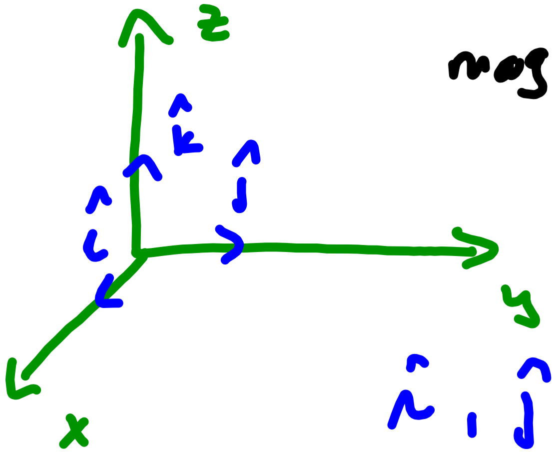
Governing equations

First: math review

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$



magnit.
- u zonal wind
- v meridional wind
- w vertical wind

$\hat{i}$, $\hat{j}$, $\hat{k}$ direction

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{d\vec{v}}{dt} = \frac{du}{dt}\hat{i} + \frac{dw}{dt}\hat{j} + \frac{dw}{dt}\hat{k} \quad \text{acceleration}$$

if we ignore derivatives of unit vectors (acceleration of Earth)

We need this for $\sum \vec{F} = m\frac{d\vec{v}}{dt}$

Partial derivatives

$$u = \frac{dx}{dt}$$

$$v = \frac{dy}{dt}$$

$$w = \frac{dz}{dt}$$

T temperature

$T(\underbrace{x, y, z,}_{\text{space}}\ t)$ ← time

$$dT = \left(\frac{\partial T}{\partial t}\right)_{x,y,z} dt + \left(\frac{\partial T}{\partial x}\right)_{y,z,t} dx + \left(\frac{\partial T}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial T}{\partial z}\right)_{x,y,t} dz$$

$\therefore \; /dt$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial T}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial T}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

$$\circledast \quad \frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

more math overview

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Del operator

apply to a scalar

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

p pressure

$$\nabla p = \underbrace{\nabla_H p}_{\text{horizontal pressure gradient}} + \frac{\partial p}{\partial z}\hat{k}$$

$$\nabla_H = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$$

points from low values to high values

$\vec{\nabla}$ apply to a vector

$$\boxed{\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

$\vec{V} \to u\hat{i} + v\hat{j} + w\hat{k}$

$\nabla \to \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\nabla \cdot \vec{V} = \nabla_H \cdot \vec{V} + \frac{\partial w}{\partial z}$$

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Application through mass continuity equation

* approximate/assume that the fluid is incompressible : $d\rho/dt = 0$ ρ density

Then mass continuity equation :

$$\nabla \cdot \vec{V} = 0$$

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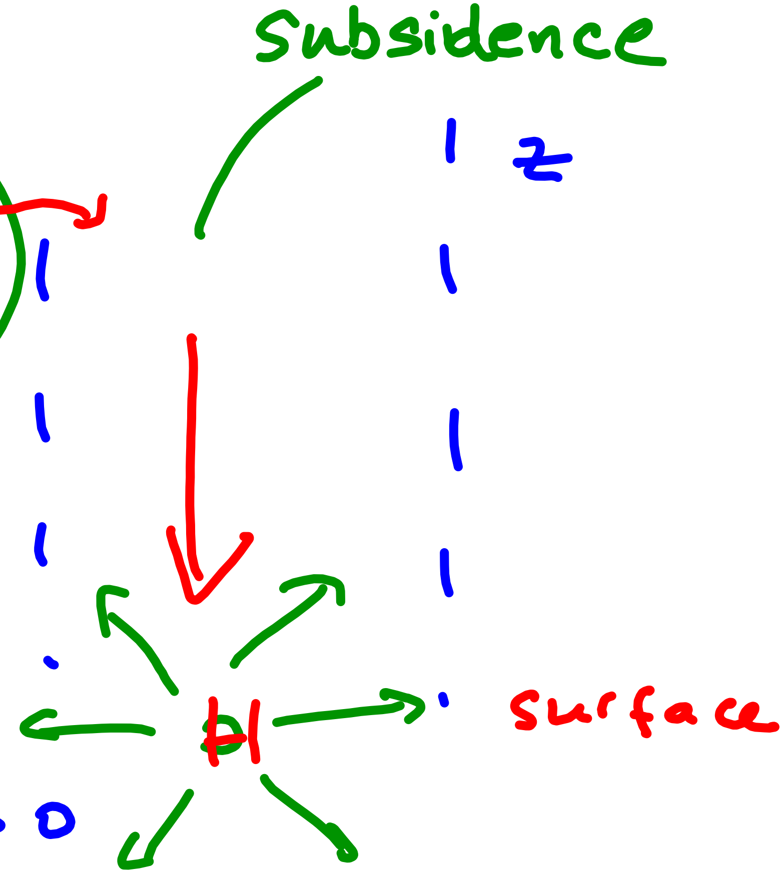
$$\underbrace{\nabla_H \cdot \vec{V}}_{\text{horiz.}} = -\frac{\partial w}{\partial z}$$

divergence term > 0

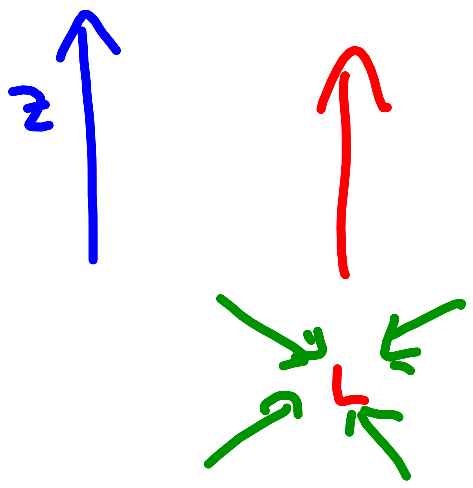
convergence < 0



$$\nabla_H \cdot \vec{V} > 0$$

$$\nabla_H \cdot \vec{V} = -\frac{\partial w}{\partial z} \rightarrow (+)$$

$$\nabla_H \cdot \vec{V} < 0$$



$\nabla_H \cdot \vec{V} < 0$

convergence in horiz

$$* \quad \frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T$$

advection

local
change of Temp. in time

Vorticity :

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

$\nabla \cdot \vec{V}$ divergence / convergence

Back to governing equations

Atmosphere is a fluid - its motion is governed by the laws of physics such as Newton's 2^{nd} law

$$m\vec{a} = \Sigma \vec{F}$$

$$m\frac{d\vec{v}}{dt} = \Sigma \vec{F}$$

valid in inertial (non-accelerating) frame of reference

not true for Earth

$$\vec{F}_{PGF} = -\frac{1}{\rho}\nabla p \qquad \rho \text{ density}$$

Pressure Gradient Force

How do we know that?

– infinitesimal parcel – drop m term

↑ mass

$$\frac{d\vec{v}}{dt} = \Sigma \vec{F}$$

Governing eqns

1 – 3

$$\frac{dT}{dt} = \text{something}$$

cons. of mass

$$\frac{d\rho}{dt} = \text{other something}$$

$\nabla \cdot \vec{v} = 0$

in

incompr. fluid

Continuity eqn

Define fields that all depend on x, y, z, t

$u(x,y,z,t)$ zonal wind (x)

$v(x,y,z,t)$ meridional wind (y)

$w(x,y,z,t)$ vertical wind (z)

$p(x,y,z,t)$ pressure

$T(x,y,z,t)$ Temperature

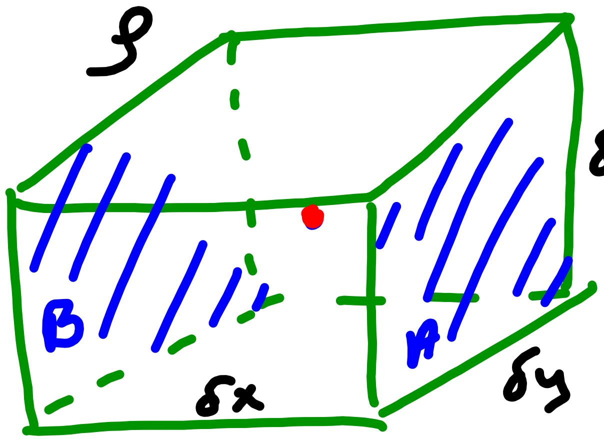
$q(x,y,z,t)$ humidity

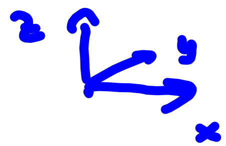
$\rho(x,y,z,t)$ density

Pressure Gradient Force

- look at infinitesimal fluid element

searching for pressure exerted on sides A & B by the atmosphere





put coordinate system in the middle

$V = \delta x \, \delta y \, \delta z$

$m = \rho \, \delta x \, \delta y \, \delta z$

$\downarrow$

$\rho \cdot V$

$$p_A = \frac{F_A}{A_A} \qquad p_B = \frac{F_B}{A_B}$$

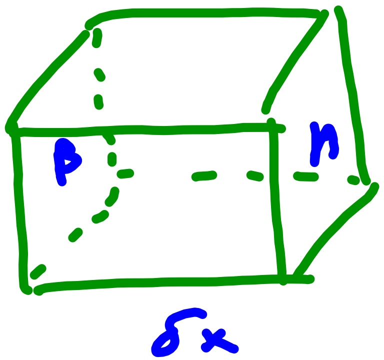
- pressure in the center p_0

Assume that pressure is continous =>
Taylor series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots$$

$$p_A = p_0 + p' \frac{\delta x}{2} = p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}$$

$$p_B = p_0 + p'\left(-\frac{\delta x}{2}\right) = p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}$$

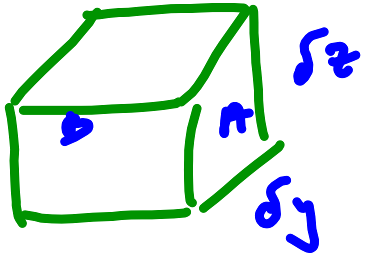


$$F_{A_x} = -\left(\cancel{p_0} + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$$

$$+\;F_{B_x} = +\left(\cancel{p_0} - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$$

F_{A_x} is pointing in negative direction

$F = p \cdot A$



F_{B_x} is positive!

m/ρ

$$F_x = F_{A_x} + F_{B_x}$$

$$F_x = -\frac{\partial p}{\partial x}\,\underbrace{\delta x\,\delta y\,\delta z}_{V}$$

$$F_x = \frac{m}{\rho}\left(-\frac{\partial p}{\partial x}\right)$$

$$\frac{F_x}{\cancel{m}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

neglecting m

$$\left.\begin{aligned} F_x &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ F_y &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ F_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \vec{F}_{PGF} = -\frac{1}{\rho}\nabla p$$

Fundamental forces

PGF $-\frac{1}{\rho}\nabla p$

Gravitational force $-g\hat{k}$

Frictional force $\vec{F}_{friction}$

Apparent forces

Coriolis

Centrifugal

Coriolis force

$$\boxed{f = 2\Omega \sin\phi}$$ Coriolis param.

$$-2\vec{\Omega} \times \vec{v} = -2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2\Omega\cos\phi & -2\Omega\sin\phi \\ u & v & w \end{vmatrix}$$

Ω rotation vector

ϕ latitude

x comp. $2\Omega\sin\phi\, v - 2\Omega\cos\phi\, w$

y $-2\Omega\sin\phi\, u$

z $2\Omega\cos\phi\, u$