceleration is necessary whenever a velocity changes. In uniform circular motion, the magnitude of the velocity (speed) is indeed constant, but the direction of the velocity is continually changing. This is why there is an acceleration term. The following demonstrates how to use Calculus to derive the magnitude of the acceleration felt in circular motion. Figure 7.4 shows a planet in a circular orbit making an angle θ with the x–axis.

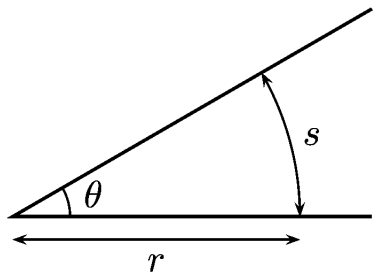


Figure 7.4: Arc-length **s** and angle are connected via $s = r\theta$.

The position $\vec{r}(\theta)$ is thus

$$\vec{r} = rcos(\theta)\,\hat{i} + rsin(\theta)\,\hat{j} \tag{7.6}$$

The arc-length formula (which is also the definition of a radian) says that $s = r\theta$ for circle of radius **r** and arc-length **s**. We also know that, for an object moving at constant speed v on an arc, $s = vt$. Combining these simple formulae, and assuming that $\theta = 0$ at time zero, we have $\theta(t) = vt/r$. Equation 7.6 becomes:

$$\vec{r} = r\ cos(vt/r)\,\hat{i} + r\ sin(vt/r)\,\hat{j} \tag{7.7}$$

Now that $\vec{r}$ is explicitly a function of t, we can differentiate twice to calculate acceleration.

$$\frac{d\vec{r}}{dt} = -r(v/r)\ sin(vt/r)\,\hat{i} + r(v/r)\,cos(vt/r)\ \hat{j} \tag{7.8}$$

$$\frac{d^2\vec{r}}{dt^2} = -r(v/r)^2\,cos(vt/r)\ \hat{i} - r(v/r)^2\,sin(vt/r)\ \hat{j} \tag{7.9}$$

Comparing 7.9 to 7.7, and then plugging in the definition of the radial unit-vector in polar coordinates $\hat{r}$, we see that 7.9 can be rewritten as:

$$\begin{aligned}\frac{d^2\vec{r}}{dt^2} &= -\frac{v^2}{r^2}\vec{r}(t) \\ &= -\frac{v^2}{r}\frac{\vec{r}}{r} \\ &= -\frac{v^2}{r}\,\hat{r}\end{aligned} \tag{7.10}$$

We identify the quantity v^2/r in equation 7.10 as centripetal acceleration, and see that it is perfectly real and that it's magnitude and direction fall directly from the definition of acceleration for the given system.

7.3.3.4 Orbits in inertial reference frames

In section 7.3.3.1, we said the Earth was not actually an inertial frame because it accelerates around the Sun. For the relatively insensitive measurements we do in most of the world"s labs, the acceleration of Earth around Sun is not noticable, thus the Earth is "approximately inertial". However, to study planetary motions, one might want to imagine oneself standing, God-like, in a fixed point in space somewhere in the solar system and watching the planets go by. As explained in our discussion of vectors, any fixed point will do, but it can be particularly convenient to assume the fixed point is the center of the sun, which we are temporarily assuming to be immobile and pinned like a giant fly to the fabric of the universe. From this vantage point, there is only one force on the earth; gravity pulling it always directly toward the sun. Since there is no balancing force to gravity, the Earth is indeed accelerating along a radial vector directed at the sun. It "stays up" for the same reason that artificial satellites in Earth orbit stay up. It has circumferential velocity which carries it around the sun as it is falling towards it. We just proved that all objects in uniform (constant velocity) circular motion have a centripetal acceleration expressed as follows:

$$a_c = \frac{v^2}{r} \tag{7.11}$$

Using Newton's second law, we set $F_{net} = m_2\, a$, and choosing a radial axis to apply it we get:

$$F_g = m_2 a_c \tag{7.12}$$

$$\frac{Gm_1m_2}{r^2} = m_2\frac{v^2}{r} \tag{7.13}$$

$$v_{circular} = \sqrt{\frac{Gm_1}{r}} \tag{7.14}$$

$$\tag{7.15}$$

This result is the same as would be obtained by the fictitious force method. Better yet, the vector form, equation 7.3 can be used for numerical calculation, and then

the acceleration components are obtained directly. For numerical calculations, it does NOT need to be assumed that the path of the planet is circular. At any given radius, if the planet is given an initial velocity $v_0 = v_{circular}$, the planet will move in a circle. If the initial velocity is chosen so that $v_0 < v_{circular}$, the planet will move in an ellipse, and if $v_0 > v_{circular}$, the planet will assume a hyperbolic orbit.

It is now a quick step to derive Kepler's period law for circular orbits.

7.3.3.5 Kepler's period law for circular orbits

We have previously shown that the semi-major axis of an ellipse **a** becomes the radius **r** when the ellipse becomes a circle. Knowing **v** for an orbit of radius **r**, we wish to find **T**, the orbital period. Since **v** is constant in magnitude:

$$\begin{aligned} T &= \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{\frac{Gm_1}{r}}} \\ T^2 &= \frac{4\pi^2 r^3}{\sqrt{Gm_1}} \\ T^2 &= \frac{4\pi^2 a^3}{\sqrt{Gm_1}} \end{aligned} \tag{7.16}$$

Alternatively, using the reciprocal relationship between ω and **T** that applies to both circular and harmonic motions:

$$\begin{aligned} v^2 = (\omega)^2[r^2] &= \left(\frac{2\pi}{T}\right)^2 [r^2] \\ \frac{4\pi^2}{T^2}[r^2] &= \frac{Gm_1}{r} \\ T^2 &= \frac{4\pi^2 r^3}{\sqrt{Gm_1}} \end{aligned} \tag{7.17}$$

We are now ready to discuss Kepler's second law.

7.3.4 Planets sweep out equal areas in equal times

Kepler's second law is a direct consequence of the conservation of angular momentum, and angular momentum conservation is a direct consequence of having a central force. Let us see how.

7.3.4.1 Rotational form of Newton's 2nd Law

Newton's second law of motion, the foundation of classical mechancics, says that a net force causes a change in momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{7.18}$$

For systems in rotational motion, there is an analogous form which can be derived by taking the derivative of the angular momentum $\vec{L}$ as defined in the introduction to this chapter:

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} \tag{7.19}$$

Using the multiplication rule for a derivative of a cross product:

$$\frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \tag{7.20}$$

The first term on the right-hand side become zero because a vector ($\vec{v}$) crossed with itself is zero.

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times \vec{p} = \vec{v} \times m\ \vec{v} = 0 \tag{7.21}$$

Plugging first equation 7.18 and then the definition of torque into equation 7.20 leaves us with

$$\frac{d\vec{L}}{dt} = \vec{\tau} \tag{7.22}$$

Equation 7.22 is the rotational form of Newton's third law. It says that angular momentum changes only in response to torque. Alternately, in the absence of torque, angular momentum is conserved. We will now use this fact to prove Kepler's second law.

7.3.4.2 Conservation of Angular Momentum

Plugging equation 7.2 in to the definition of torque $\vec{\tau} = \vec{r} \times \vec{F}$, and using the rotational form of Newton's second law:

$$\vec{r} \times \vec{F} = \vec{r} \times G m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} = \frac{d\vec{L}}{dt} \tag{7.23}$$

The relevant $\vec{r}$ on the right-hand side of equation 7.23 is the vector that joins the two bodies, in other words $\vec{r}_1 - \vec{r}_2$. 7.23 becomes

$$\frac{d\vec{L}}{dt} = (\vec{r}_1 - \vec{r}_2) \times Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} = 0 \tag{7.24}$$

However, 7.24 contains a vector crossed with itself, which is identically zero. For this reason gravity (and all central forces) exert no torques, and thus angular momentum is conserved. The statement that angular momentum is conserved is exactly equivalent to Kepler's second law, but it requires some argument to establish that this is so.

7.3.4.3 Equivalence of Kepler's second law and angular momentum conservation

Figure 7.5 shows a planet orbiting the sun. According to Kepler, the three shaded triangles should have the same area. *Planets sweep out equal areas in equal times.* The triangles as drawn do not really have the same area, and they are not triangles. Kepler's law is a statement about derivatives, e.g. $dA/dt = const.$ or $(\Delta A/\Delta t = const.)$. So in the limit of small times, the shaded regions really are triangles, the curvature at their base can be ignored. We know that $A = h\,b\,/\,2$, for **h** the height of the triangle, and **b** its base length. For the triangles drawn, **h** and the length of the longest side are not equal. In fact, for the triangles as drawn, the length of the two longest sides differs from eachother. We ignore all these complications. In the limit of small triangles, the two longer sides are both equal to eachother and equal to **h**. Thus, for small times, **h** can be assumed equal to **r**, the distance between planet and Sun. Next consider **b**. **b** is approximately equal to the arc-length **s**, thus $b \simeq v\Delta t$. For the case of the triangles drawn near aphelion and perihelion, we have the simplifying assumption that **b** and **h** are perpindicular to eachother. For the third triangle at intermediate distance from the sun, it is clear that **b** and **h** are NOT perpindicular. We must correct for this and only count the piece of **b** that is at right angles to **h**. We are ready to put this all together. For the perihelion and aphelion case (only):

$$\Delta A = hb/2 = rv\Delta t$$

For the more complicated intermediate case, we want something that will capture only the part of **b** that is perpindicular to **h**. This is what a cross-product does. Thus in general:

$$\Delta A = hb/2 = \vec{r} \times \vec{v}\Delta t$$

or

$$\frac{\Delta A}{\Delta t} = \vec{r} \times \vec{v}$$

Now we remember the definition of angular momentum:

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\ \vec{v}$$

Thus we see that:

$$\frac{\Delta A}{\Delta t} = \vec{r} \times \vec{v} = L_0/m$$

and we already know that the initial angular momentum L_0 never changes under purely central forces.

We have now put all of Kepler's empirical laws on strong physical footing. Let us see what other things about planetary motion are easy to calculate.

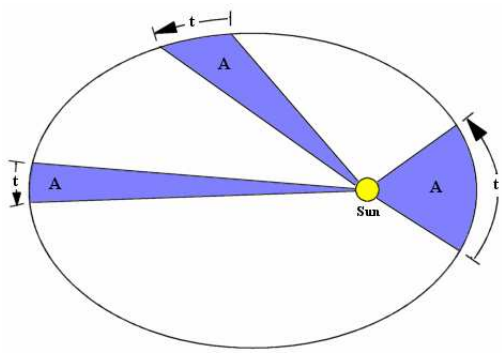


Figure 7.5: A planet in an elliptical orbit with the sun at one focus. Kepler's second law says that the three shaded triangles all have the same area. We wish to establish that this is equivalent to stating that central forces conserve angular momentum.

7.4 Gravitational Potential

We know in general that, for conservative forces, a potential (U) may be defined at follows:

$$U = -\int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \vec{F} \cdot d\vec{r}$$

For the case of gravity, this integral becomes:

$$U = -\int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \frac{Gm_1m_2\ \vec{r} \cdot d\vec{r}}{r^3} \tag{7.25}$$

Since 7.25 is a line integral, it has to apply to any path taken between points $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We can use that to our advantage, and use a path that goes directly on a

radial path between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Then the dot-product in the line integral becomes simple multiplication and we only need to evaluate:

$$U = -\int_{\mathbf{a}}^{\mathbf{b}} \frac{Gm_1m_2\,dr}{r^2} = \frac{Gm_1m_2}{r}\Bigg|_{\mathbf{a}}^{\mathbf{b}} = (Gm_1m_2)\left(\frac{1}{a} - \frac{1}{b}\right)$$

As with other potentials in physics, it is only the difference in potential between **a** and **b** that matters. It is conventional to choose point **a** at ∞, so that we are left with:

$$U = -(Gm_1m_2)\left(\frac{1}{b}\right)$$

More commonly, **b** is replaced by **r**, thus:

$$U = -\frac{Gm_1m_2}{r} \tag{7.26}$$

Note that 7.26 is always negative. It has its maximum value, zero, for $\mathbf{r} \to \infty$. This is not a problem. It is only energy differences that have a physical effect. One can just as easily subtract two negative numbers from each other as two positive numbers.

7.4.1 Escape Velocity

Using 7.26 and energy conservation, one can calculate the final speed of any projectile **m** launched from a massive object **M** as follows:

$$K_i + U_i = \frac{1}{2}mv_i^2 - \frac{GmM}{r_i} = K_f + U_f = \frac{1}{2}mv_f^2 - \frac{GmM}{r_f} \tag{7.27}$$

If **M** is a planet, one can readily use 7.27 to calculate the initial velocity needed to completely escape the influence of the planet.

7.4.1.1 Example: Calculating escape velocity from Earth

Our goal is to reach $r_f = \infty$. To just barely get there, our final velocity v_f needs only be incrementally greater than zero. Thus

$$K_f + U_f = \frac{1}{2}m(0)^2 - \frac{GmM}{\infty} = 0 - 0 = 0 \tag{7.28}$$

We can then solve for initial velocity:

$$K_i + U_i = 0 = \frac{1}{2}mv_i^2 - \frac{GmM}{r_i} \tag{7.29}$$

$$0 = \frac{1}{2}v_i^2 - \frac{GM}{r_i} \tag{7.30}$$

$$v_i = \sqrt{\frac{2GM}{r_i}} \tag{7.31}$$

Plugging in values for a projectile launched from the surface of the Earth:

$$\begin{aligned} v_i &= \sqrt{\frac{2GM}{r_i}} \\ &= \sqrt{\frac{2\left(6.67 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}\right) \cdot 5.97 \times 10^{24}\ kg}{6.4 \times 10^6\ m}} \\ v_i &= 11.1 \times 10^4\ m/s \end{aligned}$$

Note that equation 7.31 does not specify a direction for the velocity, only a magnitude. You could as well launch your projectile horizontally as vertically and it would escape to infinity (as long as it did not hit anything en route!).

7.5 Gravitation of Extended Bodies

The beautiful simplicity of the law of universal gravitation is indisputable. There is however a proverbial *fly in the ointment*. The law as expressed by equation 7.2 only applies to point particles, yet we have been blithely applying it to planets, which are definitely not points. It is a marvelous simplification that any object with spherical symmetry behaves as if all of its mass were concentrated at a point at its center.[3] This is absolutely not an obvious result. Consider standing on the Earth. There is a large amount of mass directly at your feet, a mere few meters away. There is vastly more mass 100 km from you, and still more if you consider the entire mass of the Earth, most of which is thousands of kilometers from where you stand. Yet it can be treated as if it were all $\simeq 6400$ km from you, concentrated at a single point. This is so far from obvious that Newton delayed publishing the Principia until he could prove this point (for which he also had to invent integral Calculus).

We are humbled by Newton's achievement *sui generis*, as we follow in his footsteps. Our approach will be to show that the force from a spherical shell on test-mass M_1, as sketched in figure 7.6 is the same as it would be if the shell were concentrated at a point at its center.[4].

Once we prove this, we can take an arbitrary number of shells with a common center. If each shell acts like it is concentrated at a point, then the sum of shells, a solid sphere, also must act like it is concentrated at a point. What remains then, is to show that a single spherical shell exerts the same force as a point mass at its center.

Figures 7.6 define almost all that we need. To calculate the total force from a spherical shell we will break it into circular strips like the one shown in the

[3] This is only really true if you are further from its center than its radius, but we will explain all that in what follows.

[4] Those of you who know and understand Gauss's law may say that this is obvious. You may look at the derivation that follows as a "proof" of Gauss's law. More precisely, we will show that saying a force has an inverse square distance dependence is equivalent to saying it follows Gauss's law

figure. For each strip, dF, the contribution of gravitational force from that strip, is calculated. When we have a clean expression for dF, we will integrate to get the total force. To begin:

$$dF = GM_1\, cos(\phi)\, \frac{dm}{r^2} \tag{7.32}$$

The equation looks similar to Newton's law of gravity for a point charge, except for the $cos(\phi)$ term and the mysterious construction dm. dm is physics shorthand for integrating over a set of mass elements. In this case, dm is the mass of the circular strip that contributes to force dF. Each piece of strip dm tugs in a different direction. However, by symmetry, only the components of force along line OM_1 are non-zero. The multiplier $cos(\phi)$ picks out those components along OM_1 and allows the rest of the force vectors from dm to cancel.

The next job is to calculate dm in terms of the total mass $\mathbf{m}$ of the spherical shell. For this purpose it is convenient to define σ, the mass per unit area of the shell. Thus:

$$dm = \sigma dA \tag{7.33}$$

It may seem that dA is not an improvement over dm, but dA, the area of the shaded strip, can be expressed in terms of position coordinates, what we are integrating over, whereas dm is not readily expressable in this way. What then is dA?

$$dA = 2\pi R\, sin(\theta) R\Delta\theta = 2\pi R^2\, sin(\theta)\Delta\theta \tag{7.34}$$

Equation 7.34 was obtained by considering that the area of the strip is its circumference times its width. The circumference of the strip is 2π times its radius. However its radius is not simply R, rather it is $R\, sin(\theta)$. The right panel of the figure is designed to make this "obvious". Having defined σ to help us understand how to integrate over dm, we can now get rid of it again and simplify dm.

$$dm = \sigma dA = \frac{m}{4\pi R^2} dA = \frac{m}{4\pi R^2} 2\pi R^2\, sin(\theta)\Delta\theta = \frac{m}{2}\, sin(\theta)\Delta\theta \tag{7.35}$$

Next, we need to figure out how to express r^2 for the strip shown. The figures should make clear that every element of the shaded strip is the *same* distance from M_1, and that distance is r_1. Can we express r_1 in terms of θ? By using the law of cosines, we can!

$$r_1{}^2 = r^2 + R^2 - 2rR\, cos(\theta) \tag{7.36}$$

We also need to express ϕ in terms of other variables. The law of cosines can be applied to any angle of a triangle. Thus:

$$R^2 = r^2 + r_1^2 - 2rr_1\, cos(\phi) \tag{7.37}$$

Solve for ϕ and θ.

$$cos(\theta) = \frac{r^2 + R^2 - r_1^2}{2rR} \tag{7.38}$$

$$cos(\phi) = \frac{r^2 + r_1^2 - R_1^2}{2rr_1} \tag{7.39}$$

Plugging 7.35 into 7.32 yields:

$$dF = GM_1\ cos(\phi)\ \frac{m}{2}\ \frac{sin(\theta)\Delta\theta}{r_1^2} \tag{7.40}$$

Differentiating both sides of 7.38 yields

$$sin(\theta)\Delta\theta = \frac{-2r_1 dr_1}{2rR} = \frac{r_1 dr_1}{rR} \tag{7.41}$$

Now plugging 7.41 and 7.39 into 7.40 produces:

$$dF = GM_1\ \frac{m}{2}\ \frac{dr_1}{rR}\ \left[\frac{r^2 + r_1^2 - R^2}{2rr_1^2}\right] = \frac{GM_1 m}{4r^2 R}\ \left[\frac{r^2 + r_1^2 - R^2}{r_1^2}\right] dr_1 \tag{7.42}$$

Grouping the constant terms into a convenient symbol, $\chi = \frac{GM_1 m}{4r^2 R}$, we can now integrate over r_1.

$$\int dF = \chi \int_{r-R}^{r+R} \left[\frac{r^2 + r_1^2 - R^2}{r_1^2}\right] dr_1 \tag{7.43}$$

$$F/\chi = \int \frac{r^2 - R^2}{r_1^2} dr_1 + \int dr_1 \tag{7.44}$$

$$= \frac{R^2 - r^2}{r_1} + r_1 \Big|_{r-R}^{r+R} \tag{7.45}$$

$$F = 4R\chi = \frac{GM_1 m}{r^2} \tag{7.46}$$

We did it! After much manipulation, the integral over all mass points in a sphere reduces to $F = \frac{GM_1 m}{r^2}$, exactly what would be true if the sphere was a single point concentrated at its center. This was Newton's genius, to show that replacing spheres by points is not an approximation but an exact equivalency.

Another very interesting result also falls out of this. We have implicitly assumed that $\mathbf{r} > \mathbf{R}$, that we are measuring the force from a point outside the sphere. There is nothing about the previous work that requires this however. If $\mathbf{r} < \mathbf{R}$, the lower limit of integration changes from $r - R$ to $R - r$. This corresponds to the case of measuring the force on a mass that is inside a hollow sphere, but still a distance $\mathbf{r}$ from its center. If you plug the new limits into 7.45, you will see that the force is identically zero!!

7.6 Center of Mass

There is another approximation we have been making in treating planetary motion. We have assumed that the Sun is fixed and the planets orbit it. How does this happen? Does the smaller body always orbit the larger? What if they are roughly the same mass? A little introspection reveals that it does not make sense that the smaller body "knows" it must orbit the larger.

In fact, the sun is not fixed in space. For the purpose of considering only Earth's orbit, it is reasonable to approximate it as fixed because it is so much more massive than the other planets. This is not always so. The Earth is only about eighty times as massive as the moon, and double star systems exist where both bodies have roughly equal masses. The way to handle these cases is to generalize our approach and say that planets do not orbit the sun, but rather that planets and Sun orbit their common *center of mass*.

Center of mass is defined as follows:

$$\vec{R}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \tag{7.47}$$

Here M is the total mass of the system (the sum over all the m_i). You already have a good intuition about center of mass, which is equivalent to *center of gravity*. If you imagine a humongous see-saw with the Earth and Moon placed on either end of it, the see-saw would balance if you placed the pivot at the location of the center-of-mass of the Earth-Moon system. Equation 7.47 is just the mathematics that guarantees that this is so.

7.6.1 Example: Calculating center of mass of four point-masses

Consider figure 7.8 showing four masses arranged in a square. Noting that the largest masses are on the top of the square and that the sum of the two on the left is somewhat larger than the total mass on the right of square, we expect the CM to be somewhat left of center and rather near the top of the square. The sample calculation below gives exactly that result.

$$\vec{R}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \tag{7.48}$$

$$\begin{aligned}
\vec{R}_{CM} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4} \\
&= \left(\frac{1}{1+2+4+8}\right)\left[(1)(0) + (2)(3\hat{i}) + (4)(3\hat{i} + 3\hat{j}) + (8)(3)\hat{j}\right] \\
&= \left(\frac{1}{15}\right)\left[18\hat{i} + 36\hat{j}\right] = 1.2\hat{i} + 2.4\hat{j}
\end{aligned}$$

7.6.2 Correcting Kepler's period law

We know that neither planet in an orbiting couple is stationary, so let us re-derive Kepler's period law taking this into account. Equation 7.14 now becomes two equations, one for each mass:

$$\frac{Gm_1m_2}{r^2} = m_2\frac{v_2^2}{r_2} \tag{7.49}$$

$$\frac{Gm_1m_2}{r^2} = m_1\frac{v_1^2}{r_1} \tag{7.50}$$

Here $\vec{r} = \vec{r}_1 - \vec{r}_2$, and thus $r = r_1 + r_2$, where $\vec{r}$ is the vector connecting m_2 to m_1 and r is the total distance between the bodies, r_1 is the radius of the circle made by mass one and r_2 the radius made by mass two (about the center of mass). Using 7.47, we can relate r_1, r_2, and r. Specifically:

$$\vec{R}_{CM} = (1/M)(m_1\vec{r}_1 + m_2\vec{r}_2) \tag{7.51}$$

$$0 = (1/M)(m_1\vec{r}_1 + m_2\vec{r}_2) \tag{7.52}$$

$$= m_1r_1 - m_2r_2 \tag{7.53}$$

$$m_1r_1 = m_2r_2 \tag{7.54}$$

In equation 7.51, the origin of the coordinate system was chosen to be at $\vec{R}_{CM}$. To arrive at equation 7.54, one notes that $\vec{r}_1$ and $\vec{r}_2$ point in opposite directions relative to the CM.

One can from here immediately proceed to solve equations 7.49 and 7.50 to obtain period as function of separation **r**, so long as one allows that r_1 and r_2 are unequal and *also* that v_1 and v_2 are unequal. It is somewhat easier to eliminate v_1 and v_2 from the equations. Since the angular velocity ω (about the center of mass) for a rotating pair of objects *is* equal, one can replace v_1 and v_2 with ωr_1 and ωr_2 before proceeding. Either body may then be used to finish the derivation. Let us choose m_1.

$$\frac{Gm_1m_2}{r^2} = m_1\omega^2 r_1 \tag{7.55}$$

$$r = r_1 + r_2 = r_1\left(1 + \frac{m_1}{m_2}\right)$$

$$r_1 = \left(\frac{r}{1 + m_1/m_2}\right)$$

$$\frac{Gm_1m_2}{r^2} = m_1\omega^2\left(\frac{r}{1 + m_1/m_2}\right)$$

$$= \frac{m_1m_2}{m_1 + m_2}\omega^2 r \tag{7.56}$$

$$\frac{G(m_1 + m_2)}{r^2} = \omega^2 r$$

$$= \left(\frac{2\pi}{T}\right)^2 r$$

$$\frac{GM}{r^2} = \frac{4\pi^2}{T^2}r$$

$$T^2 = \frac{4\pi^2 r^3}{\sqrt{GM}} \tag{7.57}$$

Note that equation 7.57 is identical to 7.17, except that we have replaced the mass m_1 of the "fixed" body with the total mass **M** of the two-body system.

7.6.3 Reduced Mass (μ)

An interesting construction occurs in equation 7.56. The combination of masses seen there occurs repeatedly when discussing two body problems in gravitation and in collisions. It occurs so often that it is given a name, "reduced mass", and a symbol, μ.

$$\mu = \frac{m_1m_2}{m_1 + m_2} \tag{7.58}$$

If one does a dimensional analysis, one notes that μ has dimensions of kg, like mass. As illustrated by figure 7.9, one can analyze the two-body system in which both bodies move by replacing the bodies with two fictional bodies, M and μ. Here M is fixed in space and only μ moves. μ is the same distance r from M, that m_1 is from m_2. Below is repeated the same derivation that began with equation 7.55, using the reduced mass concept. Because equation 7.59 replaces 7.55, the derivation takes three lines, rather that ten.

$$\frac{GM\mu}{r^2} = \mu\omega^2 r \tag{7.59}$$

$$\frac{GM}{r^2} = \frac{4\pi^2}{T^2} r$$

$$T^2 = \frac{4\pi^2 r^3}{\sqrt{GM}} \tag{7.60}$$

The derivation beginning with 7.59 may seem dissappointing. The reduced mass μ was introduced, and it vanishes again in the 2nd line. However μ appears with regularity in two-body problems because it falls out of conserving momentum about the center of mass. Next we will discuss three places where μ shows up, leaving the detailed derivations as exercises for the reader.

The angular momentum of a two-body system is:

$$\vec{L} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2 \tag{7.61}$$

This can also be expressed as:

$$\vec{L} = M\vec{R}_{CM} \times \vec{V}_{CM} + \mu\vec{r} \times \vec{v}. \tag{7.62}$$

Similarly, the kinetic energy of two arbitrary bodies is:

$$K = (1/2)m_1 v_1^2 + (1/2)m_2 v_2^2. \tag{7.63}$$

This is equivalent to:

$$K = (1/2)MV_{CM}^2 + (1/2)\mu v^2. \tag{7.64}$$

In the above examples, $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative position vector, $\vec{v} = \vec{v}_1 - \vec{v}_2$ is the relative velocity vector, and $\vec{V}_{CM}$ is the velocity vector of the center of mass of the system.

Figure 7.9 rather dramatically shows the Earth with a bicycle chain locking it in place. One might well wonder what it is locked to. Because all the dynamics of a two-body system can be broken into velocity/position of the center of mass of the system and relative velocities/positions. motions, one can jump into a frame moving with the center of mass, and in that frame, the body **M**, appears to be locked.

7.6.4 The Bohr Atom

To close our discussion of two-body systems, we consider the "tiny solar system" of a hydrogen atom. Atoms are held together by electic forces, not gravity, but μ shows up in this analysis as well because of its connection with momentum

conservation. The effect of using μ rather than the mass of the electron is fairly small, because a proton is 1860 times as massive as an electron, but, because experiments with light wavelengths can be so precise, one can see the effect of momentum conservation even on tiny atomic systems.

As quantum mechanics got under way physicists came to realize that the orbiting bodies in atoms were spread out into clouds. However, one can use nothing but classical mechanics and get a completely correct result for the energy of a photon emitted by an ion of a hydrogen atom.

Replacing Newton's Law of Gravitation with Coulomb's law and assuming a circular orbit:

$$\frac{kq_1q_2}{r^2} = \mu\omega^2 r \tag{7.65}$$

In equation 7.65, $q = 1.602176 \times 10^{-19}$ Coulomb is the charge of the electron and proton that make up a hydrogen atom. The constant **k** is the Coulomb constant, and has the value $k = 8.987652 \times 10^9 \quad N\ m^2/C^2$. The only bit of quantum mechanics we need tells us the radius of the orbit of an electron in a hydrogen atom in its "ground state". In classical mechanics, an orbit can have any radius, but the first realization of quantum mechanics was the orbital radii came in discrete steps, and there was a smallest possible radius. The ground-state radius of hydrogen is called the "Bohr radius", and denoted by **a**.

$$a = \frac{\hbar^2}{kq^2\mu} = 5.29178 \times 10^{-11} meters \tag{7.66}$$

The new constant $\hbar = 1.05457 \times 10^{-34}$ Joule-seconds is Planck's constant divided by 2 pi. Quantum mechanics also relates the frequency of a photon ω to its energy, E as follows:

$$E = \hbar\omega \tag{7.67}$$

In a wonderful demonstration of how well classical and quantum mechanics mesh together, if you plug the value of ω that you would calculate from equation 7.65 into equation 7.67, you obtain an energy which is exactly the atomic bond energy and the energy of a photon emitted when an electron recombines with a hydrogen ion.[5] This is even more wonderful than it seems. We have made a connection between ω the frequency of the electron orbitting the proton and ω the frequency of a photon emitted by a recombining ion. The classical theory of electromagnetic radiation explains why these two quantities in fact should be connected. For now, I hope you find it fascinating that you have just calculated the frequency of a photon from the frequency of rotation.

[5] In electron volts, this energy is 13.60569, and is also called the Rydberg energy.

7.7 Review

The beautiful simplicity of the law of universal gravitation is indisputable. Yet we are blessed by mathematical good fortune from multiple directions. The law as expressed by equation 7.2 really only applies to point particles. Our first great good fortune is the simplification which we derived in section 7.5 which says that bodies with spherical symmetry may be treated as if all their mass was concentrated at their centers. Our second fortune, derived in section 7.6.2 applies to all central forces. It allows us to ignore the fact that both bodies in a gravitational couple must move in order to conserve momentum, and to replace two moving bodies with two fictitious bodies; mass M, which is fixed and mass μ, which moves. Fundamental properties like the orbital period, energy, and angular momentum can be calculated by assuming one body is fixed. Finally, because gravity is an inverse square force, the orbits of all couples are closed, they repeat closely revolution after revolution.

7.8 End of Chapter Problems

(1) – **paper** Use equation 7.1 to determine the SI units for G. Convert the magnitude of G so it is useful for CGS units. What are the CGS units for G?

(2) – **paper** Like Cavendish, you know g and R_{Earth}, and thanks to him and others who have improved on his experiment, you know G. So "weigh the Earth". When you are done, calculate the average density of the Earth and compare it to the value Cavendish obtained. Just for fun, calculate the average density of Saturn too. (You'll have to look up its mass and radius). You have probably heard Saturn would float in a very large bathtub. Do your calculations confirm this myth? .

(3) – **paper** Make your own sketch similar to figure 7.2. On a full sheet of paper, label points "A" and "B" separated by a few centimeters. Pick *three* different origins at random (you might enlist the help of a cat or other randomizing agent). Draw vectors from the three origins to points "A" and "B" and using the graphical method of vector addition sketch the difference vectors $\vec{r}_1 - \vec{r}_2$, $\vec{r}_1\,' - \vec{r}_2\,'$, and $\vec{r}_2\,'' - \vec{r}_1\,''$. Does your work confirm that the choice of origin does not affect the calculated force?

(4) – **paper** Figure 7.3 expresses r_{min}, r_{max}, and b in terms of semi-major axis **a** and eccentricity **e**. Do the algebra to verify these relationships.

(5)– **paper** Kepler's period law can be derived exactly and easily for circular orbits beginning with equation 7.15. For simplicity, you may you assume that the larger mass is fixed in space and orbitted by the smaller mass.

(6)– **paper** Calculate the escape velocity for the Moon. Next calculate the escape velocity from the surface of the Martian moon Deimos (assuming it

is spherical, which it is not). The average density of Deimos is low, only 1.5 times that of water. Its average radius is 6.2 km. It is reputed that one could achieve escape velocity from Deimos by jumping. Do you agree?

(7)– **paper** Into what radius must you compact an object with Earth's mass in order that the escape velocity be equal to the speed of light? This is the classical way to derive the Schwarzchild radius for a black hole. What would be the (classical) Schwarzchild radius of a black hole with one solar mass? Your results are within a factor of two of the full general relativistic result – with far less math! .

(8)– **paper** In "The Moon is a Harsh Mistress", rebel lunar colonists with no weapons assaulted Earth by lobbing rocks at it with an electromagnetic catapult. Assume the catapult launches a 10000 kg rock with the minimum velocity needed to escape from the moon to the Earth. Calculate the kinetic energy liberated when the rock hits Earth. Compare this energy to the energy liberated by the explosion of 10000 kg (roughly ten kilotons) of the high explosive TNT.

(9)– **paper** If you are standing at the base of Mr. Everest, the direction of the local gravitational vector is slightly deflected from its normal direction pointing toward the center of the Earth by the presence of the large mass in front of you. Modeling Mt. Everest as a hemisphere sphere 10 km in diameter with the same average density as the Earth, and assuming you are standing 5 km from the center of that hemisphere (at the foot of the mountain), calculate the angle by which g is deflected from a radial line pointing at the center of the Earth by the presence of the mountain. Hint: This is straightforward if you use equations 7.3. For starters, assume that Everest is a full sphere with half buried in the ground. Then, if you think about it, you can simplify and allow that it is really a hemisphere.

(10)– **paper** A bit of algebra was omitted as the derivation proceeded from equation 7.43 to 7.46. Rework the problem (starting at 7.43) and fill in the missing steps. Next, assume that $\mathbf{r} < \mathbf{R}$, and again fill in the algebra to show that the force on the test mass is zero.

(11a)– **paper** Invert $\vec{r} = \vec{r}_1 - \vec{r}_2$ and equation 7.51 to express $\vec{r}_1$ and $\vec{r}_2$ purely in terms of $\vec{R}_{CM}$ and $\vec{r}$.

(11b) – **paper** Plug in your results from part (a) to 7.61 and fill in the missing algebra to derive equation 7.62 for angular momentum.

(11c) – **paper** Plug in to 7.63 and fill in the missing algebra to derive equation 7.64 for kinetic energy.

(12)– **paper** Using the published mass of Earth and Moon, calculate the length of a lunar month assuming the Earth is fixed in space and the moon orbits around it. Repeat the calculation allowing for reality that the Earth is not fixed, that the Earth/moon system can be replaced by a fixed mass M circled by a reduced mass μ. Check your results against Googled data.

(13)– **paper** Relative to the center of the sun, calculate the center of Mass of the Earth/Sun and Jupiter/Sun system. Relative to the earth, calculate the center of Mass of the Earth/Moon system.

(14a)– **paper** Look up the mass of an electron and a proton, and calculate the value of μ for a hydrogen atom to four significant figures.

(14b)– **paper** Calculate the Bohr radius (**a**) of hydrogen (to four figures) using your result from the previous part and equation 7.66, and compare it to what you would obtain if you used the mass of the electron. Compare both your answers to published values of **a**.

(15)– **paper** Calculate the orbital frequency of a hydrogen atom in its ground state to four significant figures and use your result to calculate the ionization energy in Joules of hydrogen. Convert this energy to electron volts and see that it agrees with the value given in equation 7.67.

(16a) Write a *Matlab* function defined as follows:

```
function [z]=circularOrbit(T,M)
%T [s]: Period of circular orbit
%M [kg]: Mass of large body being orbited
%z=[R,v]
%R [m]: Radius of circular orbit given T,M
%v [m/s]: Linear speed of orbiting body
```

You do not need to test this function, because you will be testing it in part b.

(16b) Attached is a text file of planetary data. Write a *Matlab* script to read this file and generate a table of orbital periods (in Earth Years) vs. average distance from the sun for the planets. Your script should use the *circularOrbit* function from part a. Look up published measured orbital periods. Include in a third column of your table the published measured orbital period.

```
%This text file contains data about planetary distances
%Column1 -- Name of planet
%Column2 -- Semimajor axis of orbit (A.U.)
Mercury     0.39
Venus       0.72
Earth       1.00
Mars        1.52
Jupiter     5.20
Saturn      9.58
Uranus      19.20
Neptune     30.05
Pluto       39.24
```

(17a) Write a script that generates N random masses between minM and maxM kg and N random velocity vectors (x, y, z components) with component magnitudes between minV and maxV. Print out the individual kinetic energies and the total kinetic energy of the eight particles. Every time you run the script, it should generate new random masses and velocities. Test the script with N=8, minM=1, maxM=10 kg, minV=-100 m/s, maxV=100 m/s.

(17b) Extend the script from part **a** to calculate the velocity, speed and kinetic energy of the center of mass of the 8-body system.

(17c) Extend the script further to calculate the kinetic energy of the masses relative to the center of mass. The part **b** and **c** results should sum to the energy from part **a**. Your script should demonstrate that this is so. Run the script 10 times and thus demonstrate numerically that total kinetic energy breaks into the kinetic energy of the center of mass + the kinetic energy relative to the center of mass.

(18) Write a *Matlab* function defined as follows:

```
1 function [R]=scharzchildRadius(M)
2 %M [kg]: Mass of large body
3 %R [m]: Schwarzchild Radius of body
```

Test your function with a script that prints out the Schwarzchild radius of the Earth, the Sun, and your own body. The Schwarzchild radius is defined in problem 6.

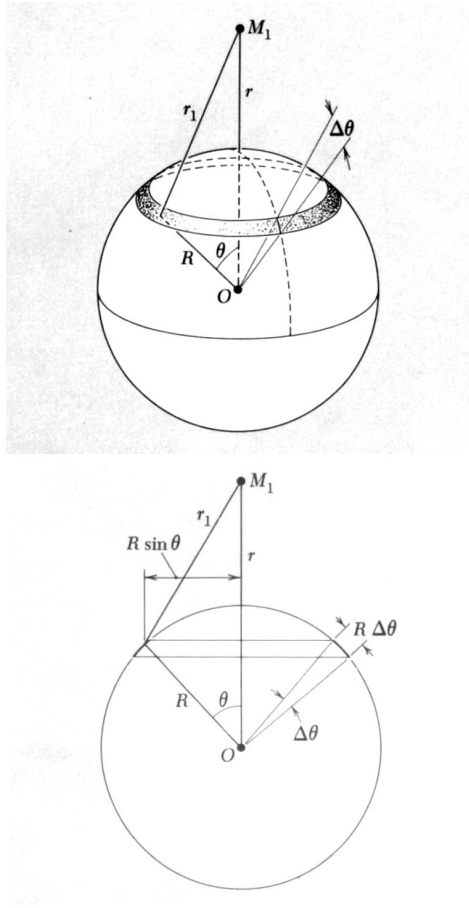


Figure 7.6: In addition to angle θ shown above, we will need angle ϕ, the angle included between line r and line r_1. On the left, a setup for integration in spherical coordinates. The spherical shell has radius **R**. The test-mass M_1 is located **r** from its center. On the right, a projection of the sphere on a plane to illustrate more clearly that the radius of the ring on the left is $R\,sin(\theta)$. .

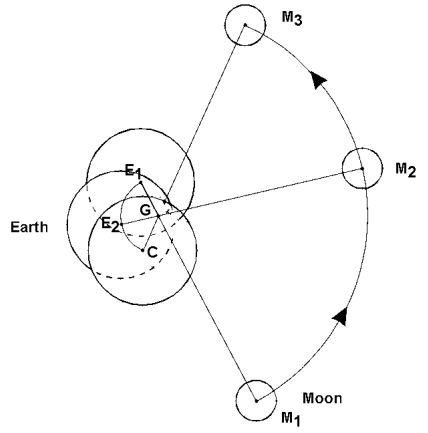


Figure 7.7: It is not completely correct to say that the Earth orbits the sun, nor that the moon orbits the Earth. As shown here, both the Earth and the moon orbit their common center of mass. Because the Earth is so much more massive than the moon, its orbital circle is much smaller, but if the Earth were truly fixed, there would be no tidal bulge on the side opposite the moon.

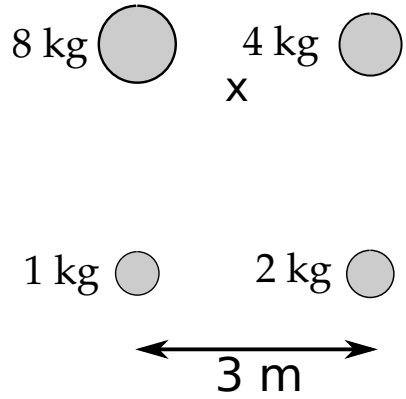


Figure 7.8: Four masses are arranged in a square as shown with the heaviest mass in the top left corner. The center of mass is indicated with an "x". Note that the large masses on top of the square draw the CM up close to them.

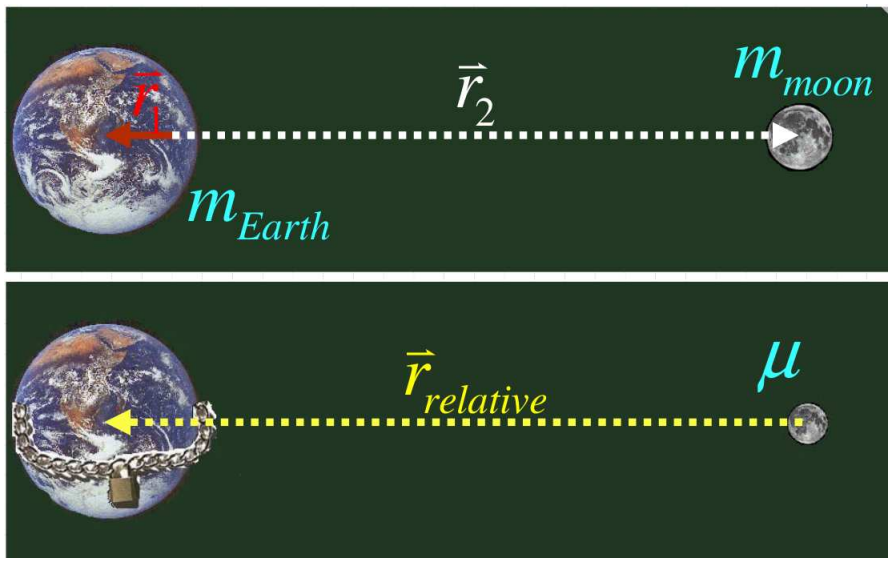


Figure 7.9: A two body interaction problem (includes collisions as well as gravity or electrostatics) can be replaced by a single body of mass μ interacting with a fixed body of mass $\mathbf{M}$.

Chapter 8

Runge-Kutta Method and Orbital Simulation

8.1 Advantages of higher-order methods

The Euler method, and other methods for solving differential equations by calculation rather than calculus are collectively referred to as "numerical methods". Numerical methods existed long before hand-held electronic calculators or personal computers. In fact the subject of this section, Runge-Kutta methods, were invented around the year 1900 in Germany. Let us return now to the projectile problem. Assume you want to solve for the trajectory of a projectile with drag over a period 3 seconds. If you set Δt=0.001 s, your computer has to do 3000 iterations to solve the problem, while if you set Δt=0.6 s, the computer only needs to do five iterations. In 2010, computers are so fast that you may not notice that the computer takes 600 times longer to solve the problem with the millisecond time step than it does with the longer step. Of course, the result with the millisecond time step is generally more accurate.[1]

Imagine solving the same problem in 1900. You would have to hire a "calculator", which at the time referred to a person who was willing and able to do repetitive calculations for hours at a stretch. This person would really notice if the solution took 3000 interations rather than five.

For Runge and Kutta, it was clearly advantageous to see if there are ways of solving differential equations numerically which can achieve the desired accuracy with fewer iterations than the Euler method. Figure 8.1 compares the performance of three numerical methods for approximating the path of a basketball

[1] It is possible to have a step that is too small. If the step size begins to approach the maximum number of digits accuracy of the computer then the solution may actually be poorer with a small step than a larger one.

with air-resistance over three seconds with Δt=0.6 s. The red $\times$ are data points calculated with the Euler method which you learned earlier. The blue + was calculated using the "2nd-order" Runge-Kutta method (RK2), while the green ★ was calculated with the "4th-order" Runge- Kutta method (RK4). The solid line gives the "correct" trajectory of the basketball. It is not surprising that the Euler method fails badly to give the correct result; as only 5 time steps were used to approximate the entire trajectory. Perhaps you will be surprised to see that the RK4 method is so efficient that it appears to perfectly conform to the correct trajectory with so few steps.

Clearly, if you ever needed to do numerical methods with a pencil and paper, you would want to use RK4. More importantly, even with powerful computers, the advantage of RK4 is so large that any serious simulation should start with this method.

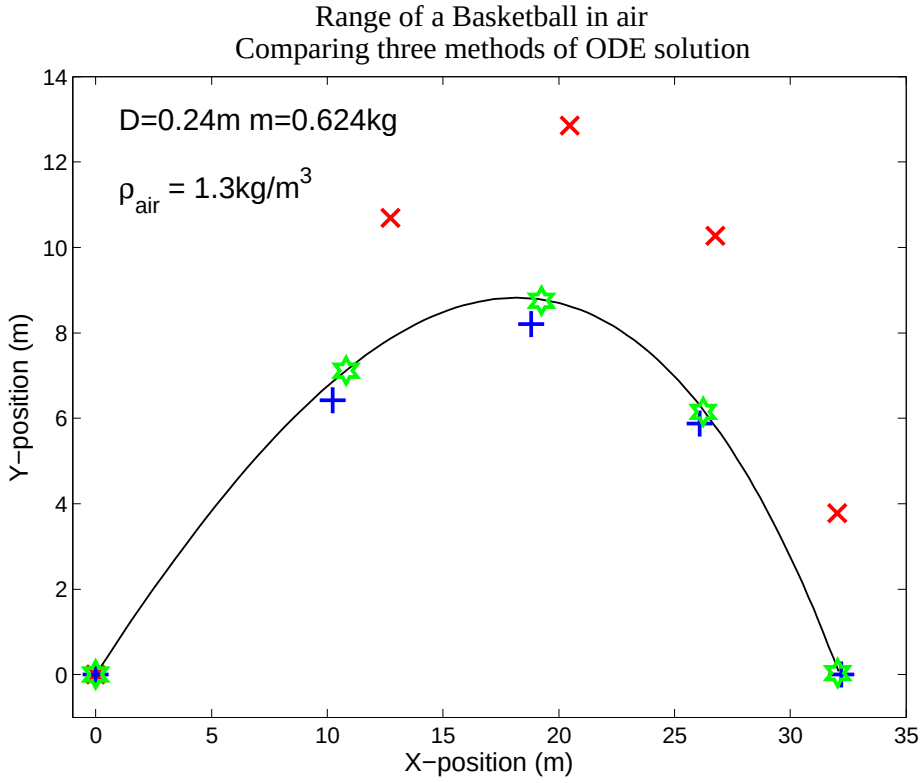


Figure 8.1: The red $\times$ shows the poor quality of an Euler method fit. The blue + shows RK2, while the green ★ shows RK4. Note the RK4 fit is very close to "correct" even with only five time points. (This plot was generated with the sphere diameter, mass, and air-density indicated in the figure as well as $v_0 = 25$ m/s and $\theta = 40°$.)

8.2 First order Runge-Kutta (RK1 or Euler method)

We already know the Euler method. We will begin with the approach we have used so far, then change the notation to be consistent with the presentation of most books on numerical methods to help us prepare for the notation we need for RK2 and RK4. Assume we are using the Euler method to calculate velocity for an object subject to linear drag. We could write

$$\Delta v = \Delta t \times (g - bv_n) \tag{8.1}$$

$$v_{n+1} = v_n + \Delta v \tag{8.2}$$

If you found the Euler method in a math book, it might be written like this.

$$k_1 = h \times f(x_n, y_n) \tag{8.3}$$

$$y_{n+1} = y_n + k_1 + O(h^2) \tag{8.4}$$

We recognize that y_{n+1} is v_{n+1} and that h is just Δt. Also, k_1 is the same as Δv. Finally $f(x_n, y_n) = g - bv_n$, where $x_n \to t_n$ and $y_n \to v_n$. [2] The term $O(\Delta h^2)$ means that the innaccuracy of the Euler approximation falls as the square of h, which in most physics problems represents Δt, the chosen time-step.

8.3 Second order Runge-Kutta (RK2)

In the classic science fiction novel "The Moon is a Harsh Mistress", author Robert Heinlein makes frequent use of the acronym *TANSTAAFL*[3]. We can apply this acronym to RK2 and RK4 and other higher order numerical methods. They are more powerful, but not quite as easy to understand thoroughly as the Euler method. Fortunately, if you organize your code clearly, the methods are not hard to use. The RK2 method is simpler than RK4, so we explain it first.

8.3.1 RK2 Algebraic Interpretation

Numerical methods texts tend to define the RK2 method like this:

$$k_1 = h\ f(x_n, y_n) \tag{8.5}$$

$$k_2 = h\ f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \tag{8.6}$$

$$y_{n+1} = y_n + k_2 + O(h^3) \tag{8.7}$$

[2] It is not necessary that our function f be a function of both t_n and v_n. In the case of drag, f only depends on v_n. Should we happen to have a function that does explicitly depend on time, this notation will accomodate it.

[3] There Ain't No Such Thing As A Free Lunch

The term $O(h^3)$ means errors decrease as Δt^3. We have already recognized that $h \to \Delta t$, $x_n \to t_n$, $y_n \to v_n$, and that f does not depend on t_n. Let's make these substitutions and rewrite.

$$k_1 = \Delta t\ f(v_n) \tag{8.8}$$

$$k_2 = \Delta t\ f(v_n + \frac{k_1}{2}) \tag{8.9}$$

$$v_{n+1} = v_n + k_2 \tag{8.10}$$

Perhaps this is still too abstract. Let us try a full blown implementation of quadratic drag in 1-D with RK2. The differential equation we want to solve is:

$$\frac{dv}{dt} = g - \frac{c}{m}v^2 \tag{8.11}$$

In RK2 notation, this is:

$$k_1 = \Delta t\ (g - \frac{c}{m}\ v_n^2) \tag{8.12}$$

$$k_2 = \Delta t\ (g - \frac{c}{m}\ (v_n + \frac{k_1}{2})^2) \tag{8.13}$$

$$v_{n+1} = v_n + k_2 \tag{8.14}$$

In RK2, we calculate k_1 just as in RK1 (Euler), however, we then use k_1 to refine v_n before plugging it back into the drag equation. This gives us k_2. Next we represent RK2 graphically to try to clarify why k_2 is defined as it is.

8.3.2 RK2 Graphical Interpretation

Figure 8.2[Left] shows the first two velocities calculated by the Euler method for a ball dropped from rest in the presence of drag. The point $v_0 = 0$ appears at the origin. The negatively sloped line extending from the origin represents the acceleration due to gravity, and at time $t = \Delta t$, the black dot represents the first predicted velocity, $v_{1\,Euler}$. The red circle at $t = 2\,\Delta t$ shows the second predicted velocity. The 2nd line is not sloped as steeply as the first because the non-zero velocity v_1 creates drag, reducing the magnitude of the acceleration for $\Delta t < t < 2\Delta t$.

Note that the first point v_1 on the graph overestimates the speed of the falling object. The velocity v_1 is an overestimate because the slope of the first line is $-g$, since it was calculated using $v_0 = 0$ and thus there is no drag between time 0 and Δt. Yet we know that for all times $0 < t < \Delta t$, there really is some drag, which is finally taken into account in the reduction of the slope of the line used to calculate v_2. The RK2 method let's us account for the non-zero velocity of the object earlier in the process. In Figure 8.2[right] we put a red $\times$ at time $\frac{\Delta t}{2}$. We then adjust the slope of the solid line, replacing it with the dashed line. The slope of the dashed line is less negative than the first solid line, but more negative than the slope of the line joining Δt and $2\Delta t$. We then ride the dashed line from $\frac{\Delta t}{2}$ to Δt to arrive at the red dot $v_{1\,RK2}$. As expected, $v_{1\,RK2}$ is less negative that

$v_{1\ Euler}$. To continue the process, we would calculate a new slope at time $t = \Delta t$ which we would ride to $t = \Delta t + \frac{\Delta t}{2}$, then adjust it again. For any given fixed timestep Δt, RK2 adjusts the slope twice as often as RK1. Equally important, each of its values of v obtained is more accurate, so the subsequent projected slopes are also more accurate. In this way, RK2 errors fall as Δt^3 rather than as Δt^2 for RK1.

8.4 Fourth order Runge-Kutta (RK4)

The notation for fourth-order Runge-Kutta is typically as follows:

$$k_1 = h\ f(x_n, y_n) \tag{8.15}$$

$$k_2 = h\ f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \tag{8.16}$$

$$k_3 = h\ f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \tag{8.17}$$

$$k_4 = h\ f(x_n + h, y_n + k_3) \tag{8.18}$$

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{2k_2}{6} + \frac{2k_3}{6} + \frac{k_4}{6} + O(h^5) \tag{8.19}$$

This can be interpreted graphically, as we did for RK2, but the graph is fairly complicated and not necessarily all that illuminating. Instead we will argue what is being done by analogy with RK1 and RK2. RK4 has four terms k_1, k_2, k_3, k_4, while RK2 had only k_1, k_2 and RK1 (Euler) has only k_1. The errors were of order h^2 for RK1, and h^3 for RK2, so we should not be surprised to see that they are of order h^5 for RK4. All of the k terms represent correction factors. In RK1, we only make one correction per step h (or per time-step Δt). In RK2, we make two corrections. In RK4 we correct our result four times before advancing to the next time step. In math, k, h, x, and y are just numbers, but in physics they represent tangible physical quantities. Till now, we have used projectile motion as our example, and have been calculating the velocity of the projectile subject to gravity and drag. In this case, all the k terms have units m/s and correspond to velocity corrections. Of course, as one is calculating velocity, one can go on to update position. In this case, the k terms will have units of m and will represent position corrections.

In RK1, we only update velocity at the end of the time interval. In RK2, we recalculate the acceleration in the middle of the time interval, using the velocity corrected by $k_1/2$. The corrected acceleration is used to calculate a new velocity correction k_2 at the end of the interval. In RK4, we calculate k_1, in *exactly the same way* as for RK1, and RK2. We use the correction k_1 to adjust the acceleration and calculate a new k_2 (just like RK2). However, we then use k_2 to correct the acceleration *again* in the middle of the time interval (equation 8.17), arriving at k_3. Finally, we calculate the new velocity at the end of the time interval, using our best guess at the final velocity, k_3 operating over the entire

time interval. Since some of the corrections tend to overestimate the velocity and some tend to underestimate it, we attempt to increase our accuracy even further by using a weighted average of the corrections (equation 8.19) to obtain the final result. This is different than RK1 and RK2. It is perhaps reasonable that the corrections in the middle of the time interval are more representative of the average behavior of the object than those at the ends, and for this reason they are weighted twice as much as the end factors. Why the weights are 2 and not some other number is beyond the scope of this book.

8.5 Applying Runge-Kutta methods to quadratic drag

In the previous chapter, you learned how to generate trajectories of objects moving under the influence of inertial (quadratic) drag using the Euler method. Below is a script implementing quadratic drag using RK1, RK2, and RK4 methods. It is hoped that seeing a concrete example for a force law that you now know well will allow you to generalize to other force laws. Note in the following script that the Euler function is documented and usage information is provided. The RK2 and RK4 functions are used in the same was as Euler. To save space in this book, I omitted the documentation for those functions in the example script.

```
1  function [z]=Euler(t,z0)
2 % Note: The initial conditions go in z0. z0 is an array, not ju
3 % For this problem (a second order ODE)
4 % z0 contains initial x, y positions and vx, vy velocities.
5 % The array t is not assumed to be evenly spaced (but it can be)
6 % The first returned data point corresponds to t(1) and the last
7 %USAGE:
8 %     global m g c
9 %     t=linspace(0,4); z0=[x0 y0 vx0 vy0];]
10 %     [z]=RK4(t,z0);
11 %     x=z(:,1); y=z(:,2);
12 %     plot(x,y);
13
14 global m g c
15 b=c/m; len=length(t);
16 z=zeros(len,4); %Initialize matrices for x, y, vx, vy
17 x=zeros(len,1);y=zeros(len,1);vx=zeros(len,1);vy=zeros(len,1);
18 x(1)=z0(1);y(1)=z0(2); %Copy initial conditions from z0
19 vx(1)=z0(3);vy(1)=z0(4);%to variables easier to remember
20
21 for n=1:len-1
22     dt=t(n+1)-t(n);
23     %k1
24     magv1=sqrt(vx(n)^2+vy(n)^2);
```

```
25      k1vx=dt*(-b*vx(n)*magv1);
26      k1vy=dt*(-g-b*vy(n)*magv1);
27      k1x=dt*vx(n);
28      k1y=dt*vy(n);
29
30      vx(n+1)=vx(n)+k1vx;
31      x(n+1)=x(n)+k1x;
32      vy(n+1)=vy(n)+k1vy;
33      y(n+1)=y(n)+k1y;
34  end
35  z(:,1)=x;   %1st column
36  z(:,2)=y;   %2nd column
37  z(:,3)=vx;
38  z(:,4)=vy;
39  end
40
41  function [z]=RK2(t,z0)
42  global m g c
43  b=c/m; len=length(t);
44  z=zeros(len,4); %Initialize matrices for x, y, vx, vy
45  x=zeros(len,1);y=zeros(len,1);vx=zeros(len,1);vy=zeros(len,1);
46  x(1)=z0(1);y(1)=z0(2);
47  vx(1)=z0(3);vy(1)=z0(4);
48   for n=1:len-1
49      dt=t(n+1)-t(n);
50      %k1
51      magv1=sqrt(vx(n)^2+vy(n)^2);
52      k1vx=dt*(-b*vx(n)*magv1);
53      k1vy=dt*(-g-b*vy(n)*magv1);
54      k1x=dt*vx(n);
55      k1y=dt*vy(n);
56      %k2
57      magv2=sqrt((vx(n)+k1vx/2)^2+(vy(n)+k1vy/2)^2);
58      k2vx=dt*(-b*(vx(n)+k1vx/2)*magv2);
59      k2vy=dt*(-g-b*(vy(n)+k1vy/2)*magv2);
60      k2x=dt*(vx(n)+k1vx/2);
61      k2y=dt*(vy(n)+k1vy/2);
62
63      vx(n+1)=vx(n)+k2vx;
64      x(n+1)=x(n)+k2x;
65      vy(n+1)=vy(n)+k2vy;
66      y(n+1)=y(n)+k2y;
67  end
68  z(:,1)=x;   %1st column
69  z(:,2)=y;   %2nd column
70  z(:,3)=vx;
```

```
z(:,4)=vy;
end

function [z]=RK4(t,z0)
global m g c
b=c/m;
len=length(t);
z=zeros(len,4); %Initialize matrices for x, y, vx, vy
x=zeros(len,1);y=zeros(len,1);vx=zeros(len,1);vy=zeros(len,1);
x(1)=z0(1);y(1)=z0(2);
vx(1)=z0(3);vy(1)=z0(4);
for n=1:len-1
    dt=t(n+1)-t(n);
    %k1
    magv1=sqrt(vx(n)^2+vy(n)^2);
    k1vx=dt*(-b*vx(n)*magv1);
    k1vy=dt*(-g-b*vy(n)*magv1);
    k1x=dt*vx(n);
    k1y=dt*vy(n);
    %k2
    magv2=sqrt((vx(n)+k1vx/2)^2+(vy(n)+k1vy/2)^2);
    k2vx=dt*(-b*(vx(n)+k1vx/2)*magv2);
    k2vy=dt*(-g-b*(vy(n)+k1vy/2)*magv2);
    k2x=dt*(vx(n)+k1vx/2);
    k2y=dt*(vy(n)+k1vy/2);
    %k3
    magv3=sqrt((vx(n)+k2vx/2)^2+(vy(n)+k2vy/2)^2);
    k3vx=dt*(-b*(vx(n)+k2vx/2)*magv3);
    k3vy=dt*(-g-b*(vy(n)+k2vy/2)*magv3);
    k3x=dt*(vx(n)+k2vx/2);
    k3y=dt*(vy(n)+k2vy/2);
    %k4
    magv4=sqrt((vx(n)+k3vx)^2+(vy(n)+k3vy)^2);
    k4vx=dt*(-b*(vx(n)+k3vx)*magv4);
    k4vy=dt*(-g-b*(vy(n)+k3vy)*magv4);
    k4x=dt*(vx(n)+k3vx);
    k4y=dt*(vy(n)+k3vy);
    %%
    vx(n+1)=vx(n)+k1vx/6+k2vx/3+k3vx/3+k4vx/6;
    x(n+1)=x(n)+k1x/6+k2x/3+k3x/3+k4x/6;
    vy(n+1)=vy(n)+k1vy/6+k2vy/3+k3vy/3+k4vy/6;
    y(n+1)=y(n)+k1y/6+k2y/3+k3y/3+k4y/6;
end
z(:,1)=x;   %1st column
z(:,2)=y;   %2nd column
z(:,3)=vx;  z(:,4)=vy;
```

```
117 end
```

8.6 Applying Runge-Kutta methods to gravitation

In the previous chapter we expressed the law of gravitation in vector component form in equations 7.3. This is the most useful form for simulating orbits of satellites and planets. For the case of a small planet orbiting a nearly fixed central body, the equations can be simplified further, because we set $\vec{r}_1 = 0$. Then we can also replace $\vec{r}_2$ by simply $\vec{r}$, obtaining:

$$F_x = Gm_1m_2\frac{-x}{|\vec{r}|^3} \tag{8.20}$$

$$F_y = Gm_1m_2\frac{-y}{|\vec{r}|^3} \tag{8.21}$$

8.7 Sample listing for Runge-Kutta2 applied to gravitation

The following listing is fairly long. Note however that the first third of the code consists of the **%Document** and **%Define** sections. This is rather typical. In fact, it is only because the **%Derive** section has for the first time become non-trivial that these housekeeping sections only take up a third of the script. The **%Derive** section starts on line 27 with a **case** statement to allow one to pick which form of Runge-Kutta method to use. This function does not have a **%Display** section, but rather is designed to be called by another script whose sole purpose is to display. This approach allows flexibility. You can have several different **%Display** scripts that access the same model.

We finally get down to business on line 44 with the RK2 function. Note that the RK2 function also devotes several lines to defining variables before doing the actual RK2 method. You see that lines 55-60 correspond to equation 8.5, 61-66 correspond to 8.6, and 68-69 correspond to 8.7.

This script does two things you have not seen before. It solves not just for v(t) but also for x(t). Also, it solves for v_x, v_y, x and y all at once. While we only need k_1 and k_2 in the RK2 method, we need to keep our k's straight. The correction factor for v_x is obviously not the same as the correction factor for v_y, and likewise not the same as those for x and y. Thus we have four lines of script corresponding to one line of algorithm. Lines 56 and 62 are not strictly needed. I define an intermediate variable, *magr* to calculate the magnitude of r, the denominator in 8.20. The calculation on line 56 could have been done as part of lines 57 and 58. I gave the *magr* calculation its own line to avoid the repetition of doing it on

lines 57 and 58. Also, even moderately complex expressions like *magr* make the code look cluttered and hard to understand.

Note on line 63 that $k2vx$ depends on x and $k1x$ whereas for all the drag problems, $k2vx$ would only depend vx and $k1vx$ (and perhaps vy and $k1vy$ as well. The RK methods allow the use *any* of the variables in simulations, based on what the physics requires. One needs only to keep track of what order k you are using and not (for example) to plug a k_1 estimate into the calculation of k_3 (for RK4).

```
1  function [time, x,y ]=orbital_model6(v0,R0,M,solvertype,npts)
2  %===**DOCUMENT**=== by R.Sonnenfeld, NMT Physics: Version 1.3
   10/24/2009.
3  %USAGE  r0=150E9; v0=30000; Msun=1.99E30;
4  %[time, x1,y1 ]=orbital_model6(v0,r0,Msun,solver,npts);
5
6  %Implements 2D Runge-Kutta methods to solve orbital mechanics pr
7  %If solvertype==1, Use Euler solvertype==2, use RK2, solvertype=
8  %===**DEFINE**===
9  %SET INITIAL CONDITIONS
10 x0=R0; y0=0; vx0=0; vy0=v0;  %R0 (m): Initial distance from sun -
11 vx0=0; vy0=v0;  %vx0, vy0 (m/s): Initial velocity components
12 % Note: The initial conditions go in the array z0.
13 z0=[x0 y0 vx0 vy0];
14 %================
15 %SET CONSTANTS
16 global GM
17 G=6.67E-11;
18 GM=G*M;    %Calculate constants
19 %================
20 %SET SIMULATION DURATION
21 years=5;
22 t0=0;  tf=years*365*86400;
23 %===**DERIVE**===
24 %SOLVE ODE
25 t=linspace(0,tf,npts);
26
27 switch solvertype
28     case 1
29         [t,z]=Euler(t,z0);
30     case 2
31         [t,z]=RK2(t,z0);
32     case 3
33         [t,z]=RK4(t,z0);
34     otherwise
35     error('Invalid solver type %d.',solvertype)
36 end
37 %================
```

```
38 %Unpack the ODE solutions into column vectors with meaningful names
39 x=z(:,1); y=z(:,2);
40 vx=z(:,3); vy=z(:,4);
41 time=t;
42 end
43
44 function [t,z]=RK2(t,z0)
45 global GM
46 len=length(t);
47 z=zeros(len,4); %Set up a matrix for x, y, vx, vy
48 x=zeros(len,1);y=zeros(len,1);vx=zeros(len,1);vy=zeros(len,1);
49 %Plug initial conditions into convenient variables
50 x(1)=z0(1);y(1)=z0(2);
51 vx(1)=z0(3);vy(1)=z0(4);
52 for n=1:len-1
53     %The below makes the method work even if the time values are not
54     dt=t(n+1)-t(n);
55     %k1
56     magr1=sqrt((x(n)^2+y(n)^2))^3;
57     k1vx=dt*(-GM*x(n)/magr1);
58     k1vy=dt*(-GM*y(n)/magr1);
59     k1x=dt*vx(n);
60     k1y=dt*vy(n);
61     %k2
62     magr2=sqrt(((x(n)+k1x/2)^2+(y(n)+k1y/2)^2))^3;
63     k2vx=dt*(-GM*(x(n)+k1x/2)/magr2);
64     k2vy=dt*(-GM*(y(n)+k1y/2)/magr2);
65     k2x=dt*(vx(n)+k1vx/2);
66     k2y=dt*(vy(n)+k1vy/2);
67     %%
68     vx(n+1)=vx(n)+k2vx;   x(n+1)=x(n)+k2x;
69     vy(n+1)=vy(n)+k2vy;   y(n+1)=y(n)+k2y;
70 end
71 z(:,1)=x;  %1st column
72 z(:,2)=y;  %2nd column
73 z(:,3)=vx; z(:,4)=vy;
74 end
75
76 function [t,z]=Euler(t,z0)
77 error('This function has not been implemented yet'); end
78
79 function [t,z]=RK4(t,z0)
80 error('This function has not been implemented yet'); end
```

Lines 76-77 and 79-80 are often referred to as "stub" functions. They exist and they "work", they just do not do anything. In this case, if the Euler or RK4

method is specified, you get an error message reminding you to write the code first.

8.8 End of Chapter Problems

(1a – paper) In sections 8.2 and 8.3.1, the RK1 and RK2 methods are layed out and then applied to the case of one-dimensional motion with quadratic drag. Using the definition of RK4 given in 8.4, apply it to the case of 1-D quadratic drag.

(1b – paper) For a body moving only along the x-axis and subject to the force $F = -sx - bv$, write the Runge-Kutta equations for v(t) and x(t) for the RK1, RK2, and RK4 methods.

(2) In section 8.5 a script is provided implementing functions for RK1, RK2, and RK4 for a sphere moving under quadratic drag. Using these functions, write a script that generates figure 8.1. (All the parameters you need to reproduce the plot are either included in the labels or figure caption.)

Hint – The example functions provided use the **global** command in order to pass the parameters **m**, **g**, and **c** from your calling program. Your calling program must also use the **global** command.

(3) The example from section 8.6 provides working code for RK2, but does not include the **%Display** section of code. Write a separate function in a separate file which will call the RK2 function provided and display a circular orbit of the Earth around the Sun. Your script should replicate in so far as possible figure 8.3. Your function should be passed the parameters *npts* and *solvertype* as defined below. After you have generated the figure, try *solvertype* = 3. You should get an error message. Here is the documentation section of your function:

```
function display_orbit(npts,solver)
%DISPLAY_ORBIT display_orbit(npts,solver)
%Calls Orbital model and plots the orbit.
%
%USAGE: display_orbit(50,3)
% npts number of data points used between
% begin and end times set by model.
% if solvertype=1 - Use Euler method
% if solvertype=2      Use Runge-Kutta2 method
% If solvertype=3      Use Runge-Kutta4 method
%
%USES: [time, radius, theta]=orbital_model6(41800,0,0,solver);
```

(4) The example from section 8.6 provides working code for RK2, but does not provide code for RK4 or RK1 (Euler). Expand the "stubs" for the RK4 and Euler functions to make working code. Use this plus the script you developed in the previous problem to reproduce figures 8.4b, c.

(5) Note that figures 8.3 and 8.4 have in the title the method used to do the calculation. Further, they indicate the initial velocity and the number of

simulation points per Earth year. Upgrade your script from problem 2 so that it automatically prints the correct values when you change *npts* or *solvertype* in your calling function. Demonstrate that it works by generating a figure with the call `display_orbit(500,1)`

(6) Now that you can easily document any plot with its parameters (because you have completed the first three problems) See how many data points per year are needed to make orbits appear stable over ten years for

A. RK4

B. RK2

C. Euler

(7) In the previous problem you estimated how many data points you needed to make simulation accurate over 10 years. First, do an analytical calculation of what initial velocity is needed to have a perfectly circular orbit at a radius of exactly 150 million kilometers. Now use this radius and define "stable orbit" as having a radius that changes by $< 1\%$ over 10 years. Create a script that allows you to tell precisely how many points are needed to get an orbit stable to just under $< 1\%$. You may get the result by running the script many times, or run it in a loop to get the result automatically.

(8) Write a new display script that will show four potential orbits for an Earthlike planet. Obtain plots and submit figures for all three solver types, assuming 1000 data points per Earth year. Run the simulation for 18 Earth years. All orbits start at Earth's normal radius but have the following initial velocities.

A. 24 km/s

B. 30 km/s

C. 36 km/s

D. 40 km/s

(9) Write a script for a planet beginning at Earth's orbit that will determine the minimum velocity needed so that the planet reaches an aphelion of 1000 A.U. Check the value you obtain against escape velocity as calculated analytically using energy conservation.

(10) Simulate decay of an orbit of a satellite encountering air resistance. Begin with a 1 kg satellite with diameter $D = 5\ m$ orbitting the Earth with a period of 350 minutes. Assume that the air-density at that altitude is $\rho = 1 \times 10^{-10}\ \ kg/m^3$. The satellite is subject to to square law drag. Simulate this using the RK4 method including both gravitation and air-resistance. Your plot should look appear as in figure 8.5.

(11) Write a display script that will animate the motion of a single planet. It should use your RK4 function. The planet should be at Earth's average distance from the sun and start with the following circumferential velocities:

A. 24 km/s

B. 30 km/s

C. 36 km/s

D. 40 km/s

(Hint –) Recall that Chapter 5 discusses animation, and that if you solved problem 5.5 you probably have an animation script laying around.

(12) Write a display script that will animate the motion of four planets. It should use your RK4 function. All planets should begin at Earth's average distance from the sun and start with the following circumferential velocities:

A. 24 km/s

B. 30 km/s

C. 36 km/s

D. 40 km/s

(13) Problem 1b represents an oscillator with linear drag. Select a mass and a spring constant and a value for b that is 0.1*k and plot x vs. t for 10 cycles of the oscillator.

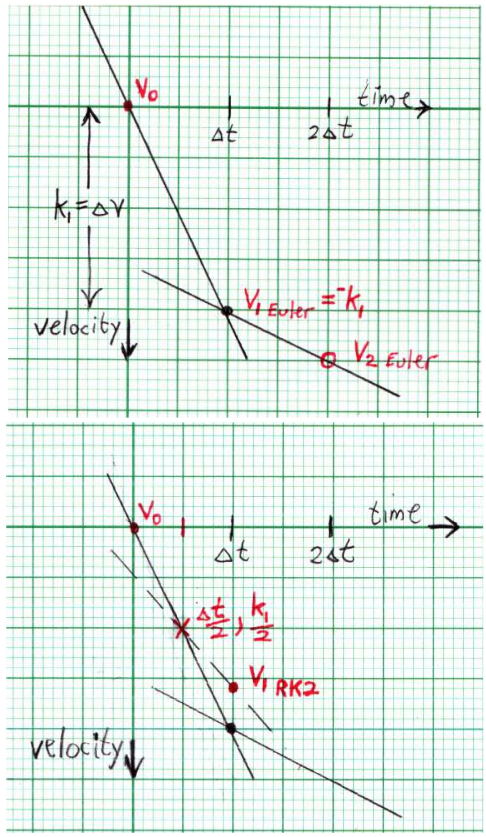


Figure 8.2: (*Left:*) Graphical interpretation of the Euler method for a ball falling with air resistance. v_0 represents the ball at rest. During the first time interval, the ball falls with an acceleration of $-g$ and achieves $v_{1\ Euler}$. For the second time interval the magnitude of acceleration is smaller, because air-drag now affects the ball. This method overestimates the speed of a ball dropped from rest, because the increase in velocity dependent drag is not calculated until the end of the time interval. (*Right:*) Graphical interpretation of the RK2 method for a ball beginning at rest. The RK2 method adjusts the acceleration twice during each time interval, to better account for the effect of changing speed on acceleration as it happens. The dotted line is intermediate in slope between the two solid lines.

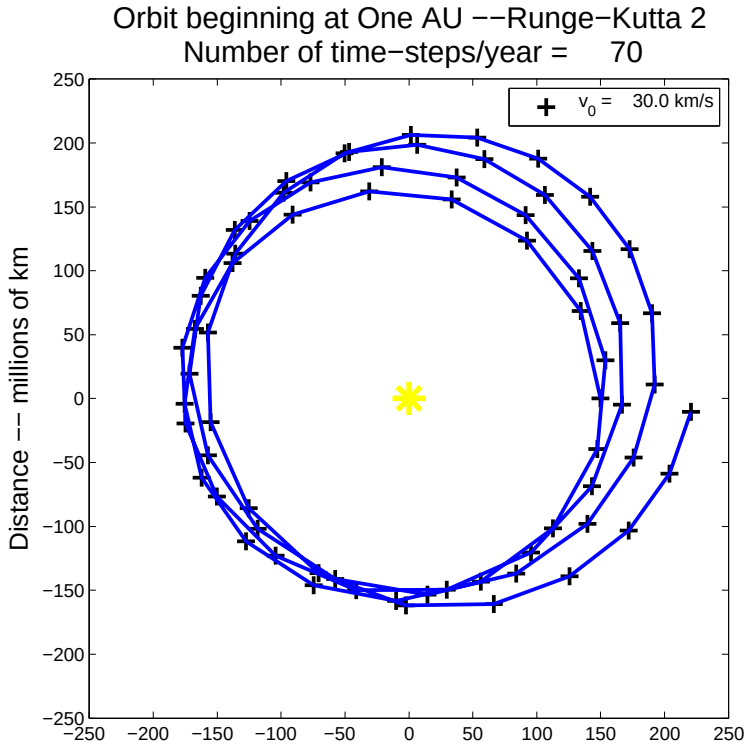


Figure 8.3: Five earth-years of simulation time with the RK2 method shows a simulated Earth orbiting the sun roughly five times. However the orbit is not reproducible, and would in fact become unstable if the simulation were allowed to run longer. This shows that 70 points/year (roughly one every 5^o is inadequate time resolution with the RK2 method) .

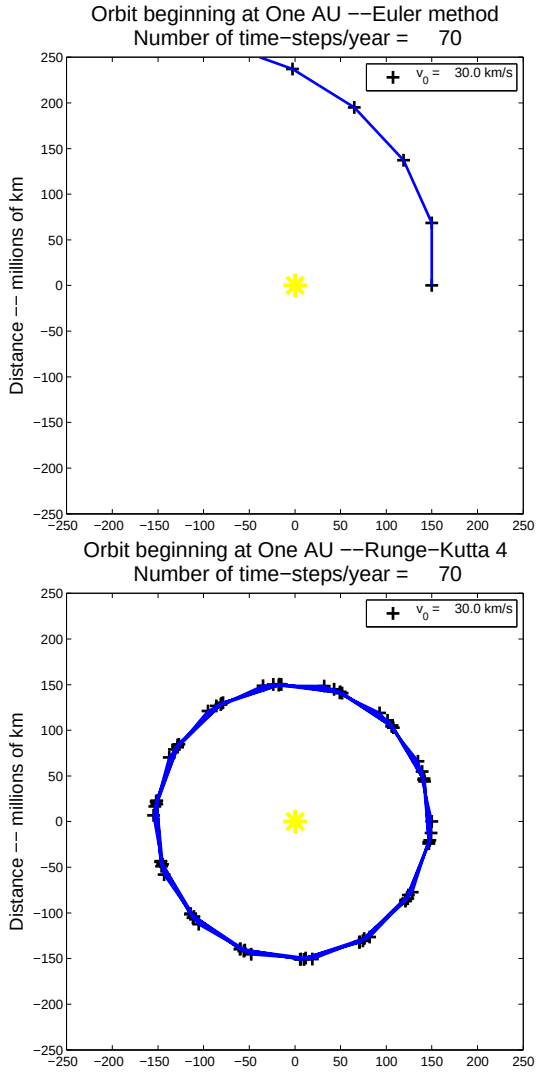


Figure 8.4: The RK2 method in figure 8.3 did not work so well. It is not surprising that the Euler method in the left panels fails after only a very few data points, with our planet flying out of orbit. On the other hand, the RK4 method in the right panel demonstrates its power by produceing nice stable orbits with only 70 simulation points. .

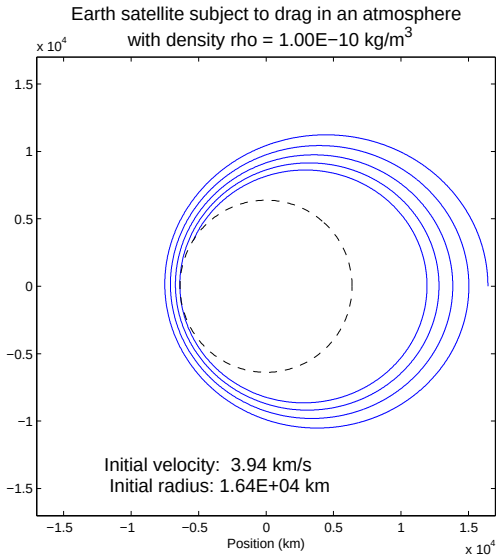


Figure 8.5: Plot of a satellite orbit decaying. See problem 6.8

Appendices

.1 Possible Projects for Physics 241

.1.1 Preproposal

I encourage you to propose your own project. Pick anything that interests you. The point of the preproposal is to think about what you might like to do so we can discuss what is realistic. Please prepare three pre-proposals. A preproposal is a description no longer than the ones I include below. You may simply select three of mine, but I encourage you to come up with your own, or some variation on my proposals.

.1.2 Proposal

The full proposal is due two weeks after the preproposal. A full proposal consists of:

1. A two or three sentence description of the general topic.
2. A discussion of what new code you are going to have to write and what existing code you can use or modify.
3. A discussion of a part of the problem that can be solved analytically. (E.g., location of Lagrange points, parameters of a Keplerian orbit, behavior of a linear oscillator, the relation between speed and acceleration for circular motion, the kinetic derivation of the ideal gas law).
4. A discussion of possible sources for more information about the problem (for example a web address that seems to have relevant information, or a book or article.)

.1.3 The Projects

.1.3.1 Modeling

Model any system that is interesting to you:

(1) Lunar Lander: Matlab has sufficient tools that you can make a simple game out of landing a space ship on the moon. Your graphics will be boring, but you can land a dot on a rough surface using the arrow keys to control the direction and thrust of maneuvering jets. Note, you will have to create a matlab "timer" object to allow the real-time control of your plot that you need. You will also need to learn about figure callbacks. This is an advanced project, but quite rewarding.

(2) Bouncing Ball: Animate a bouncing ball, then give is a small horizontal velocity,then give it a coefficient of restitution so it loses energy on each bounce. You can take it further, model the interaction between ball and surface and compression of the ball, or measure the frequency vs. time as the ball loses energy.

(3) Collisions in 1 and 2D: You can simulate hard-sphere collisions for arbitrary off angles (impact parameters) and arbitrary masses.

(4) Tides: A little reflection on tidal forces can allow you to predict the timing and strength of tides. Much can be done here without the need for programming, but as your model gets more sophisticated you can generate a tide chart for any location and time period on Earth. You can compare your results with published tide charts. You can get quite close to reality with a simple model.

(5) Circular Motion Simulator: Take a data set of arbitrary angle vs. time data and illustrate with animation the orientation of position, velocity, and acceleration vectors for circular motion. Start with uniform circular motion but be capable of handling circular motion at varying speeds.

(6) Orrery: An orrery is a scale model of a solar system (or part of it). You know enough to do a full model of the solar system. You can predict the motions, periods, velocities and eccentricities of all the planets based solely on initial conditions. Check your model against published solar system data and compare the results.

(7) Gravity Assist: Have a planet orbitting the sun and a spaceship encountering the planet. See how it can acquire additional speed by stealing some energy from the planet

(8) Nonlinear oscillator: Use RK4 to model a 1-Dimensional oscillator. After that works, add non-linear terms. Do a fourier transform and observe second harmonic generation for different non-linear terms. Animate.

(9) Restricted three body problem: Analyze motion of three bodies subject only to mutual gravitation. Confine all three bodies to a single plane (this makes it restricted.) This problem is so rich that I have broken it into several projects. Each one is complete by itself. Just pick one not all.

(9a) Lagrange points: These are points of stable equilibrium for three bodies. Simulate a Lagrange system and study oscillations about Lagrange points. Try ever larger perturbations from equilibrium and see how far from a Lagrange point the orbit becomes unstable.

(9b) Orbit of Neptune: The orbit of Neptune was predicted based on perturbations of the orbit of Uranus. Use the standard orbital parameters for Uranus and Neptune. How much does Neptune perturb Uranus? If you just had the perturbed orbit of Uranus (and a computer), could you deduce where Neptune was and find its mass?

(9c) Three body system Chaos: The motion of even a single free body subject to gravity between two fixed bodies is probably chaotic. Study this system.

(10) Transfer orbits: Write program to get spacecraft from one planet to another by arbitrary paths. How much fuel does it take for a particular transfer orbit? Animate.

(11) Roche limit: Study tidal forces and the Roche limit. Model a planet as a collection of a small number (4-100) of planetesimals stuck together with a sticking force corresponding to the yield stress of rock. Allow planet to come close to a larger planet. Observe at what radius the planet breaks up. Study ring formation?

(12) Ising Model: The Ising model is a very simple and rich model of magnetism and phase transitions. A line of interacting masses only interact with their nearest neighbors. They can all go into the same state as you vary the strength of their interactions. Many articles exist to help you set up an understand this model.

(13) Kinetic theory of gasses I: Model an ideal gas as a collection of small solid spheres. You can yield the ideal gas law analytically. With enough spheres, you can simulate a gas and increase the temperature by increasing particle velocity. You can animate sphere motion and demonstrate the temperature dependence of pressure. Although in a real gas particles constantly bounce into eachother, you can learn a lot here if you just have a collection of particles that interact with the walls of the enclosure only

(14) Kinetic theory of gasses II: If you work out how to handle collisions between atoms, you can arrive at the Maxwell-Boltzmann distribution of energies. Starting with masses all at the same velocity, let the interact. You will see that the acquire different average velocities, consistent with the Maxwell-Boltzmann distribution and equipartition of energy.

.1.3.2 Data Analysis

(15) Analyze any data set that is interesting to you: The national climate data center has infinite supplies of weather data. There are other public scientific data bases, and many major scientific projects publish their raw data on the web, along with tools to access it. Walking around the departement and talking to your favorite professors will probably lead to some interesting data you can analyze and plot.

(16) Accelerometer and GPS: I have raw GPS and Accelerometer data for a vehicle driving around campus and accelerating on straightaways. There are two projects here. One is artifact removal from the GPS data to plot position vs. time. The other is integration of the accelerometer data. The two data sets should agree about the vehicle position and and acceleration.

(17) Lightning radio Emission: I have many data files of lightning propagation in clouds that gives the location and time of lightning source points. Visualizing this data is an interesting 3D graphing and animation problem.

(18) Electric fields from lightning: I have raw data of electric fields from lightning. Every lightning flash causes a step in electric field. You should process the raw data and write a program that automatically measures the height of electric field steps.

(19) Fourier Transforms: A fourier transform converts time-domain data to the frequency domain. It is incredibly useful throughout physics. Matlab has fft built in. So you can use it as part of another data analysis project. OR If you really want to learn how it works, you can write your own Discrete fourier transform program (the FFT is just a "faster" way of doing a DFT) and compare your results to the built-in Matlab functions using some sample data which you can generate.

(20) Fourier Synthesis: Arbitrary waveforms may be built up from sums of sines and cosines. This is called fourier synthesis, and it is the opposite of a fourier transform. To understand one you really need to understand the other, but your project may focus on synthesis or on transforms.

(21) Music Analysis: You have probably seen musical screen savers that fourier analyze and graphically display songs in real time. The real-time part is beyond you, but you can write a program that visualizes 30 seconds of your favorite song.

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