Chapter 2

Mass, Momentum, and Transport Equations

In physics we are used to focusing on the behavior of closed parcels of matter. In fluid dynamics it is more productive to consider what happens at fixed locations through which multiple parcels are continuously passing. Thus, we can define, for instance, the density of mass as a function of position and time, $\rho = \rho(x, y, z, t)$. This specifies the density of whatever parcel happens to be at point (x, y, z) at time t. At some later time it gives the density of some other parcel which happens to be at that point at that time. Similarly, the velocity of parcels which pass through this point are specified by the fluid velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$. The physical laws governing fluids are expressed through partial differential equations which impose constraints on quantities like ρ and $\mathbf{v}$. A good reference on fluid dynamics is Kundu (1990).

2.1 Mass continuity

The conservation of mass is expressed in terms of the mass continuity equation, which we now derive. Consider the factors controlling the mass M inside the cube shown in figure 2.1.

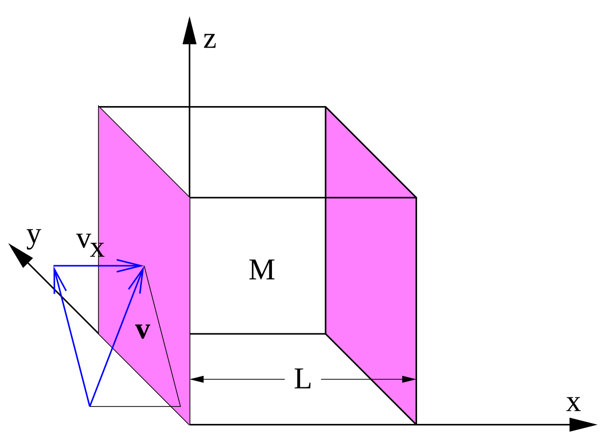

Figure 2.1: Schematic for consideration of fluid mass conservation inside a cube.

Since we are assuming that mass is conserved, mass increases when fluid flows into the cube, and it decreases when mass flows out. If the x component of the velocity on the left face of the cube is $v_x(x)$ and the density there is $\rho(x)$, then the mass per unit time flowing into the cube through this face is $\rho(x)v_x(x)L^2$. The mass flowing into the right face of the cube can similarly be computed as $-\rho(x+L)v_x(x+L)L^2$, where the minus sign comes from the fact that a positive value of v_x there corresponds to mass flowing out of the cube. The net inflow of mass through these two faces can therefore be written

$$\rho(x)v_x(x)L^2 - \rho(x+L)v_x(x+L)L^2 = -L^3\frac{\rho(x+L)v_x(x+L) - \rho(x)v_x(L)}{L} \approx -L^3\frac{\partial \rho v_x}{\partial x}. \quad (2.1)$$

Similar considerations applied to the y and z faces gives us the net flow of mass into the cube per unit time, which we equate to the time rate of change of mass inside the cube. Relating the mass to the density by $M = \rho L^3$, we get

$$\frac{\partial \rho L^3}{\partial t} = -L^3\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right), \quad (2.2)$$

which may be written more compactly as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (2.3)$$

An alternate form of the mass continuity equation may be obtained by applying the product rule to the divergence term $\nabla \cdot (\rho \mathbf{v}) = \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}$:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (2.4)$$

The total derivative is interpreted in this case as the time derivative of the density following a specific fluid parcel. By the chain rule

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \ldots = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho, \quad (2.5)$$

where dx/dt is the x component of the velocity v_x, etc., in this interpretation. The total time derivative in fluid dynamics is sometimes called the *material derivative* for this reason.

An *incompressible fluid* is one for which parcel densities do not change with time, i. e.,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0. \quad (2.6)$$

A corollary of the incompressibility condition is that the divergence of the velocity field is zero:

$$\nabla \cdot \mathbf{v} = 0. \quad (2.7)$$

An incompressible fluid may have spatially varying density. Imagine, for instance, ocean water with variable temperature and salinity. Each parcel maintains its own density, but as parcels move around, or are *advected*, the spatial distribution of density changes. Equation (2.6) allows one to predict these changes if the velocity field $\mathbf{v}$ is known.

For a homogeneous, incompressible fluid the situation is even simpler. Equation (2.6) becomes trivially satisfied and only equation (2.7) must be considered.

2.2 Momentum equation

An equation for the velocity field may be derived in a similar way. The physical principle used here is the conservation of momentum. The quantity $\rho\mathbf{v}$ is the momentum per unit volume, so consideration of the changes in the momentum inside the cube shown in figure 2.1 yield

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \mathbf{F}, \tag{2.8}$$

where $\mathbf{F}$ is the force per unit volume on the fluid. Physically, this states that the time rate of change of the momentum in the box equals the rate at which fluid carries momentum in and out of the box plus the external force exerted on the fluid.

A word must be said about the combination $\mathbf{v}\mathbf{v}$. This combination of vectors is neither a dot product nor a cross product. Instead, it is called a *dyadic product.* In matrix form it is written

$$\mathbf{v}\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} = \begin{pmatrix} v_xv_x & v_xv_y & v_xv_z \\ v_yv_x & v_yv_y & v_yv_z \\ v_zv_x & v_zv_y & v_zv_z \end{pmatrix}. \tag{2.9}$$

The combination $\nabla\cdot(\rho\mathbf{v}\mathbf{v})$ thus becomes a row vector

$$\nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \begin{pmatrix} A_x & A_y & A_z \end{pmatrix} \tag{2.10}$$

where

$$A_x = \frac{\partial\rho v_xv_x}{\partial x} + \frac{\partial\rho v_yv_x}{\partial y} + \frac{\partial\rho v_zv_x}{\partial z}, \tag{2.11}$$

etc.

Several different forces contribute to $\mathbf{F}$, the pressure gradient force, the gravitational force, and an inertial force due to the fact that the reference frame of the earth is rotating:

$$\mathbf{F} = \mathbf{F}_p + \mathbf{F}_g + \mathbf{F}_i. \tag{2.12}$$

Here we ignore viscous forces because they are of little direct significance to motions in the atmosphere and ocean.

Referring again to figure 2.1, the pressure exerts an inward normal force on each face of the cube equal to the pressure times the area of the face. The x component of the force on the cube is therefore the sum of the contributions from the left and right faces, and is equal to $L^2p(x) - L^2p(x+L) = -L^3\partial p/\partial x$. Considering contributions from all the faces and dividing by the volume L^3 yields the pressure force per unit volume:

$$\mathbf{F}_p = -\nabla p. \tag{2.13}$$

The gravitational force per unit volume is just the mass density times the gravitational field:

$$\mathbf{F}_g = \rho\mathbf{g} \tag{2.14}$$

The inertial force is somewhat more tricky to derive. Figure 2.2 shows a Cartesian coordinate system with the rotation vector of the earth Ω aligned with the z axis. A point $\mathbf{r}$ fixed to the earth moves with velocity $\Omega\times\mathbf{r}$. That the direction of this vector is correct

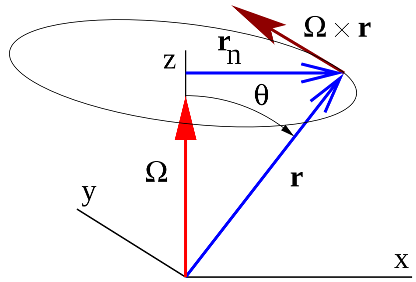

Figure 2.2: Sketch for understanding a rotating reference frame.

can be verified by examination of figure 2.2 – $\Omega \times \mathbf{r}$ is tangent to a circle centered on the z axis. The magnitude of this velocity is $|\Omega \times \mathbf{r}| = \Omega r \sin\theta = \Omega r_n$ as expected, where r_n is the radius of this circle. Thus, the velocity in the inertial frame $\mathbf{v}_I$ of something moving relative to the earth's reference frame is

$$\mathbf{v}_I = \mathbf{v}_R + \Omega \times \mathbf{r}, \tag{2.15}$$

where $\mathbf{v}_R$ is the velocity relative to the earth. A similar relationship can be verified for the time derivative of any vector $\mathbf{B}$:

$$\left(\frac{d\mathbf{B}}{dt}\right)_I = \left(\frac{d\mathbf{B}}{dt}\right)_R + \Omega \times \mathbf{B}. \tag{2.16}$$

To verify this, simply replace $\mathbf{r}$ by $\mathbf{B}$ in figure 2.2.

Let us now consider the acceleration of an object at position $\mathbf{r}$. From equation 2.16 the acceleration in the inertial reference frame may be written

$$\mathbf{a}_I = \left(\frac{d\mathbf{v}_I}{dt}\right)_I = \left(\frac{d\mathbf{v}_I}{dt}\right)_R + \Omega \times \mathbf{v}_I. \tag{2.17}$$

Substitution of equation 2.15 yields after rearrangement

$$\mathbf{a}_I = \mathbf{a}_R + 2\Omega \times \mathbf{v}_R + \Omega \times (\Omega \times \mathbf{r}), \tag{2.18}$$

where $\mathbf{a}_R = (d\mathbf{v}_R/dt)_R$ is the acceleration of the object relative to the rotating earth. The quantity $2\Omega \times \mathbf{v}_R + \Omega \times (\Omega \times \mathbf{r})$ is therefore the acceleration of the reference frame of the object. The term $\Omega \times (\Omega \times \mathbf{r})$ may also be written (using a standard vector identity) as $-\Omega^2 \mathbf{r}_n$, where $\mathbf{r}_n$ is defined in figure 2.2. This is nothing more than the standard centripetal acceleration of an object rotating with the earth. The other term accounts for the fact that the object may be moving relative to the rotating earth. The inertial force per unit mass is simply minus the additional terms beyond $\mathbf{a}_R$ on the right side of equation (2.18). The inertial force per unit volume of fluid is therefore the fluid density times this:

$$\mathbf{F}_i = -\rho(2\Omega \times \mathbf{v}_R - \Omega^2 \mathbf{r}_n). \tag{2.19}$$

We drop the subscripted R from the velocity for brevity of notation since this is generally the only velocity we deal with in geophysical fluid dynamics.

The gravitational force and the second term in equation (2.19) are both derivable from a scalar potential. It is customary to combine these two terms into an "effective gravity" with a potential Φ called the *geopotential.* The net force per unit volume from these two terms is therefore

$$\mathbf{F}_e = -\nabla\Phi = \rho(\mathbf{g} + \Omega^2\mathbf{r}_n) = \rho\mathbf{g}_e. \tag{2.20}$$

The centrifugal force term is small compared to the gravitational force, so for most work it is generally ignored and $\mathbf{g}_e$ is simply replaced by $\mathbf{g}$.

Substituting these forces into equation (2.8) results in the *flux form* of the momentum equation for geophysical fluid dynamics:

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p + \rho\nabla\Phi + 2\rho\Omega\times\mathbf{v} = 0. \tag{2.21}$$

The *advective form* of the momentum equation may be obtained by noting that the first two terms of this equation can be split using the product rule into

$$\frac{\partial\rho}{\partial t}\mathbf{v} + \rho\frac{\partial\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v})\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v}. \tag{2.22}$$

The first and third terms of this expression cancel out by virtue of the continuity equation (2.3), allowing us to rewrite equation (2.21) as

$$\frac{d\mathbf{v}}{dt} + \frac{\nabla p}{\rho} + \nabla\Phi + 2\Omega\times\mathbf{v} = 0. \tag{2.23}$$

2.3 Atmosphere

2.3.1 Dry entropy

In an atmosphere without moisture the ideal gas law for dry air is

$$\frac{p}{\rho} = R_d T \tag{2.24}$$

where p is the pressure, ρ is the air density, T is the absolute temperature, and $R_d = R/m_d$, R being the universal gas constant and m_d the molecular weight of dry air. If moisture is present there are minor modifications to this equation, which we ignore here. The dry entropy is

$$s_d = C_p\ln(T/T_R) - R_d\ln(p/p_R) \tag{2.25}$$

where C_p is the mass (not molar) specific heat of dry air at constant pressure, T_R is a constant reference temperature (say 300 K), and p_R is a constant reference pressure (say 1000 hPa). A variable related to the dry entropy is the potential temperature θ, which is defined

$$\theta = T_R\ln(s_d/C_p). \tag{2.26}$$

The potential temperature is the temperature air would have if it were compressed or expanded (without condensation of water) in a reversible adiabatic fashion to the reference pressure p_R.

We ignore heat conduction and the irreversible production of entropy, which means that latent heat release and radiation (solar and thermal) are the only sources of entropy. The governing equation for dry entropy is therefore

$$\frac{ds_d}{dt} = \frac{Q}{T_R}, \tag{2.27}$$

where $Q = Q_l + Q_r$ is the heating rate per unit mass and is composed of latent and radiative heating parts. Strictly speaking the denominator on the right side should contain the actual temperature, but we have replaced it by the constant reference temperature for the sake of a consistent approximation. The latent heating at this level of approximation is

$$Q_l = -L\frac{dr_v}{dt} \tag{2.28}$$

where L is the latent heat of condensation, assumed constant here, and r_v is the mixing ratio of water vapor.

2.3.2 Moist entropy

The moist entropy is defined at our level of approximation as

$$s = s_d + \frac{Lr_v}{T_R} \tag{2.29}$$

and the governing equation is

$$\frac{ds}{dt} = \frac{Q_r}{T_R}. \tag{2.30}$$

The radiative heating per unit volume ρQ_r is generally supplied by atmospheric radiation models. A variable related to the moist entropy is the equivalent potential temperature, defined

$$\theta_e = T_R \ln(s/C_p). \tag{2.31}$$

2.3.3 Water

If condensation is occurring, the water vapor mixing ratio r_v is generally quite close to the saturation mixing ratio r_s. The saturation mixing ratio may be written in terms of the ratio of the saturation vapor pressure $e_s(T)$ of water divided by the partial pressure of dry air, approximated as the total pressure:

$$r_s = \frac{m_w}{m_d}\frac{e_s(T)}{p} \tag{2.32}$$

where m_w is the molecular weight of water vapor.

Condensation results in the production of tiny water droplets typically of order 10^{-5} m in diameter. These droplets are small enough to be carried along with the flow. If these *cloud droplets* are not converted into precipitation, the sum of the vapor mixing ratio and the cloud droplet mixing ratio r_c is conserved in parcels. This quantity $r_t = r_v + r_c$ is called the *total cloud water mixing ratio.* Conversion into precipitation is the coalescence of cloud droplets into larger drops which fall relative to the air. Precipitation can also be converted into water vapor if it evaporates in sub-saturated air. If P is the net conversion rate per unit mass of total advected water into precipitation (with negative values corresponding to evaporation of precipitation), then the total advected water obeys the equation

$$\frac{dr_t}{dt} = -P. \tag{2.33}$$

The governing equation for precipitation mixing ratio r_r reflects its sedimentation relative to the air. In flux form it is written

$$\frac{\partial \rho r_r}{\partial t} + \nabla \cdot [\rho(\mathbf{v} - w_t \mathbf{k}) r_r] = \rho P \tag{2.34}$$

where $-w_t\mathbf{k}$ is the velocity of the precipitation relative to the air. In reality, different sized raindrops fall at different speeds, and separate equations are needed to follow the evolution of each rain category. In this case one must also account for the transfer of raindrops from category to category as they grow and shrink. Adding ice to the mix further complicates the treatment. Texts on cloud physics such as Byers (1965) and Young (1993) should be consulted for more details on this subject. For our purposes we will consider only one category of precipitation. The governing equation for precipitation may be written in advective form as

$$\frac{dr_r}{dt} - \frac{1}{\rho}\frac{\partial \rho w_t r_r}{\partial z} = P. \tag{2.35}$$

It is often useful to use the moist entropy and the total advected water mixing ratio as dependent variables in numerical models. A problem with this approach is that one must also infer the dry entropy and the water vapor mixing ratio from these variables. This is not possible to do directly when the air is saturated. In this case an iterative approach using a numerical technique such as Newton's method is needed.

2.3.4 Density

For atmospheric calculations the density must be expressed in terms of the above atmospheric thermodynamic variables. We now see in particular how the density can be written in terms of the potential temperature and the pressure. Given the ideal gas law and the definition of potential temperature θ, we can easily derive

$$\rho = \frac{p}{R_d \theta (p/p_R)^{R_d/C_p}}. \tag{2.36}$$

From this we obtain

$$\frac{\nabla p}{\rho} = \theta \nabla \left[C_p (p/p_R)^{R_d/C_p} \right] \equiv \theta \nabla \Pi. \tag{2.37}$$

The quantity Π is called the *Exner function.* Since the potential temperature can be written in terms of the dry entropy, substitution of this equation into equation (2.23) results in a form of the momentum equation appropriate for atmospheric flows.

2.4 Ocean

We make the approximation that ocean water is incompressible. Under this condition the temperature of ocean water only changes when heating occurs, resulting in the governing equation

$$C_l \frac{dT}{dt} = Q, \tag{2.38}$$

where Q is the heating rate per unit mass in the ocean. In the interior of the ocean the only contributor to this is solar radiation. At the surface evaporation can cool the ocean, but as this is a surface phenomenon, it requires special treatment.

To the extent that surface changes in salinity are ignored, the salinity (which is essentially a mixing ratio or mass concentration for salt) obeys a very simple equation:

$$\frac{dS}{dt} = 0. \tag{2.39}$$

The density of ocean water is in general a complex function of temperature, salinity, and pressure. For purposes of exposition we can ignore the dependence of this density on pressure, thus allowing ocean water to be treated as an incompressible fluid:

$$\rho = \rho(T, S) \tag{2.40}$$

Ocean density[1] varies over the range 1024 kg m^{-3} $< \rho <$ 1031 kg m^{-3}.

2.5 References

Byers, H. R., 1965: *Elements of cloud physics.* University of Chicago Press, Chicago, 191 pp.

Kundu, P. K., 1990: *Fluid Mechanics.* Academic Press, San Diego, 638 pp.

Young, K. C., 1993: *Microphysical processes in clouds.* Oxford University Press, Oxford, 427 pp.

2.6 Problems

1. Suppose the ocean salinity field has the form $S = S_0 + \Gamma(x - v_x t)$ where S_0 and Γ are positive constants. The velocity field is given by $\mathbf{v} = (v_x, 0, 0)$, where v_x is a positive constant.

   (a) Verify that $dS/dt = 0$.

   (b) Compute $\partial S/\partial x$ and use this to discuss the spatial structure of salinity.

   (c) Compute $\partial S/\partial t$ and explain physically why this is not equal to dS/dt.

[1] Actually, potential density, which is the density of ocean water raised adiabatically to sea level.

2. Consider a wave (type unspecified) in an incompressible fluid, with spatial dependence of the velocity field $\mathbf{v} = \mathbf{v}_0 \sin(\mathbf{k} \cdot \mathbf{x})$. The vector $\mathbf{v}_0$ is a constant as is the wave vector $\mathbf{k}$. What condition does mass continuity impose on $\mathbf{v}_0$?

3. Given $\mathbf{v} = (Ax, -Ay, 0)$ in a homogeneous fluid, where A is a constant:

   (a) Verify that this flow is incompressible.

   (b) Sketch the vector field in the $x - y$ plane by drawing vectors of the appropriate relative magnitude and direction on a grid of points.

   (c) Compute the quantity $\rho\mathbf{v}\mathbf{v}$ as a 3×3 matrix.

   (d) Compute $\mathbf{X} = -\nabla \cdot (\rho\mathbf{v}\mathbf{v})$ and sketch this vector field.

4. Suppose the vertical velocity field in the atmosphere at some height $z = h$ is $v_z = W \sin(kx)$ and the entropy field is $s = S \sin(kx + \phi)$, where W, S, ϕ, and k are constants. the upward flux of entropy through $z = h$ is given by $F_s = \overline{\rho v_z s}$ where the overbar indicates averaging in x. Compute F_s and explore how it varies with ϕ. You may assume that the density ρ is constant for this calculation.