Predictability and Chaos

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Predictability

- Predictability is ability to estimate the future state of a dynamical system knowing the involved physical processes and initial conditions of the system. In other words, the predictability of the system refers to the degree of accuracy with which it is possible to predict the future state of the system.
- Intuitively – The best example is the predictability of weather and climate

Prediction

- The state of atmosphere is continuously evolving under a set of physical laws.
- Dynamical prediction

  The process of predicting the future state of the atmosphere is based on temporal extrapolation of the present state using the physical laws.
- Statistical prediction

  The extrapolation rules are determined empirically based on past states of the system.

Causes of imperfect predictions

- Incomplete (imperfect) knowledge of the initial state of the system (atmosphere)
- Incomplete (imperfect) knowledge of the boundary conditions (atmosphere)
- Imperfect methods (models) by which the temporal and spatial extrapolation was performed due to incomplete knowledge of the physical laws.
- Imperfect numerical representations and computation techniques.

Great discovery – predictability limits

- Early studies: Poincare (1903) – "Although we know all physics laws exactly, we could know initial conditions (IC) only approximately. Small changes in the IC produce very great ones in the final phenomena. A small error in the former can produce an enormous error in the latter and prediction becomes impossible."
- Until the late 1950s scientists thought that better computers and better input data would always lead to better and better accuracy of weather forecasts.
- First challenge – Ed Lorenz (around 1956). He designed a model – a set of 12 equations representing certain atmospheric conditions and solved it numerically on an available computer. After one of the completed runs, he repeated the same run, which was interrupted for some reason in the middle. He took the current outputs and inserted them as inputs to continue the interrupted run.
- After some time into the simulation, he noticed that the results of the new run started to differ and then completely diverge from the original run.
- At first he suspected a problem with the computer, but the repeated original run produced the expected results.
- After some analysis and thinking, he discovered that the problem was in the precision of the interrupted output. The print out had less significant digits than the precision of the computer.
- He noticed that the small difference between something retained to six decimal places and rounded off to three had amplified in the course of two months. Eventually, the differences became as big as the values itself.
- Lorenz concluded that we cannot make forecasts two months ahead even if we have a perfect model. Small errors (or even uncertainties) would amplify until they became too large. For example, if an initial temperature entered in the computer is 12.235C instead of 12.23528C, that would imply the growth of differences between the two runs that would eventually lead to a completely different forecast at the end of the simulation.

Lorenz discovery – predictability limits

- Lorenz’ discovery led to rapid development of theories of how deterministic systems such as weather forecasts can lead to predictability break up and chaotic behavior.
- Current studies show that increased complexity of the physical system actually can reduce the level of chaotic behavior.
- CHAOS: Aperiodic, long-term behavior of a bounded, deterministic system that exhibits sensitive dependence on initial conditions and control parameters.
- *** Chaos (greek): Origin of the Universe (Great Emptiness); also: State of a system without order.

Predictability problem –Lorenz' famous system of thermal convection in the atmosphere (in 1960s)

$$\frac{dX}{dt} = sY - sX$$

$$\frac{dY}{dt} = -XZ + rX - Y$$

$$\frac{dZ}{dt} = XY - bZ$$

$$s = 10; r = 28; b = \frac{8}{3}$$

X – size of the convective motion

Y – proportional to the temperature difference between ascending and descending fluids

Z – proportional to the deviation of the vertical temperature profile from a linear function.

s – Prandtl number

$$r = \frac{R_a}{R_c}$$

R_a – Rayleigh number; R_c – critical Rayleigh number

$$b = \frac{4}{1+a^2}$$

a – constant in the critical Rayleigh number

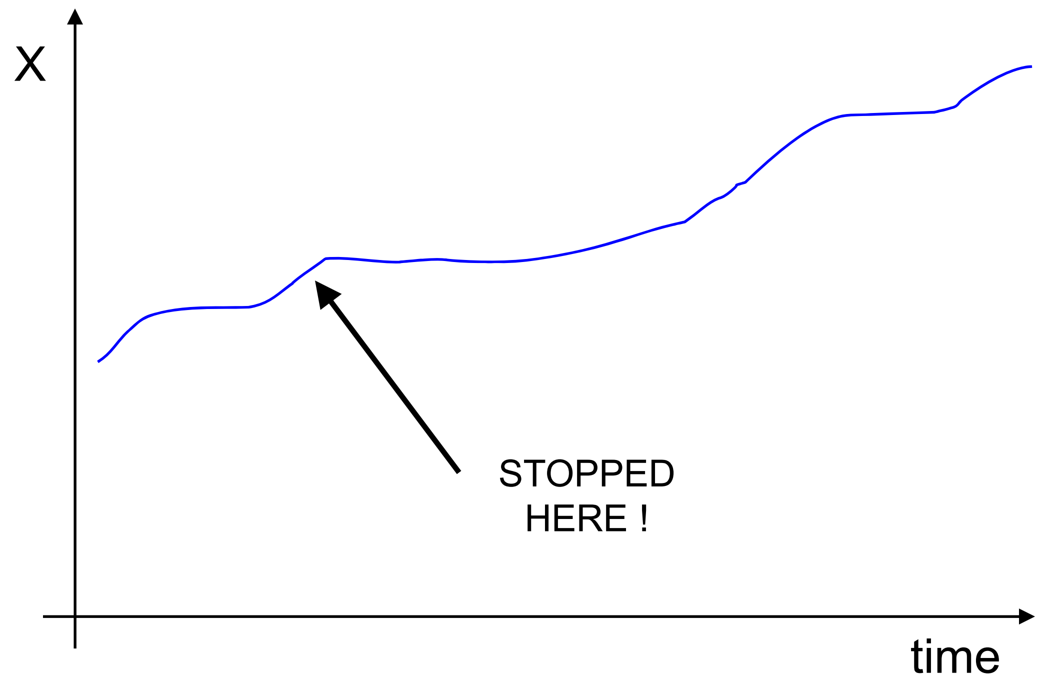
Lorenz’ Model

Output results (blue)

Where model stopped

X=0.432
Y=9.122
Z=0.011

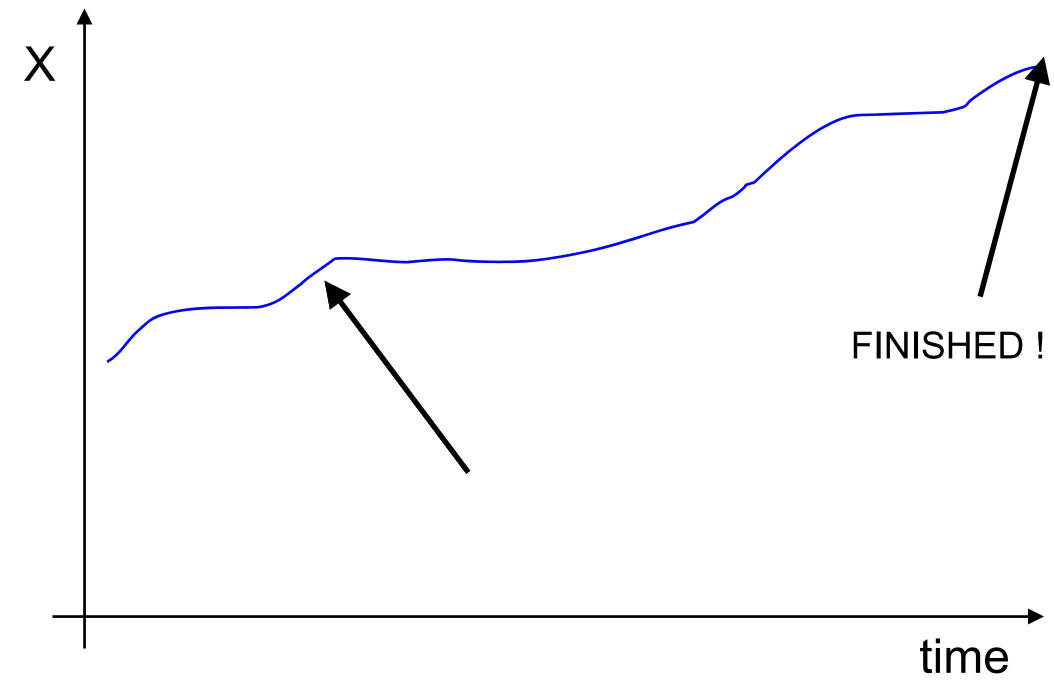
Lorenz used these results to continue the run



Lorenz' Model

Last output used as input data to continue the model run (blue)

X=0.432
Y=9.122
Z=0.011



Lorenz' Model

New run from the beginning (red)
(No round off)

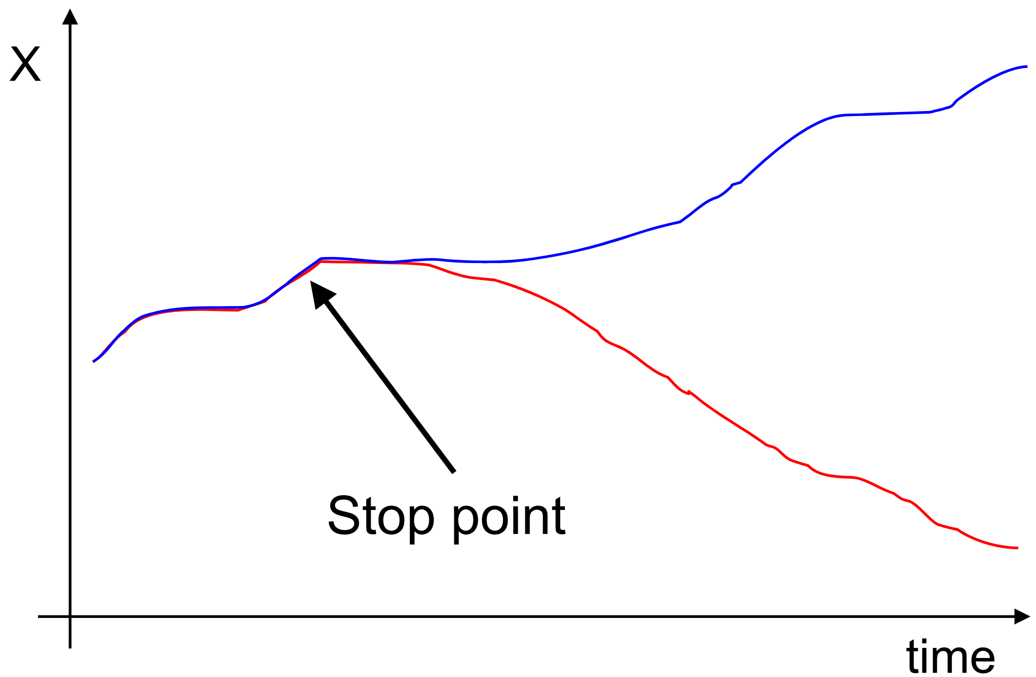
At the stop point:

X=0.431922
Y=9.122325
Z=0.010908

"Broken" run - Input Data (blue)

At the stop point:

X=0.432
Y=9.122
Z=0.011



Characteristics of predictability

- Weather and climate models are complex systems with known problems in accurate predictability. However, even extremely simple systems can show the lack of predictability.
- Let us take an example of a system that should be easy to understand.

Predictability – simple linear iteration formula

Bank – simple interest formula

$$X_{n+1} = r\,X_n$$

Xn+1 – New value

R – Interest ratio

Xn – Initial value

After successive multiplications

$$X_{n+1} = r\,X_n = r\,(r\,X_{n-1}) = r\,r\,(r\,X_{n-2}) = \ldots\ldots = r^{n+1}\,X_0$$

So, if we know the initial value and the interest rate, we can easily calculate the new value

The new value can be uniquely calculated for any combination of r, n, and X_0

Simple nonlinear system – Logistic feedback iterator

- Assume a species of cell living in a contained environment (e.g., fish tank) with constant food supply and temperature. There will be a maximum population of size N that can be supported by the environment
- P_n is the actual population at time n
- If P_n is smaller than N, we expect population to grow
- If P_n is greater than N, we expect population to decrease
- A growth rate r can be defined as

$$r = \frac{P_{n+1} - P_n}{P_n}$$

Logistic Feedback Iterator – Verhulst Model (1845)

- Verhulst assumed that the growth rate at time n should be proportional to 1-P_n (the fraction of the environment that is not yet used by the population at time n):

$$\frac{P_{n+1} - P_n}{P_n} \propto 1 - P_n$$

Introduce a suitable constant r (e.g., speed of process)

$$\frac{P_{n+1} - P_n}{P_n} = r\,(1 - P_n)$$

Solving this equation for P_{n+1}, yields the population model

$$P_{n+1} = P_n + r\,P_n\,(1 - P_n)$$

- This is called the LOGISTIC MODEL – needs to be iterated (nonlinear expression)

Logistic Feedback Iterator – Verhulst Model (1845)

$$P_{n+1} = P_n + r\,P_n\,(1 - P_n)$$

- Iteration:

  Take P0=0.01 (initial population is 1% of the maximum population)

  r=3

$$P_1 = P_0 + r\,P_0\,(1 - P_0)$$

$$P_2 = P_1 + r\,P_1\,(1 - P_1)$$

$$P_3 = P_2 + r\,P_2\,(1 - P_2)$$

…….

$P_0 = 0.01$
$P_1 = 0.0397$
$P_2 = 0.15407173$
$P_3 = 0.545072626044…$
……….

Observe: Continued iteration requires higher and higher computational accuracy if we insist on exact results. Although this seems to be trivial, it leads to serious problems.

Predictability – iteration formula

$$X_{n+1} = X_n + r\, X_n\,(1 - X_n)$$

This still looks simple

Let us take r=3, n=100, and X_0=0.01

X_{n+1}

1. 0.039700000000000
2. 0.154071730000000
3. 0.545072626044421
4. 1.288978001188801
5. 0.171519142109176
6. 0.597820120107099
7. 1.319113792413797
8. 0.056271577646257
9. 0.215586839232630
10. 0.722914301179573

20\. 0.596529312494691

50\. 1.313996746606757

100\. 0.393788595636378

Predictability – iteration formula

$$X_{n+1} = X_n + r\,X_n\,(1 - X_n)$$

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6.	0.597820120107099
7.	1.319113792413797
8.	0.056271577646257
9.	0.215586839232630
10.	0.722914301179573

What if we stop at the 10th iteration and truncate the result to only 3 decimal places

0.722914301179573 => 0.722

Compare the "old" and the "new" iterations:

NEW	OLD
20. 1.309731022679916	20. 0.596529312494691
50. 1.084204314601272	50. 1.313996746606757
100. 1.230459200984260	100. 0.393788595636378

Predictability – iteration formula

$$X_{n+1} = X_n + r\,X_n\,(1 - X_n)$$

What if we stop at 10th iteration and truncate the results to only 3 decimal places

0.722914301179573 => 0.722

Compare the "old" and the "new" iterations:

Striking result: Even at 20th iteration – the results are not correlated any more

X_{n+1}

1. 0.0397000000000000
2. 0.154071730000000
3. 0.545072626044421
4. 1.288978001188801
5. 0.171519142109176
6. 0.597820120107099
7. 1.319113792413797
8. 0.056271577646257
9. 0.215586839232630
10. 0.722914301179573

NEW	OLD
20. 1.309731022679916	20. 0.596529312494691
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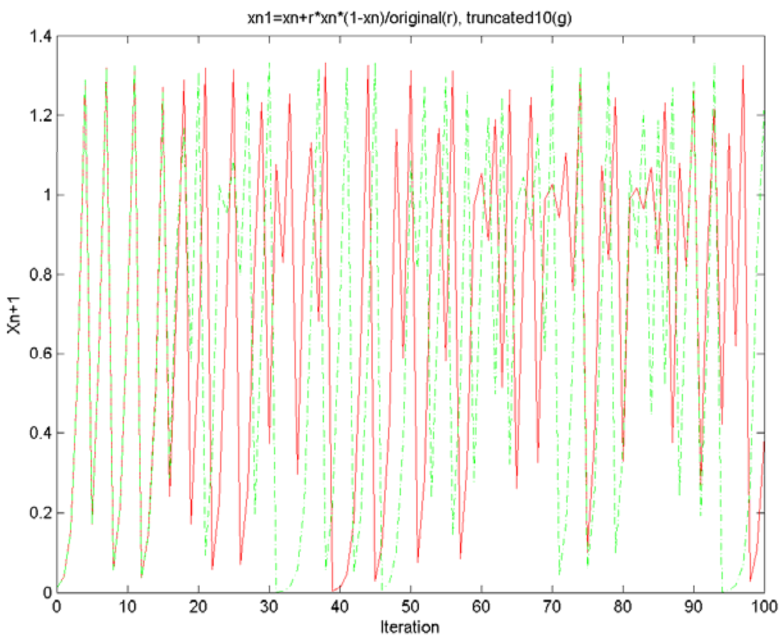
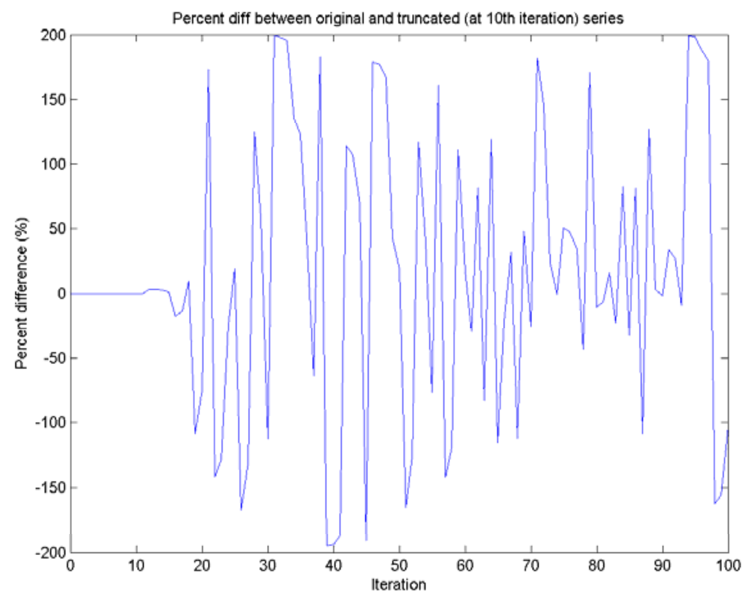
100. 1.230459200984260	100. 0.393788595636378

Predictability – iteration formula

Compare the "old" and the "new" iterations:

Striking result: After 20th iteration – the results are not correlated





Time series:

Original (red), truncated (green) expressions

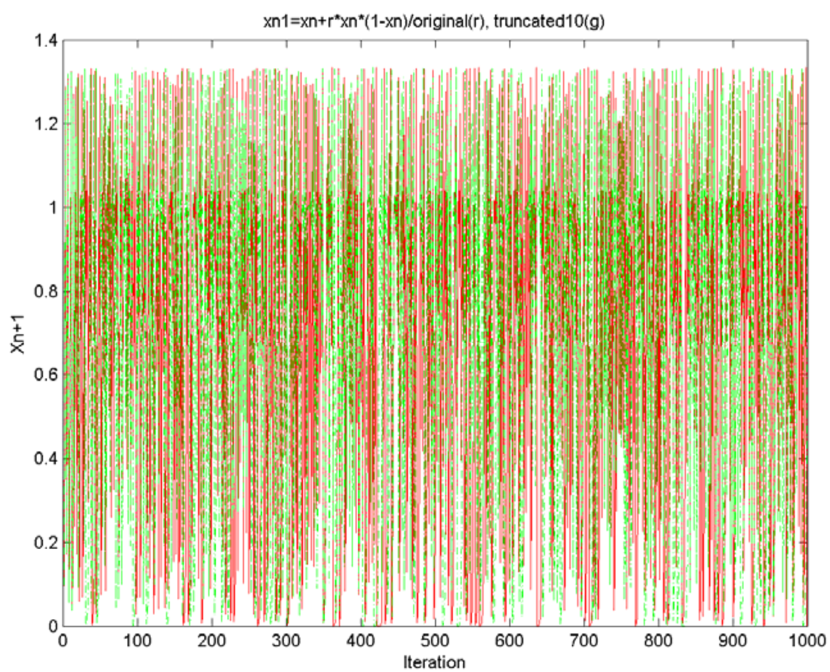
Time series:

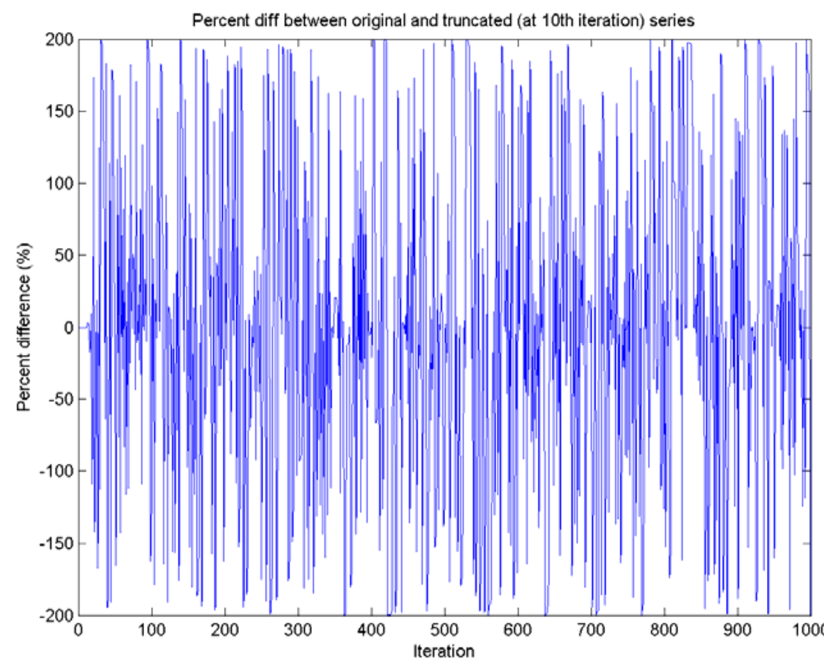
% difference between original and truncated expressions

Predictability – iteration formula

Compare the "old" and the "new" iterations:

Striking result: After 1000th iteration – the results are not correlated at all





Time series:

Original (red), truncated (green) expressions

Time series:

% difference between original and truncated expressions

Predictability – iteration formula

What if we just re-write the same expression differently?

$$X_{n+1} = X_n + r\,X_n\,(1 - X_n) =$$

$$= X_n + r\,X_n - r\,X_n^{\,2} =$$

$$= (1 + r)\,X_n - r\,X_n^{\,2}$$

Notice that

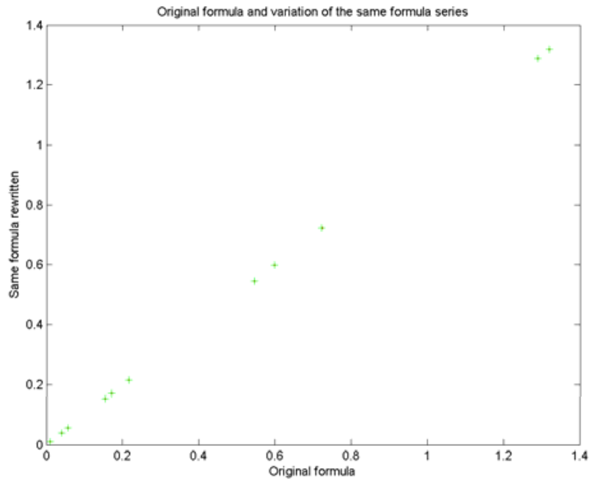
$$X_{n+1} = X_n + r\,X_n\,(1 - X_n) \;\&$$

$$X_{n+1} = (1 + r)\,X_n - r\,X_n^{\,2}$$

are algebraically completely identical,

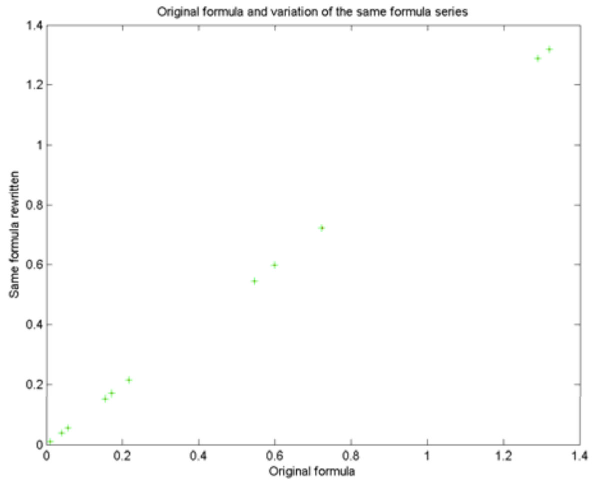
but computers think differently!!!

Iteration	$X_{n+1} = X_n + r\, X_n\, (1 - X_n)$	$X_{n+1} = (1+r)\, X_n - r\, X_n^{\ 2}$	Difference
1	0.010000000000000	0.010000000000000	0.000000000000000
10	0.215586839232630	0.215586839232638	-0.000000000000007
20	0.171084846701943	0.171084846695175	0.000000000006768
30	1.232112462387190	1.232112456898180	0.000000005489003
40	0.002909156902851	0.002908166812190	0.000000990090661
50	0.586382615268778	0.575607525195148	0.010775090073631
60	0.972495402397394	0.988950671746734	-0.016455269349340
70	0.986032164226998	0.588027498266818	0.398004665960180
80	1.245281926676870	0.746693837160232	0.498588089516638
90	0.821069312375225	1.177948649654320	-0.356879337279099
100	0.107040381336610	1.294133025295300	-1.187092643958690



After 10 iterations

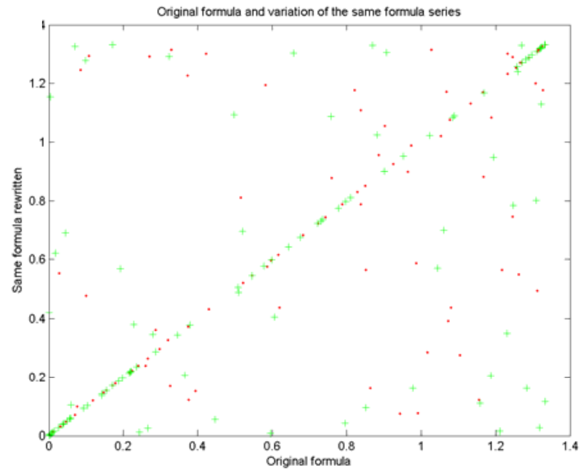
Scatter plots: Same formula rewritten in two equivalent ways



After 10 iterations

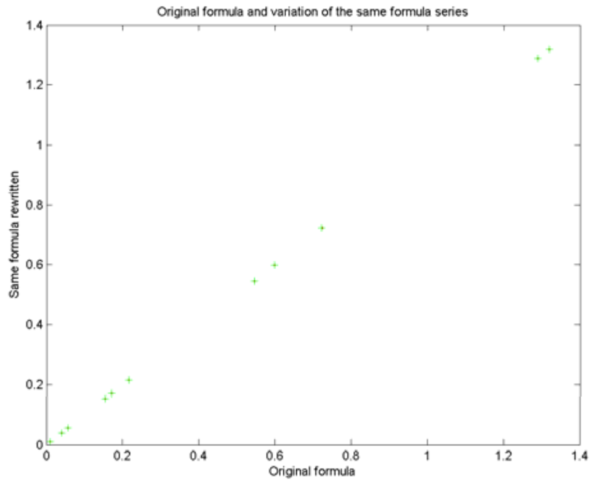
Scatter plots: Same formula rewritten in two equivalent ways

After 100 iterations



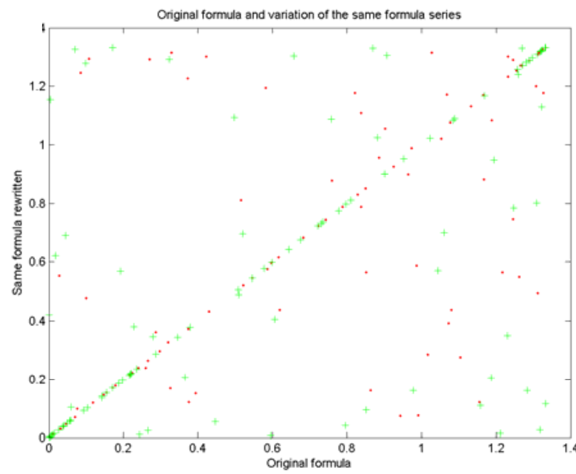
CHAOS !!!

Scatter plots: Same formula rewritten in two equivalent ways

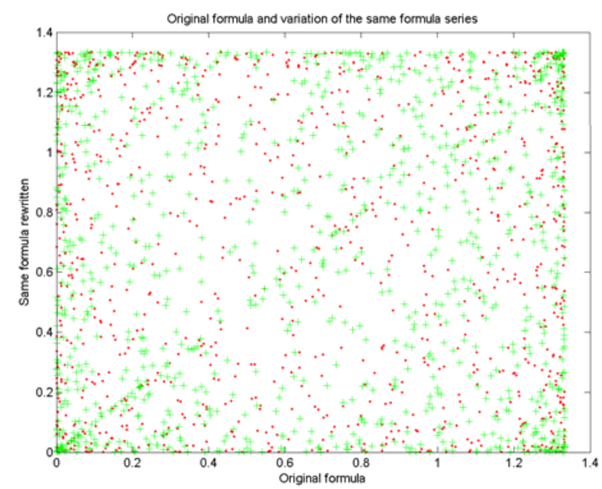


After 10 iterations

After 100 iterations



After 1000 iterations

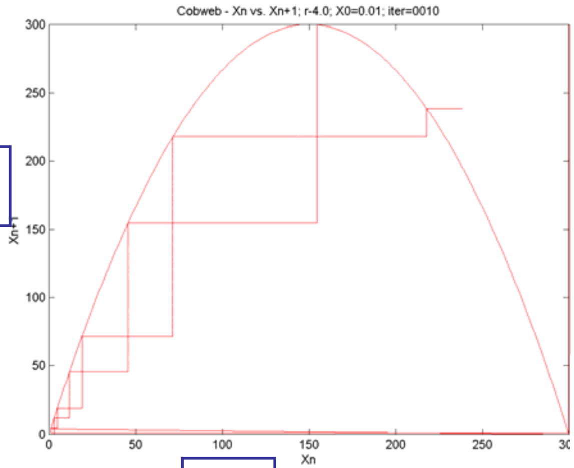


CHAOS !!!

Cob web diagram – graphical presentation of the iterative formula

$r = 4; \; X_0 = 0.01$

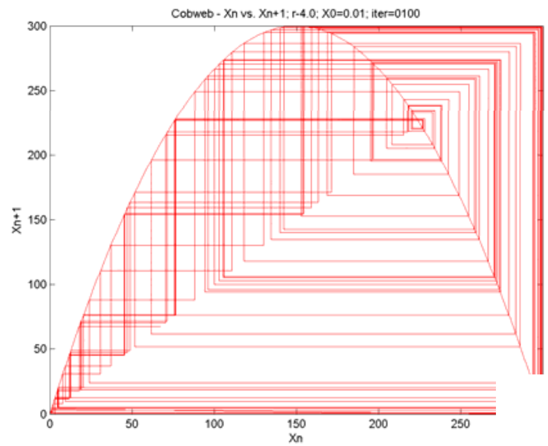
X_{n+1}



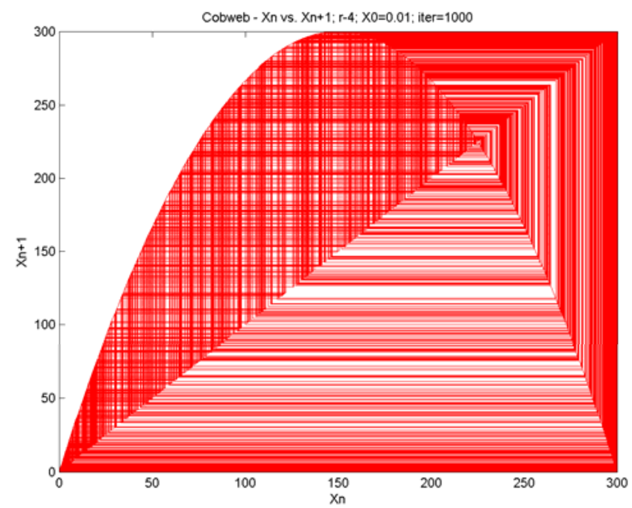
X_n

Max iter=10

Max iter=100



Max iter=1000



Measure of Chaos – Ljapunov Exponent

- Assume a small arbitrary initial error E_0

$$Total\ amplification\ factor: \left|\frac{E_n}{E_o}\right|$$

$$\left|\frac{E_n}{E_o}\right| = \left|\frac{E_n}{E_{n-1}}\right|\left|\frac{E_{n-1}}{E_{n-2}}\right|.....\left|\frac{E_1}{E_o}\right|$$

$$From\ the\ error\ growth\ of\ the\ linear\ system\ \ X_{n+1} = c\,X_n$$

$$The\ error\ growth\ is: \left|\frac{E_n}{E_o}\right| = \frac{c^n\ X_0}{X_0} = c^n\ \ i.e.,$$

$$\ln\left|\frac{E_n}{E_o}\right| = n\ln c\ \ => \ \ln c = \frac{1}{n}\ln\left|\frac{E_n}{E_o}\right|$$

$$\ln c = \frac{1}{n}\ln\left|\frac{E_n}{E_{n-1}}\frac{E_{n-1}}{E_{n-2}}.....\frac{E_1}{E_o}\right| = \frac{1}{n}\sum_{k=1}^{n}\ln\left|\frac{E_k}{E_{k-1}}\right|$$

Measure of Chaos – Ljapunov Exponent

- Approximate a small arbitrary error ε as the previous error and:

$$E_{k+1} = f(x_k + \varepsilon) - f(x_k)$$

where $f(x) = r\,x(1-x)$

$$\ln c = \frac{1}{n}\sum_{k=1}^{n} \ln\left|\frac{E_k}{\varepsilon}\right| = \frac{1}{n}\sum_{k=1}^{n} \ln\left|\frac{f(x_{k-1}+\varepsilon) - f(x_{k-1})}{\varepsilon}\right| = \frac{1}{n}\sum_{k=1}^{n} \ln\left|f'(x_{k-1})\right|$$

For $n \to \infty$ we obtain the Ljapunov exponent λ

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln\left|f'(x_{k-1})\right|$$

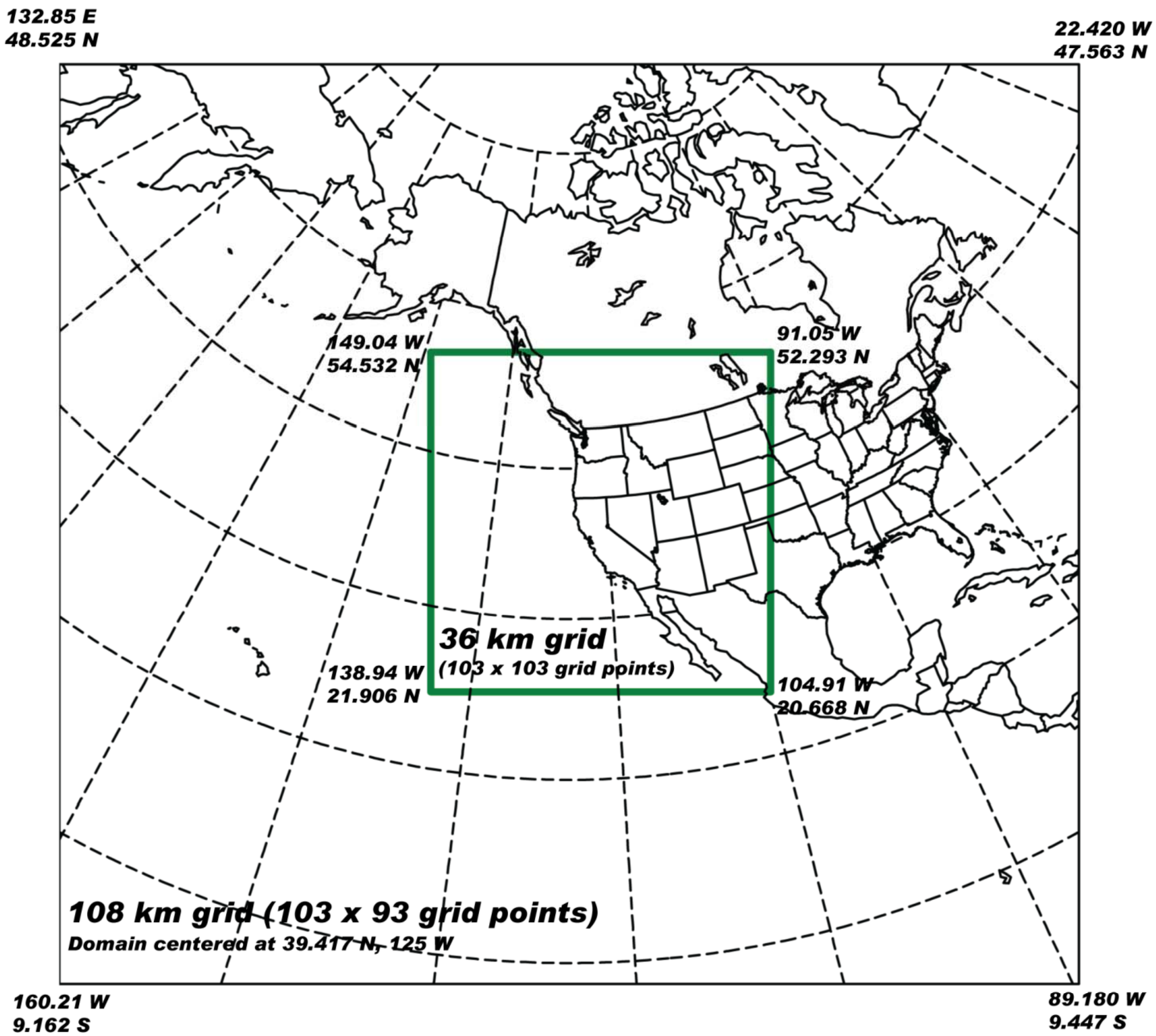
For λ>0 => system is chaotic

For λ<0 or λ=0 => system is stable

Improving weather and climate forecasts – ensemble forecasting

- Epstein (1969) – Since the skill of the single deterministic forecast decreases in time, stochastic (probabilistic) forecasts should be considered. The only feasible way is “ensemble forecasting” – several (many) model forecasts are performed by introducing perturbations in initial conditions (IC) and model’s physics.
- IC perturbations: determining fastest growing errors of the forecasts
- NCEP – “Breeding growing perturbations” – start with initial perturbations and after 1.5 days use differences between the model and forecasts to scale perturbations for a new run.
- ECMWF – “Singular vectors” – use linear tangent model to obtain fastest-growth errors.
- Canadian Center – using an ensemble of data assimilation systems (considering observational errors and adding random numbers to the observations) and including different parameters in the physical parameterizations of the model in different ensembles to create IC.
- Additional way – “Super-ensemble” – joint probabilistic forecast from two or more models (single forecasts or ensembles).

Chaos – Example: Regional/mesoscale models MM5 and WRF



50 MM5 ensembles + 50 WRF ensembles = 100 ensemble members

- MM5 – Mesoscale Model 5 (Grell et al. 1994; NCAR).
- WRF – Weather and Research Forecasting model (Skamarock et al. 2005)
- Simultaneous runs of MM5 and WRF for 15 days (12 – 27 Dec 2008)
- Initial/Boundary conditions: GFS (0.5 x 0.5 deg grid resolution for 0-168 hours, 2.5 x 2.5 deg for > 168 h – 360 hours)
- Domain setup: 2-domains (I: 108 km grid; 103 x 93 grid points; II: 36 km grid; 103 x 103 grid points)

MM5 simulations	PBL scheme	Cloud Microphysics	Cumulus Scheme	Radiation
Control	Eta M-Y	Reisner 2	Kain-Fritsch	RRTM (FRAD=4)
1	Burk-Thompson	Reisner 2	Grell	CCM2 (FRAD=3)
2	Burk-Thompson	Reisner 2	Betts-Miller	CCM2 (FRAD=3)
3	Eta M-Y	Goddard (GFSC)	Betts-Miller	RRTM (FRAD=4)
4	Burk-Thompson	Goddard (GFSC)	Betts-Miller	RRTM (FRAD=4)
5	Gayno-Seaman	Schultz	Betts-Miller	CCM2 (FRAD=3)
6	MRF	Simple ice (Dudhia)	Kain-Fritsch	CCM2 (FRAD=3)
7	Gayno-Seaman	Goddard (GFSC)	Betts-Miller	CCM2 (FRAD=3)
8	MRF	Simple ice (Dudhia)	Grell	RRTM (FRAD=4)
9	Burk-Thompson	Reisner 2	Grell	CCM2 (FRAD=3)
10	MRF	Goddard (GFSC)	Grell	CCM2 (FRAD=3)
11	MRF	Goddard (GFSC)	Kain-Fritsch	RRTM (FRAD=4)
12	Eta M-Y	Reisner 2	Grell	CCM2 (FRAD=3)
13	MRF	Reisner 2	Kain-Fritsch	Dudhia (FRAD=2)
14	Gayno-Seaman	Reisner 2	Grell	Dudhia (FRAD=2)
15	Burk-Thompson	Reisner 2	Betts-Miller	Simple cloud (FRAD=1)
16	Gayno-Seaman	Simple ice (Dudhia)	Grell	Simple cloud (FRAD=1)
17	Gayno-Seaman	Reisner 2	Betts-Miller	Dudhia (FRAD=2)
18	Burk-Thompson	Goddard (GFSC)	Grell	Dudhia (FRAD=2)
19	Burk-Thompson	Schultz	Kain-Fritsch	Dudhia (FRAD=2)
20	MRF	Simple ice (Dudhia)	Kain-Fritsch	Simple cloud (FRAD=1)
21	Eta M-Y	Reisner 2	Kain-Fritsch	RRTM (FRAD=4)
22	Burk-Thompson	Reisner 2	Grell	RRTM (FRAD=4)
23	Burk-Thompson	Simple ice (Dudhia)	Kain-Fritsch	CCM2 (FRAD=3)
24	Burk-Thompson	Simple ice (Dudhia)	Betts-Miller	Simple cloud (FRAD=1)
25	Gayno-Seaman	Goddard (GFSC)	Kain-Fritsch	Dudhia (FRAD=2)
26	Burk-Thompson	Schultz	Grell	RRTM (FRAD=4)
27	Eta M-Y	Simple ice (Dudhia)	Grell	Dudhia (FRAD=2)
28	Burk-Thompson	Reisner 2	Kain-Fritsch	Dudhia (FRAD=2)
29	Burk-Thompson	Reisner 2	Betts-Miller	RRTM (FRAD=4)

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WRF Simulation	PBL-Scheme	Cloud Microphysics	Cumulus Scheme	Radiation (SW/LW)
Control	Mellor-Yamada-Janjic	Thompson	Kain-Fritsch	Dudhia/RRTM
1	Mellor-Yamada-Janjic	Goddard microphysics	Betts-Miller	GFDL/GFDL
2	Mellor-Yamada-Janjic	Goddard microphysics	Kain-Fritsch	GFDL/GFDL
3	Mellor-Yamada-Janjic	Lin et al.	Kain-Fritsch	Goddard/RRTM
4	Mellor-Yamada-Janjic	Eta microphysics	Kain-Fritsch	GFDL/GFDL
5	Mellor-Yamada-Janjic	Eta microphysics	Betts-Miller	CAM/CAM
6	Mellor-Yamada-Janjic	Eta microphysics	Kain-Fritsch	CAM/CAM
7	Mellor-Yamada-Janjic	Thompson	Betts-Miller	Dudhia/RRTM
8	Mellor-Yamada-Janjic	Goddard microphysics	Grell-Devenyi	GFDL/GFDL
9	Mellor-Yamada-Janjic	Goddard microphysics	Betts-Miller	CAM/CAM
10	Mellor-Yamada-Janjic	Thompson	Betts-Miller	CAM/CAM
11	Mellor-Yamada-Janjic	Thompson	Betts-Miller	Goodard/RRTM
12	Mellor-Yamada-Janjic	Lin et al.	Grell-Devenyi	Goddard/RRTM
13	Mellor-Yamada-Janjic	Lin et al.	Betts-Miller	GFDL/GFDL
14	Mellor-Yamada-Janjic	Goddard microphysics	Betts-Miller	GFDL/RRTM
15	Mellor-Yamada-Janjic	Lin et al.	Kain-Fritsch	Dudhia/GFDL
16	Mellor-Yamada-Janjic	Eta microphysics	Kain-Fritsch	Dudhia/CAM
17	Mellor-Yamada-Janjic	WRF-single mom (6)	Betts-Miller	Goddard/RRTM
18	Mellor-Yamada-Janjic	WRF-single mom (3)	Kain-Fritsch	Dudhia/RRTM
19	YSU (new MRF)	Eta microphysics	Kain-Fritsch	GFDL/GFDL
20	YSU (new MRF)	Lin et al.	Betts-Miller	GFDL/GFDL
21	YSU (new MRF)	Goddard microphysics	Betts-Miller	Goddard/RRTM
22	YSU (new MRF)	Lin et al.	Kain-Fritsch	CAM/CAM
23	YSU (new MRF)	Lin et al.	Betts-Miller	CAM/CAM
24	YSU (new MRF)	Goddard microphysics	Betts-Miller	Dudhia/RRTM
25	YSU (new MRF)	Thompson	Grell-Devenyi	GFDL/GFDL
26	YSU (new MRF)	Eta microphysics	Betts-Miller	Goddard/RRTM
27	YSU (new MRF)	Eta microphysics	Kain-Fritsch	CAM/CAM
28	YSU (new MRF)	Morrison	Kain-Fritsch	Goddard/RRTM
29	YSU (new MRF)	WRF-single mom(6)	Kain-Fritsch	Goddard/RRTM
30	YSU (new MRF)	WRF-single mom(3)	Betts-Miller	Dudhia/RRTM
31	Pleim-Xiu	Eta microphysics	Betts-Miller	Goddard/RRTM
32	Pleim-Xiu	Lin et al.	Betts-Miller	Goddard/RRTM
33	Pleim-Xiu	Eta microphysics	Grell-Devenyi	GFDL/GFDL
34	Pleim-Xiu	Goddard microphysics	Grell-Devenyi	Dudhia/RRTM

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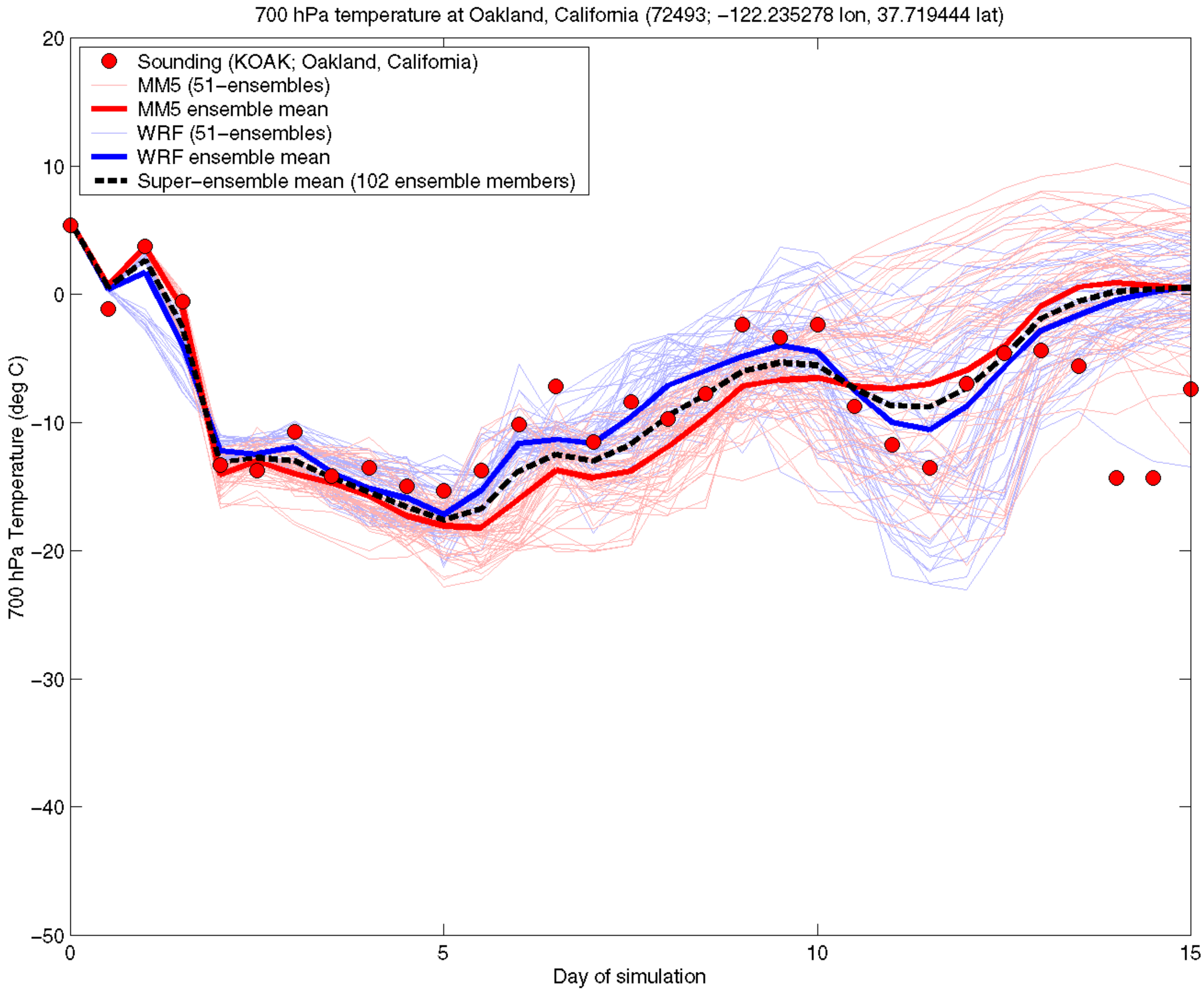
MM5 – 50 ensemble runs

WRF – 50 ensemble runs

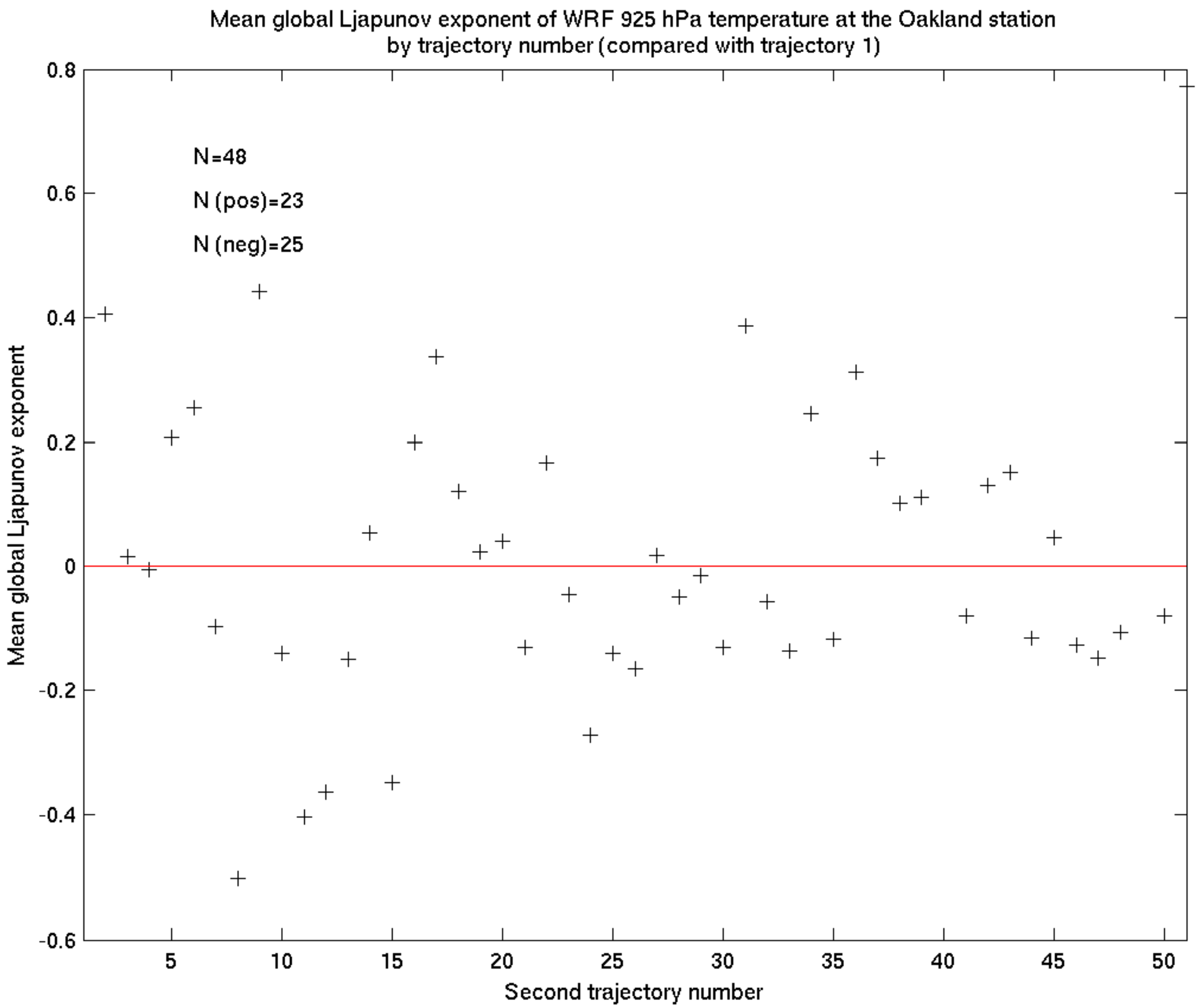
Physics parameterization options (PBL, cloud mic., Cu, Rad)

Super-ensemble – MM5 & WRF



50 ensemble trajectories for MM5 (red) and 50 ensemble trajectories for WRF (blue)

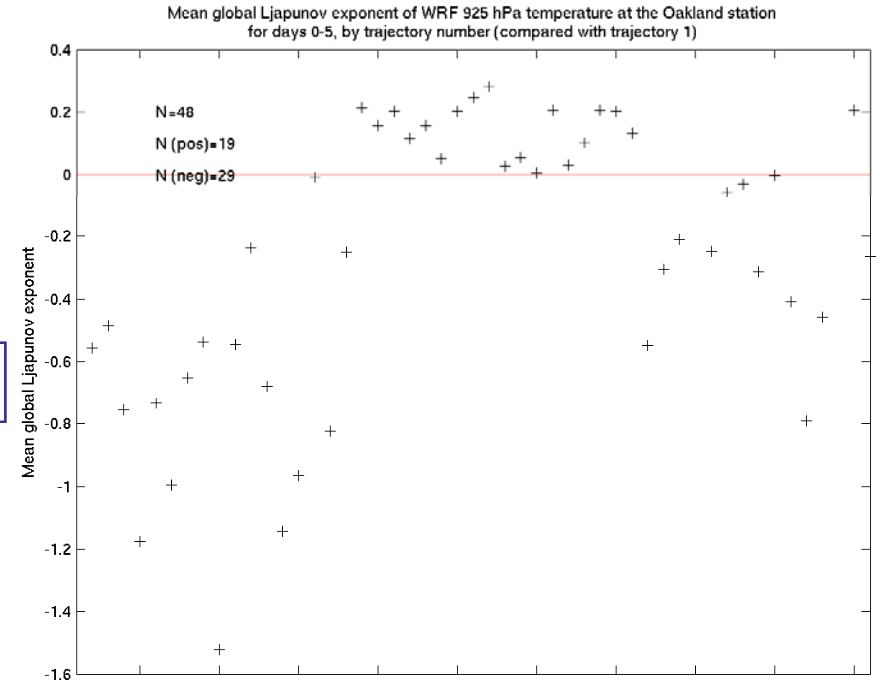
WRF – Global Ljapunov Exponent (LJE) – Mean LJE for each of 50 trajectories



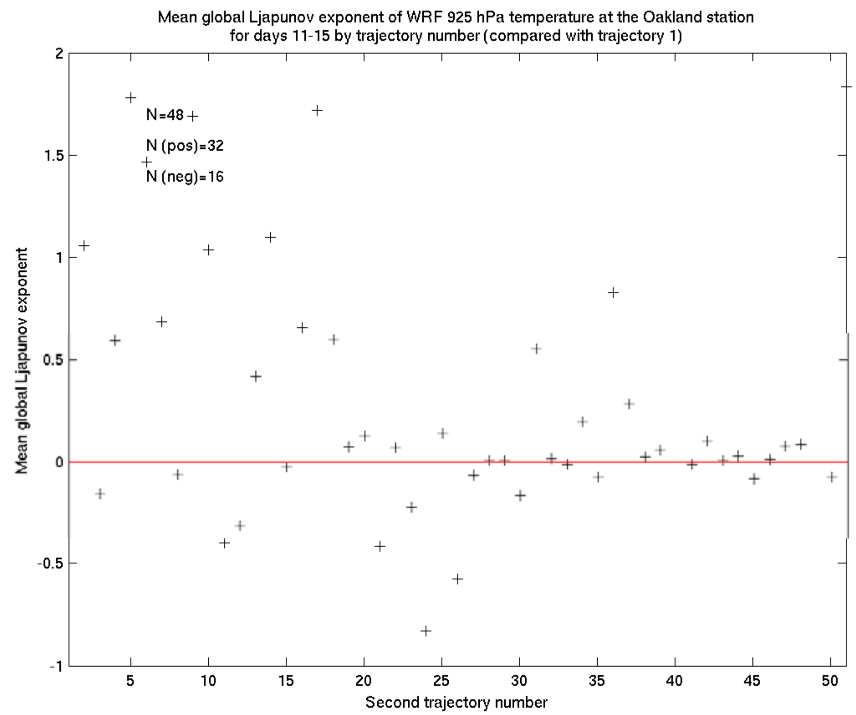
LJE < 0 -> Stable behavior
LJE > 0 -> Chaotic behavior

WRF – Global Ljapunov Exponent (LJE)
LJE for each of 50 trajectories
at 12-hr intervals

Forecast 0-5 days

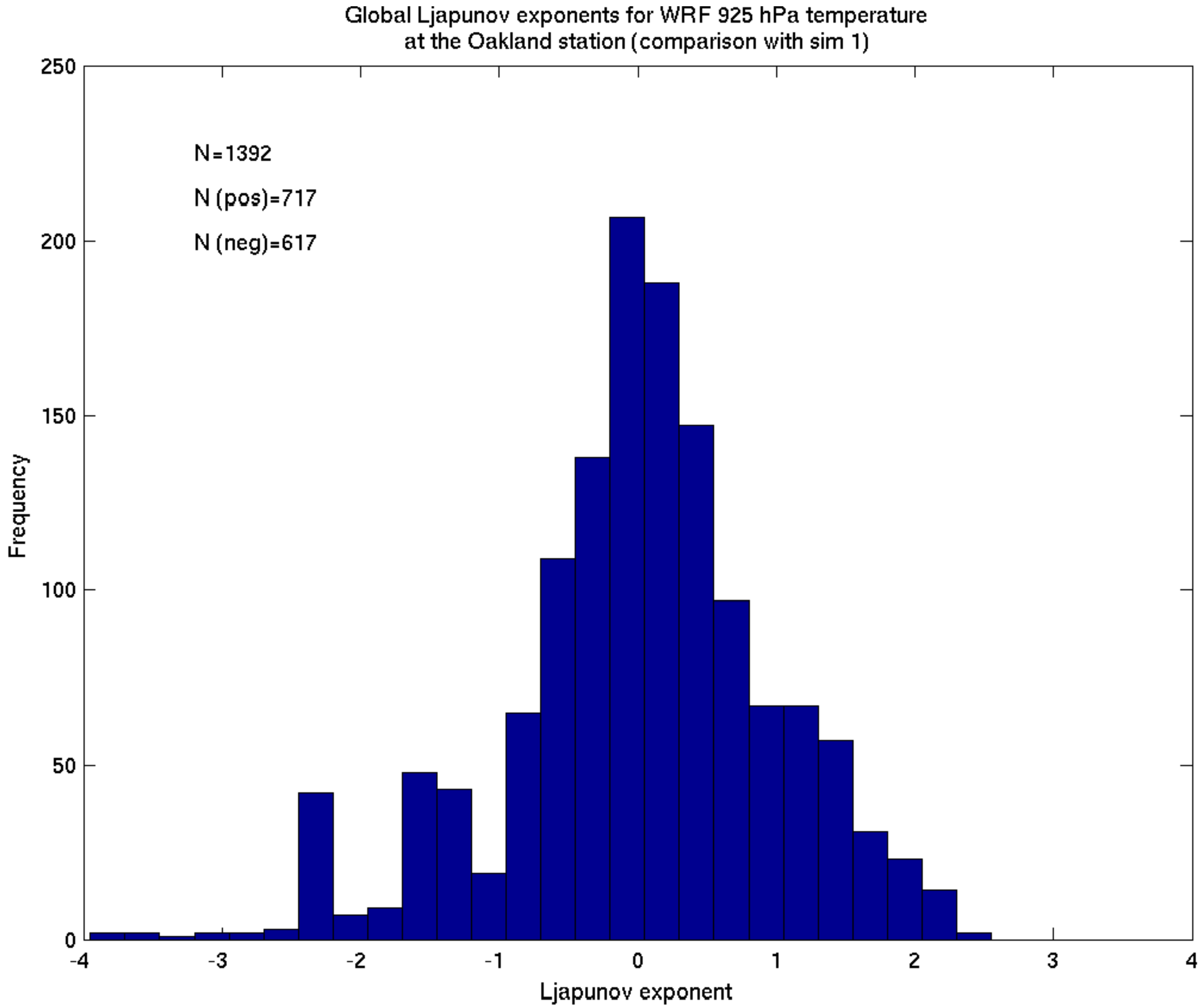


Forecast 10-15 days



LJE < 0 -> Stable behavior
LJE > 0 -> Chaotic behavior

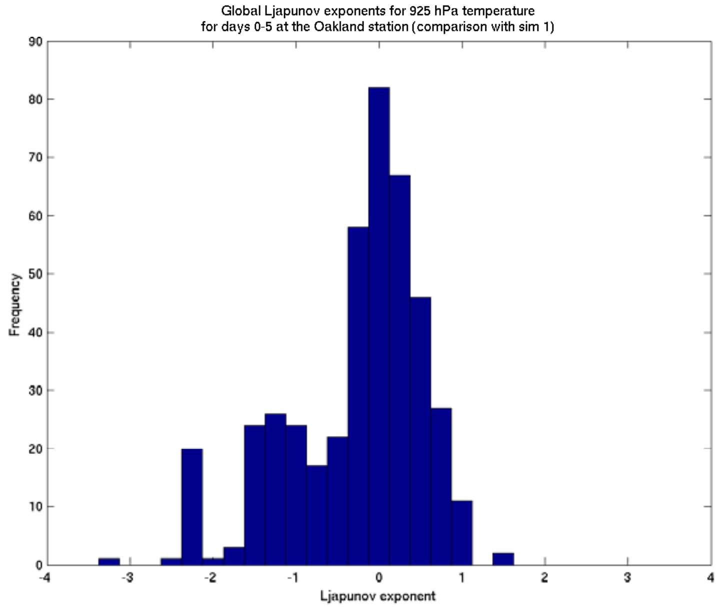
WRF – Local Ljapunov Exponent (LJE) – LJE in successive steps for each of 50 trajectories at 12-hr intervals



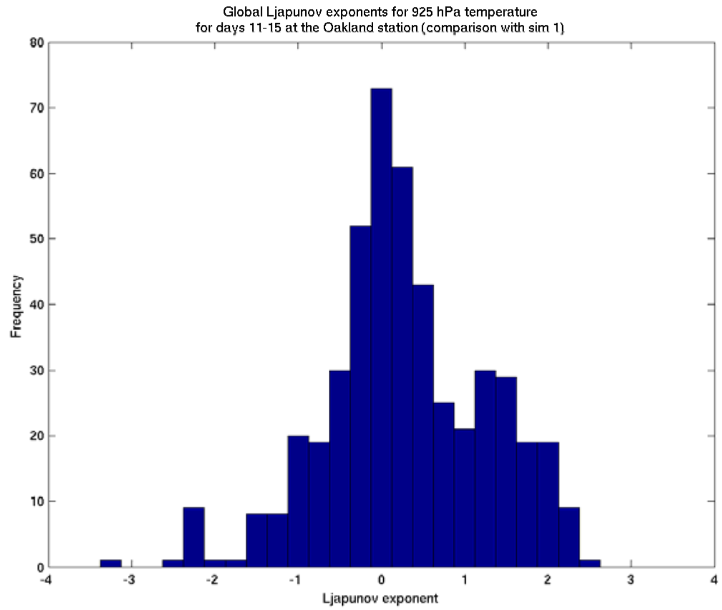
LJE < 0 -> Stable behavior
LJE > 0 -> Chaotic behavior

WRF – Local Ljapunov Exponent (LJE) – LJE in successive steps for each of 50 trajectories at 12-hr intervals

Forecast 0-5 days



Forecast 10-15 days



LJE < 0 -> Stable behavior
LJE > 0 -> Chaotic behavior

Chaos – What did we learn?

- Chaos is present in many simple and complex models and algorithms.
- Chaos is the aperiodic, long-term behavior of a bounded, deterministic system that exhibits sensitive dependence on initial conditions and algorithm parameters.
- In essence, the computational error of a parameter grows and readily exceeds the value of the iterated (predicted) parameter. Consequently, chaos represents a break in the predictability in dynamical systems.
- The roots of chaos are intrinsically linked to general number representation and the limitations of any computers in precision and algebraic operations.
- Positive Ljapunov exponent is one of the measures of chaotic behavior.

Summary characteristics of chaotic systems

- The governing equations of these systems are nonlinear
- The chaotic systems are aperiodic
- They have sensitive dependence on initial conditions
- They have sensitive dependence on boundary conditions
- They are governed by one or more control parameters, a small change in which can cause the chaos to appear or disappear