Statistical Mechanics

PHYS 508 — Spring 2015

Problem Assignment # 8

due 04-24-15

1. **Chandrasekhar limit** (14 points)

   A white dwarf star can be modeled as a relativistic electron gas ($\epsilon_p = c\sqrt{m^2c^2 + p^2}$) in an ion background. Assume that the ions only provide charge neutrality and gravitational energy. Typical numbers for the electron density, temperature, and mass of a white dwarf are $n = p_F^3/3\pi^2\hbar^3 = 10^{30}$ cm^{-3}, $T = 10^7$ K, and $M = 10^{33}$ g, respectively.

   (a) Show that the electron gas is almost completely degenerate, so that we can put $T = 0$ to a good approximation.

   (b) Calculate the internal energy U and the pressure p of the electron gas in both the nonrelativistic limit ($p_F << mc$) (as a check), and the ultrarelativistic limit ($p_F >> mc$).
   hint: $n \propto 1/V$, so p_F is V-dependent!

   (c) Calculate the internal energy as a function of the star's radius R in both the nonrelativistic and the ultrarelativistic limits.
   hint: $n \approx \rho/\mu m_p$, with m_p the proton mass and μ the average molecular weight per electron, and $\rho = M/(4\pi/3)R^3$ the mass density.

   (d) Consider the total energy $E = U + E_G$, with $E_G = -GM^2/R$ the gravitational energy, as a function of R. Show that in the ultrarelativisitic limit the white dwarf is unstable for masses $M > M_c$, and determine the critical mass M_c. (What happens to stars whose core mass exceeds M_c when they run out of thermonuclear fuel?)

2. **Thermodynamics of the photon gas** (5 points)

   Consider a gas of photons as discussed in ch. 2, section 4.

   (a) Show that the free energy is given by

   $$F = -\frac{4\sigma}{3c} V T^4 \quad ,$$

   with the Stefan-Boltzmann constant σ given in terms of fundamental constants. Determine the numerical value of σ.

   (b) Find the entropy, the heat capacity, the internal energy, and the pressure, respectively, of the photon gas. Why is the equation of state different from the one derived in chapter 2, section 2.1?

3. **Black bodies** (9 points)

   Consider a sphere of gas with radius R whose central temperature is held by some mechanism (e.g., nuclear fusion) at a temperature T. Assume that you can treat the sphere as a photon gas in equilibrium.

   (a) Show that the luminosity of the system, i.e. the total power radiated, is given by

   $$L = 4\pi R^2 \sigma T^4 \quad ,$$

   with σ the Stefan-Boltzmann constant from problem 1.

hint: Consider the internal energy of a shell of thickness dR, the rate at which energy is transported of a distance R, integrate, and take into account that only the radiation directed in the outward direction escapes.

(b) The sun can be considered a black body with radius $R \approx 700,000$ km held at a temperature $T \approx 5,800$ K. Find the solar constant S, i.e., the radiative power incident ont he earth's atmosphere (in kW/m^2). Also find the wavelength at which the solar spectrum has its maximum intesity.

(c) Approximate the earth as a black body with no intrinsic energy source at a distance $R = 1$ a.u. from the sun. Calculate the temperature of the earth.

(d) An exploding nuclear bomb can be considered (For a short time) as a spherical black body with a radius $R \approx 10$ cm, and a temperature $T \approx 10^6$ K. Find the wavelength corresponding to the maximum fo the spectrum, and the radiative flux (in kW/m^2) at a distance $r = 1$ km from the center of the explosion, and compare your result with the solar constant.