

# Quantum critical behavior in disordered itinerant ferromagnets: Instability of the ferromagnetic phase

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An effective field theory is derived, which describes the quantum critical behavior of itinerant ferromagnets, as the transition is approached from the ferromagnetic phase. This complements a recent study of the critical behavior on the paramagnetic side of the phase transition, and investigates the role of the ferromagnetic Goldstone modes near criticality. We find that the Goldstone modes have no direct impact on the critical behavior, and that the critical exponents are the same as determined by combining results from the paramagnetic phase with scaling arguments.

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## I. INTRODUCTION

In two recent papers,<sup>1,2</sup> hereafter denoted by I and II, respectively, an effective field theory has been developed, which describes the quantum ferromagnetic transition in disordered itinerant electron systems, and the exact critical behavior was determined for all spatial dimensions  $d > 2$ . Soft modes in addition to the order parameter fluctuations, viz., diffusive particle-hole excitations (“diffusons”), were found to play a crucial role in the determination of the critical behavior. This theory studied the transition, as it is approached from the paramagnetic phase. The critical behavior in the ferromagnetic phase, in particular, the exponent  $\beta$ , was obtained from scaling arguments. This raises the following question: In the ferromagnetic phase of a Heisenberg magnet, the existence of spin waves, which are the Goldstone modes that result from the spontaneous breaking of the rotational symmetry in spin space, changes the soft-mode spectrum compared to the paramagnetic phase. This change is twofold; the Goldstone modes appear in addition to the soft modes present in the paramagnetic phase, while the transverse spin diffusons, which are soft in the paramagnetic phase, acquire a mass.<sup>3</sup> Given that I and II demonstrated the importance of all of the soft modes that couple to the order parameter fluctuations, whether or not they are directly related to them, will this change invalidate the scaling arguments based on a theory for the paramagnetic phase?

A related question follows from the fact that the Goldstone modes, contrary to the soft particle-hole excitations, exist at nonzero temperature as well as at  $T=0$ . Therefore, if there was an observable whose leading critical behavior depended on the Goldstone modes, then one would expect that observable to exhibit classical critical behavior on the ferromagnetic side of the phase transition, even at temperatures low enough for the system to otherwise exhibit quantum critical behavior. In such a case, the scaling arguments used in II might still be formally correct, but their region of validity would vanish.

In the present paper we investigate the validity of the scaling arguments used before. We will find that they were

correct, and the critical behavior obtained in II was indeed exact, even on the ferromagnetic side of the transition. To this end, we develop the ferromagnetic analog of the theory described in I and II. The general strategy is to derive a theory that expresses the system in terms of the order parameter fluctuations, and any other soft modes that couple to them. We then analyze the resulting theory to determine the critical behavior of the system, as it approaches the transition from the ferromagnetic phase.

This paper is organized as follows. In Sec. II we present a simple generalized mean-field theory that allows an explicit description of the ferromagnetic phase based on the theory developed in I. Its results agree with those obtained in II, except that it does not reproduce the logarithmic corrections to power-law scaling found in the latter paper. In Sec. III we present and motivate an effective action for itinerant quantum ferromagnets that describes the instability of the ferromagnetic phase. This action is analyzed by means of renormalization-group methods in Sec. IV. It is shown that the generalized mean-field theory yields indeed the exact critical behavior except for logarithms. In Sec. V we summarize and discuss our results. Several technical points related to a derivation of the effective action from a microscopic model are relegated to an appendix.

## II. GENERALIZED MEAN-FIELD THEORY

In this section we present a simple way to obtain an approximate theory for the ferromagnetic phase from the formalism of I. It will turn out that this simple theory produces essentially the correct result, and the technically more involved development in the later sections will serve to show this.

### A. A simplified effective action

We recall that the effective action  $\mathcal{A}_{\text{eff}}$  in I took the form of a Landau-Ginzburg-Wilson (LGW) functional of the order parameter field  $\mathbf{M}$ , a generalized nonlinear  $\sigma$ -model for the soft fermionic degrees of freedom  $q$ , and a term that couples these fields,

$$\mathcal{A}_{\text{eff}} = \mathcal{A}_{\text{LGW}}[\mathbf{M}] + \mathcal{A}_{\text{NL}\sigma\text{M}}[q] + \mathcal{A}_{\text{c}}[\mathbf{M}, q]. \quad (2.1)$$

We simplify the expressions given in I for these terms by making a mean-field approximation for the order parameter field, i.e., we replace the fluctuating vector field  $\mathbf{M}$  by its expectation value, which is characterized by a number  $m$ ,

$$\begin{aligned} \mathbf{M}^\alpha(\mathbf{x}, \tau) &\approx \langle \mathbf{M}^\alpha(\mathbf{x}, \tau) \rangle = m \hat{z}, \\ {}^i M_n^\alpha(\mathbf{k}) &\approx \delta_{i3} \delta_{n0} \sqrt{V/T} m. \end{aligned} \quad (2.2)$$

Here  $(\mathbf{x}, \tau)$  denotes the dependence of  $\mathbf{M}$  on real space and imaginary time,  $\mathbf{k}$  and  $n$  are the wave vector and the Matsubara frequency index, respectively, of the Fourier transform of  $\mathbf{M}$ ,  $\alpha$  is a replica index,  $V$  denotes the system volume, and we have assumed that the magnetization orders in the  $z$  or  $3$  direction. With this approximation, the LGW part of the action becomes simply a Landau theory,

$$\mathcal{A}_{\text{LGW}} \approx \frac{V}{T} \left[ \frac{t}{2} m^2 + \frac{u}{4} m^4 + O(m^6) \right]. \quad (2.3)$$

The coupling term in the action [Eq. (2.25c) in I] simplifies to

$$\begin{aligned} \mathcal{A}_{\text{c}} &\approx m \sqrt{\pi K_t} \int d\mathbf{x} \sum_{\alpha} \sum_{r=0,3} (\sqrt{-1})^r \\ &\times \sum_m \text{tr}[(\tau_r \otimes s_3) \hat{Q}_{mm}^{\alpha,\alpha}(\mathbf{x})]. \end{aligned} \quad (2.4)$$

Here  $K_t$  is the spin-triplet interaction amplitude of the underlying fermionic model; the spin-triplet interaction term has been decoupled by introducing the Hubbard-Stratonovich field  $\mathbf{M}$ .  $\hat{Q}$  is a composite field<sup>4</sup> that originates from the fermion variables,<sup>16</sup>

$$Q_{12} \equiv \frac{i}{2} \begin{pmatrix} -\psi_{1\uparrow} \bar{\psi}_{2\uparrow} & -\psi_{1\uparrow} \bar{\psi}_{2\downarrow} & -\psi_{1\uparrow} \psi_{2\downarrow} & \psi_{1\uparrow} \psi_{2\uparrow} \\ -\psi_{1\downarrow} \bar{\psi}_{2\uparrow} & -\psi_{1\downarrow} \bar{\psi}_{2\downarrow} & -\psi_{1\downarrow} \psi_{2\downarrow} & \psi_{1\downarrow} \psi_{2\uparrow} \\ \bar{\psi}_{1\downarrow} \bar{\psi}_{2\uparrow} & \bar{\psi}_{1\downarrow} \bar{\psi}_{2\downarrow} & \bar{\psi}_{1\downarrow} \psi_{2\downarrow} & -\bar{\psi}_{1\downarrow} \psi_{2\uparrow} \\ -\bar{\psi}_{1\uparrow} \bar{\psi}_{2\uparrow} & -\bar{\psi}_{1\uparrow} \bar{\psi}_{2\downarrow} & -\bar{\psi}_{1\uparrow} \psi_{2\downarrow} & \bar{\psi}_{1\uparrow} \psi_{2\uparrow} \end{pmatrix}. \quad (2.5a)$$

Here  $\psi$  and  $\bar{\psi}$  are the fermionic, i.e., Grassmann-valued fields that provide the basic description of the electrons,<sup>5</sup> and all fields are understood to be taken at position  $\mathbf{x}$ . The indices 1, 2, etc. denote the dependence of the Grassmann fields on fermionic Matsubara frequencies  $\omega_{n_1} = 2\pi T(n_1 + 1/2)$  and replica indices  $\alpha_1$ , etc., and the arrows denote the spin projection. It is convenient to expand the  $4 \times 4$  matrix in Eq. (2.5a) in a spin-quaternion basis,<sup>3</sup>

$$Q_{12} = \sum_{r,i=0}^3 (\tau_r \otimes s_i) {}^i_r Q_{12}, \quad (2.5b)$$

with  $\tau_0 = s_0 = \mathbb{1}_2$ , the  $2 \times 2$  unit matrix, and  $\tau_j = -s_j = -i\sigma_j$  ( $j=1,2,3$ ), with  $\sigma_{1,2,3}$  the Pauli matrices. In this basis,  $i=0$  and  $i=1,2,3$  describe the spin-singlet and spin-triplet degrees of freedom, respectively. The  $r=0,3$  compo-

nents correspond to the particle-hole channel (i.e., products of  $\bar{\psi}\psi$ ), while  $r=1,2$  describe the particle-particle channel (i.e., products of  $\bar{\psi}\bar{\psi}$  or  $\psi\psi$ ). For our purposes the latter are not important, and we therefore drop  $r=1,2$  from the spin-quaternion basis, as was done in I and II. In terms of the remaining fields, the spin density can be expressed as

$$\begin{aligned} n_s^i(\mathbf{x}, i\Omega_n) &= \sqrt{T} \sum_m \sum_{ab} \bar{\psi}_{m,a}(\mathbf{x}) \sigma_{ab}^i \psi_{m+n,b}(\mathbf{x}) \\ &= \sqrt{T} \sum_m \sum_{r=0,3} (\sqrt{-1})^r \text{tr}[(\tau_r \otimes s_i) Q_{m,m+n}(\mathbf{x})], \end{aligned} \quad (2.5c)$$

and the magnetization couples linearly to the spin density, cf. Eq. (2.4). The difference between  $Q$  and  $\hat{Q}$  is that the former contains all fermionic degrees of freedom, while the latter contains the soft fermion excitations only. The latter can be parametrized in terms of a matrix field  $q$ ,<sup>1,4</sup>

$$\hat{Q} = \begin{pmatrix} \sqrt{1-qq^\dagger} & q \\ q^\dagger & -\sqrt{1-q^\dagger q} \end{pmatrix}. \quad (2.5d)$$

The elements of this block matrix, clockwise from upper left, denote those elements of  $\hat{Q}$  that have frequency labels  $n_1, n_2 > 0$ ;  $n_1 > 0, n_2 < 0$ ;  $n_1, n_2 < 0$ ; and  $n_1 < 0, n_2 > 0$ , respectively.

The nonlinear  $\sigma$  model can be expressed in terms of  $\hat{Q}$  as<sup>1,3,4,6,7</sup>

$$\begin{aligned} \mathcal{A}_{\text{NL}\sigma\text{M}} &= \frac{-1}{2G} \int d\mathbf{x} \text{tr}(\nabla \hat{Q}(\mathbf{x}))^2 + 2H \int d\mathbf{x} \text{tr}(\Omega \hat{Q}(\mathbf{x})) \\ &+ \mathcal{A}_{\text{int}}^{(s)}[\hat{Q}], \end{aligned} \quad (2.6a)$$

with  $\Omega$  a frequency matrix with elements

$$\Omega_{12} = (\tau_0 \otimes s_0) \delta_{12} \omega_{n_1}. \quad (2.6b)$$

The coupling constants  $G$  and  $H$  are proportional to the inverse conductivity and the specific heat coefficient, respectively.  $\mathcal{A}_{\text{int}}^{(s)}$  is the spin-singlet interaction,

$$\begin{aligned} \mathcal{A}_{\text{int}}^{(s)} &= \frac{\pi T K_s}{4} \int d\mathbf{x} \sum_{r=0,3} (-1)^r \sum_{n_1, n_2, m} \sum_{\alpha} \{ \text{tr}[(\tau_r \otimes s_0) \\ &\times \hat{Q}_{n_1, n_1+m}^{\alpha\alpha}(\mathbf{x})] \} \{ \text{tr}[(\tau_r \otimes s_0) \hat{Q}_{n_2+m, n_2}^{\alpha\alpha}(\mathbf{x})] \}, \end{aligned} \quad (2.6c)$$

with a coupling constant  $K_s$ . At the level of the bare effective action,  $\mathcal{A}_{\text{NL}\sigma\text{M}}$  does not contain any spin-triplet interaction term, since the Hubbard-Stratonovich decoupling turned the latter into the terms containing  $\mathbf{M}$ . For later reference we note, however, that under renormalization a spin-triplet interaction will be generated as long as  $K_s \neq 0$ . We therefore add to the action a term

$$\mathcal{A}_{\text{int}}^{(t)} = \frac{\pi T \tilde{K}_t}{4} \int d\mathbf{x} \sum_{r=0,3} (-1)^r \sum_{n_1, n_2, m} \sum_{\alpha} \sum_{i=1}^3 \{ \text{tr}[(\tau_r \otimes s_i) \times \hat{Q}_{n_1, n_1+m}^{\alpha\alpha}(\mathbf{x})] \} \{ \text{tr}[(\tau_r \otimes s_i) \hat{Q}_{n_2+m, n_2}^{\alpha\alpha}(\mathbf{x})] \}, \quad (2.6d)$$

with  $\tilde{K}_t$  the spin-triplet interaction constant generated by renormalization.

### B. Generalized Landau theory

We now can obtain a generalized mean-field or Landau free energy density  $f(m)$  that takes into account the effects of the fermionic soft modes by formally integrating out  $q$ ,

$$f(m) = \lim_{N \rightarrow 0} \frac{-T}{VN} \ln \int D[q] e^{\mathcal{A}_{\text{eff}}}, \quad (2.7)$$

with  $N$  the number of replicas. Since  $\mathcal{A}_{\text{eff}}$  contains  $q$  to all orders, the integral in Eq. (2.7) can be performed only in terms of a loop expansion. The lowest-order term is obtained by expanding  $\hat{Q}$  to second order in  $q$ . The integration can then be done exactly. The Gaussian propagators that are needed for this calculation have been given in I (the generalization to the case  $\tilde{K}_t \neq 0$  is straightforward, and can also be found in Ref. 8), and we therefore only give the result,

$$f(m) \approx f(m=0) + \frac{t}{2} m^2 + \frac{u}{4} m^4 + \frac{2}{V} \times \sum_k T \sum_{n=1}^{\infty} \ln \left( \frac{[k^2 + G(H + \tilde{K}_t)\Omega_n]^2 + \pi G^2 K_t m^2}{(k^2 + GH\Omega_n)^2 + \pi G^2 K_t m^2} \right). \quad (2.8)$$

By minimizing  $f(m)$  with respect to  $m$ , we obtain a generalized mean-field equation of state. If we introduce suitable units, drop nonessential constants, and add an external magnetic field  $h$ , the latter can be written

$$h = tm + um^3 - m \frac{\text{const}}{V} \sum_k T \sum_{n=1}^{\infty} \frac{(k^2 + \Omega_n)\Omega_n}{[(k^2 + \Omega_n)^2 + m^2]^2}, \quad (2.9a)$$

with  $\text{const} > 0$ . An inspection of the integral shows that the leading term for small  $m$  is finite for  $d > 2$ , and that the leading nonanalytic  $m$  dependence is given by  $m^{(d-2)/2}$ . The generalized mean-field equation of state thus reads

$$h = tm + v m^{d/2} + um^3, \quad (2.9b)$$

with  $v > 0$ .

We see that for  $2 < d < 6$  the nonanalytic term dominates over the  $m^3$  contribution. The exponents  $\beta$  and  $\delta$ , defined by  $m(h=0) \propto (-t)^\beta$  and  $m(t=0) \propto h^{1/\delta}$ , respectively, for  $2 < d < 6$  in this approximation are

$$\beta = 2/(d-2), \quad \delta = d/2. \quad (2.10)$$

These values agree with those found in II, apart from logarithmic corrections to scaling that the generalized mean-field theory misses. Equation (2.9a) is also very similar to the

effective equation of state that was obtained in Ref. 9, and it has the same qualitative properties. Notice, however, that the derivation of Eq. (2.9a) did not involve any divergent integrals, while the equivalent result in Ref. 9 was obtained by resumming an infinite series of divergent terms. We will discuss the relation between these theories in more detail in Sec. V.

While the above procedure provides a fast and simple way to incorporate soft-mode effects into the equation of state, and the agreement with prior results is encouraging, our mean-field approximation is of course uncontrolled. Furthermore, since it neglects the order parameter fluctuations, it does not shed any light on the role played by the Goldstone modes. In the following two sections we will therefore construct an effective field theory for the ferromagnetic phase, and perform a renormalization-group analysis to study the critical behavior. We will find that the results of the generalized mean-field theory are indeed exact apart from logarithmic corrections to scaling.

### III. EFFECTIVE FIELD THEORY FOR FERROMAGNETS

In this section we present an effective field theory for the ferromagnetic phase in analogy to that for the paramagnetic phase given in I. The structure of such a theory can be deduced without a calculation by combining various existing results for disordered itinerant ferromagnets. In the interest of brevity and clarity, we pursue this route. We have also checked the result by means of a derivation from a microscopic model, using the methods of I and Ref. 10. We summarize the salient points of this derivation in Appendix A; a complete account has been given in Ref. 11.

In I it was shown how to derive an effective long-wavelength and low-frequency field theory for a disordered metal that approaches a ferromagnetic instability. We now give the corresponding action for a ferromagnetic metal, which we will motivate below. We denote the fluctuations of the longitudinal part of the order parameter field  $\mathbf{M}$  about its expectation value  $m$  by  $\delta M_\ell$ , and the transverse part by  $M_t$ . Similarly, we denote the spin-singlet and the longitudinal spin-triplet components of the fermionic field  $q$  by  $q_\ell$ , and the transverse spin-triplet components by  $q_t$ . The action then has the general form

$$\mathcal{A}_{\text{eff}}[\mathbf{M}, q] = \mathcal{A}_\ell^G[\delta M_\ell, q_\ell] + \mathcal{A}_t^G[M_t, q_t] + \Delta \mathcal{A}[\delta M_\ell, M_t, q_\ell, q_t]. \quad (3.1a)$$

Here the superscript G denotes the action at the Gaussian level, where the longitudinal and transverse fields decouple, separated into longitudinal and transverse contributions.  $\Delta \mathcal{A}$  represents contributions of higher than bilinear order in the fields. We now specify the various terms in this action in the same schematic notation as is in I [cf. Eq. (3.8) of I], suppressing everything that is not necessary for power counting, and considering all fields as functions of real space position  $\mathbf{x}$  and Matsubara frequencies. The Gaussian contributions are then given by

$$\begin{aligned} \mathcal{A}_\ell^G = & \int d\mathbf{x} [ |t| + a_{\ell,d-2} \partial_x^{d-2} + a_{\ell,2} \partial_x^2 ] (\delta M_\ell)^2 \\ & + \int d\mathbf{x} \left[ \frac{1}{G_\ell} \partial_x^2 + H_\ell \Omega + TK \right] q_\ell^2 + c_{\ell,1} \sqrt{T} \int d\mathbf{x} \delta M_\ell q_\ell, \end{aligned} \quad (3.1b)$$

$$\begin{aligned} \mathcal{A}_t^G = & \int d\mathbf{x} [ a_{t,d-2} m^{(d-4)/2} \partial_x^{d-2} + a_{t,2} \partial_x^2 ] M_t^2 \\ & + \int d\mathbf{x} \left[ \frac{1}{G_t} \partial_x^2 + H_t \Omega + Km \right] q_t^2 + c_{t,1} \sqrt{T} \int d\mathbf{x} M_t q_t. \end{aligned} \quad (3.1c)$$

Here  $K$  is a generic coupling constant that is equal to  $K_s$  in the spin-singlet channel,  $\tilde{K}_t$  in the longitudinal spin-triplet channel, and  $K_t$  in the transverse spin-triplet channels. We now motivate this form of the Gaussian effective action.

First, the magnetic order breaks the rotational symmetry in spin space and forces us to distinguish between the longitudinal and transverse components of all fields. The soft-mode structures in the longitudinal channels are similar to those in a paramagnet. The spin-singlet and the longitudinal spin-triplet fermionic fields are known to be unaffected by the presence of a nonzero magnetization (see, e.g., Ref. 8); the vertex for  $q_\ell$  thus has the same form as in I. The same is true for the longitudinal order parameter fluctuations. The distance from criticality,  $t$ , is actually magnetization dependent, but this is irrelevant for power counting purposes. The coupling between  $\delta M_\ell$  and  $q_\ell$  is also unaffected. Notice that this coupling provides the leading frequency dependence of the  $\delta M_\ell$  propagator, viz., a term proportional to  $\Omega/k^2$ , with  $\mathbf{k}$  a wave vector. We will refer to this propagator as the longitudinal magnon. That is, in the Fourier space and for small wave numbers and frequencies the longitudinal magnon propagator has the form

$$\begin{aligned} \langle \delta M_\ell(\mathbf{k}) \delta M_\ell(-\mathbf{k}) \rangle \\ = \frac{1}{|t| + a_{\ell,d-2} |\mathbf{k}|^{d-2} + a_{\ell,2} \mathbf{k}^2 + G_\ell c_{\ell,1} \Omega / \mathbf{k}^2}. \end{aligned} \quad (3.2)$$

For this reason we have left the less leading direct frequency dependence ( $\propto \Omega$ ) out of the  $\delta M_\ell$  vertex.

Equations (3.1) and (3.2) demonstrate an important point that was discussed in detail in I and II; namely, there is more than one time scale in the problem, and hence more than one dynamical exponent  $z$ .<sup>17</sup> The diffusive dynamics of the fermions are described by an exponent  $z=2$ , while the critical  $z$ , which describes the critical dynamics of the order parameter field, has a different value. This will be important in what follows.

The soft-mode structure in the transverse spin-triplet channel is qualitatively different compared to the paramagnetic phase. The transverse fermion fields acquire a mass proportional to  $m$ ; see, Ref. 8 or Eq. (2.4). The order parameter vertex, on the other hand, is massless due to the presence

of ferromagnetic Goldstone modes. The structure of the fermionic vertex suggests that the magnetization can scale like a gradient squared. For the transverse magnetization vertex, we expect terms that represent the classic dispersion relation,<sup>12</sup>

$$\Omega \sim m \mathbf{k}^2, \quad (3.3a)$$

as well as terms that reflect the nonanalytic magnetization dependence of the magnon effective mass,<sup>13</sup>

$$\Omega \sim m^{(d-2)/2} \mathbf{k}^2. \quad (3.3b)$$

Again the leading dynamics come from the coupling to the fermions, which here produces a term  $\Omega/m$  due to the mass in the  $q_t$  vertex. Equations (3.3) therefore show that we must include a simple gradient-squared term, as well as a gradient-squared term with a coefficient proportional to  $m^{(d-4)/2}$  in the  $M_t$  vertex in order to accurately represent the Goldstone modes. The latter reflects the same nonanalyticity as the  $\partial_x^{d-2}$  term in the longitudinal channel (recall that  $m \sim \partial_x^2$ ). The transverse magnon or the Goldstone mode thus has the form

$$\langle M_t(\mathbf{k}) M_t(-\mathbf{k}) \rangle = \frac{1}{a_{t,d-2} m^{(d-4)/2} \mathbf{k}^2 + a_{t,2} \mathbf{k}^2 + c_{t,1} \Omega / m}. \quad (3.4)$$

Beyond the Gaussian order, the transverse and singlet/longitudinal spin channels are coupled. By analyzing the spin structure of the action as given in I, we find, schematically,

$$\begin{aligned} \Delta \mathcal{A} = & c_{2,1} \sqrt{T} \int d\mathbf{x} \delta M_\ell q_\ell^2 + c_{2,2} \sqrt{T} \int d\mathbf{x} \delta M_\ell q_t^2 \\ & + c_{2,3} \sqrt{T} \int d\mathbf{x} \delta M_t q_\ell q_t + u_1 T \int d\mathbf{x} (\delta M_\ell)^4 \\ & + u_2 T \int d\mathbf{x} (\delta M_\ell)^2 (\delta M_t)^2 + u_3 T \int d\mathbf{x} (\delta M_t)^4 \\ & + \int d\mathbf{x} \left[ \frac{1}{G_{4,1}} \partial_x^2 + H_{4,1} \Omega \right] q_\ell^4 + \int d\mathbf{x} \left[ \frac{1}{G_{4,2}} \partial_x^2 + H_{4,2} \Omega \right. \\ & \left. + m \right] q_t^2 q_t^2 + \int d\mathbf{x} \left[ \frac{1}{G_{4,3}} \partial_x^2 + H_{4,3} \Omega + m \right] q_t^4. \end{aligned} \quad (3.5)$$

The first and the third class of terms are the  $M q^2$  and  $q^4$  vertices, respectively, that were found to be important in describing the critical behavior as the transition is approached from the paramagnetic phase. The  $M q^2$  vertices would generate the nonanalyticities in the  $M^2$  vertices if they had not been included in the bare action, and together with the  $q^4$  vertices they give rise to log-lognormal corrections to power laws. We also include an  $M^4$  vertex since its coupling constant  $u$  is a dangerously irrelevant operator with respect to the magnetization.<sup>14</sup>

#### IV. RENORMALIZATION-GROUP ANALYSIS

We now conduct a power-counting analysis of the action given in Sec. III. The purpose of this exercise is to determine

the role of the Goldstone modes in the ferromagnetic transition, and thus compare the critical behavior on the ferromagnetic side of the transition with the results from the paramagnetic theory.

For the following analysis, we assign lengths  $L$  a scale dimension of  $[L] = -1$ . Rescaling lengths by a factor  $b$  under a renormalization-group transformation will change all other quantities according to  $A \rightarrow b^{[A]}A$ , with  $[A]$  the scale dimension of  $A$ . In particular, the temperature  $T$  and frequency  $\Omega$  have a scale dimension  $[T] = [\Omega] = z$ . The scale dimensions of the fields we characterize as usual by means of exponents  $\eta$ ,<sup>14</sup>

$$[\delta M_\ell] = (d-2 + \eta_\ell)/2, \quad (4.1a)$$

$$[M_t] = (d-2 + \eta_t)/2, \quad (4.1b)$$

$$[q_{\ell,t}] = (d-2 + \eta'_{\ell,t})/2. \quad (4.1c)$$

As in Sec. III we consider all fields functions of frequency and real space position. Throughout this section we use Ma's technique of choosing scale dimensions, and then checking self-consistently whether this choice leads to a physical fixed point.<sup>14</sup>

#### A. Stable fixed point

Before considering the critical fixed point that describes the ferromagnetic transition, it is illustrative to discuss the stable fixed point that corresponds to the system deep in the ferromagnetic phase. We start with the longitudinal degrees of freedom.

The natural choice for  $t$  is to be marginal at the stable fixed point.<sup>14</sup> This choice determines the scale dimension of  $\delta M_\ell$ , so we have

$$[t] = 0, \quad \eta_\ell = 2. \quad (4.2a)$$

$q_\ell$  we expect to be unaffected by the magnetization, so we choose its scale dimension to be consistent with diffusive behavior,

$$\eta'_\ell = 0. \quad (4.2b)$$

By the same argument, the dynamics of  $q_\ell$ , i.e., the scale dimensions of  $\Omega$  and  $T$  in the  $q_\ell$  vertex, should be governed by a dynamical exponent  $z=2$ . We further choose the coupling  $c_{\ell,1}$  to be marginal, which implies that the dynamics of  $\delta M_\ell$ , as represented by the factor  $\sqrt{T}$  in the last term in Eq. (3.1b), are also governed by  $z=2$ . We thus have

$$[c_{\ell,1}] = 0, \quad z_{\text{diff}} = z_\ell = 2. \quad (4.2c)$$

From Eq. (3.1b) we then find that  $G_\ell$ ,  $H_\ell$ , and  $K$  are all marginal,

$$[G_\ell] = [H_\ell] = [K] = 0, \quad (4.2d)$$

while  $a_{\ell,d-2}$  and  $a_{\ell,2}$  are irrelevant with scale dimensions  $[a_{\ell,d-2}] = -(d-2)$  and  $[a_{\ell,2}] = -2$ , respectively.

We now turn to the transverse degrees of freedom.  $q_t$  is expected to be a massive fluctuation deep in the ferromagnetic phase, which means we choose

$$\eta'_t = 2. \quad (4.3a)$$

This, together with the marginality of  $K$ , implies that  $m$  is marginal, as one would expect on physical grounds, and so is  $h$ , since  $[h] = [m] + [t]$ , see the equation of state, Eqs. (2.9),

$$[m] = [h] = 0, \quad (4.3b)$$

while  $1/G_t$  and  $H_t$  are irrelevant with scale dimensions  $[1/G_t] = [H_t] = -2$ . We further choose  $a_{t,2}$  and  $c_{t,1}$  to be marginal. This renders  $a_{t,d-2}$  marginal as well,

$$[a_{t,2}] = [a_{t,d-2}] = [c_{t,1}] = 0, \quad (4.3c)$$

and it leads to

$$\eta_t = 0, \quad (4.3d)$$

and to a transverse time scale governed by a dynamical exponent

$$z_t = 2. \quad (4.3e)$$

We see that at the stable fixed point the various time scales coincide, and there is only one dynamical exponent,  $z=2$ . This reflects the fact that, deep in the ferromagnetic phase, the diffusive electron dynamics, the frequency dependence of the longitudinal magnon propagator, Eq. (3.2), and the dispersion of the Goldstone modes, Eqs. (3.3), all lead to the same scaling of the frequency with the wave number. From Eq. (3.5) it is then easily seen that all nonGaussian terms in the action are irrelevant with respect to the stable fixed point, with the least irrelevant operators having scale dimensions equal to  $-(d-2)/2$ . Temperature and frequency are relevant operators, of course. This establishes the description of the ferromagnetic phase within our formalism.

#### B. Critical fixed point

We now turn to the critical fixed point that describes the ferromagnet-to-paramagnet transition. Since this is a symmetry-restoring transition, any sensible candidate for the critical fixed point must have the feature that the longitudinal and transverse fields have the same scale dimensions. We demand that the fermion fields be diffusive,

$$\eta'_\ell = \eta'_t = 0, \quad (4.4a)$$

with a time scale given by

$$z_{\text{diff}} = 2. \quad (4.4b)$$

Equations (3.1b) and (3.1c) then imply

$$[G_\ell] = [G_t] = [H_\ell] = [H_t] = [K] = 0. \quad (4.5)$$

We furthermore require that  $c_{1,\ell}$  and  $c_{1,t}$  are marginal, which implies

$$[\delta M_\ell] = [M_t] = 1 + (d-z)/2. \quad (4.6)$$

Here  $z$  is the dynamical exponent associated with the factor of  $\sqrt{T}$  in either of the coupling terms in Eqs. (3.1b) and (3.1c). As was explained in I, this  $z$  can equal to  $z_{\text{diff}}$ , e.g., in terms in perturbation theory where the longitudinal magnon

propagator is convoluted with a diffusive one, which makes the frequency in Eq. (3.2) scale like a wave number squared, making the longitudinal magnon effectively massive. The critical longitudinal magnon, on the other hand, we expect to be massless. In this case, we choose  $a_{\ell,d-2}$  to be marginal for  $2 < d < 4$ , and  $a_{\ell,2}$  for  $d > 4$ . This implies a critical time scale

$$z_c = \begin{cases} d & \text{for } 2 < d < 4 \\ 4 & \text{for } d > 4, \end{cases} \quad (4.7)$$

as well as a critical exponent  $\eta \equiv \eta_\ell = \eta_t$ ,

$$\eta = \begin{cases} 4-d & \text{for } 2 < d < 4 \\ 0 & \text{for } d > 4. \end{cases} \quad (4.8)$$

$t$  is by definition the only relevant operator (apart from the temperature/frequency and the external magnetic field) at a physical critical fixed point, and its scale dimension determines the correlation length exponent  $\nu = 1/[t]$ . We thus have

$$\nu = \begin{cases} 1/(d-2) & \text{for } 2 < d < 4 \\ 1/2 & \text{for } d > 4. \end{cases} \quad (4.9)$$

We now turn to the scale dimension of  $m$ . From Eq. (3.1c), in conjunction with Eqs. (4.4a) and (4.5), we have

$$[m] = 2. \quad (4.10a)$$

However, in order to determine the physical order parameter exponents  $\beta$  and  $\delta$ , we need to take into account that the critical behavior of the magnetization is affected by dangerous irrelevant variables. From the equation of state, Eq. (2.9b), we see that  $m(h=0) \propto v^{-2/(d-2)}$  for  $2 < d < 6$ , and  $m(h=0) \propto u^{-1/2}$  for  $d > 6$ .<sup>18</sup> The effective scale dimension of  $m$  is therefore  $[m]_{\text{eff}} = [m] + 2[v]/(d-2)$  for  $2 < d < 6$ , and  $[m]_{\text{eff}} = [m] + [u]/2$  for  $d > 6$ . But from the equation of state we have  $[u] = [t] - 2[m] = -2$  and  $[v] = [t] - (d-2)[m]/2$ . This yields  $[u] = -2$  for  $d > 6$ ,  $[v] = 0$  for  $2 < d < 4$ , and  $[v] = 4-d$  for  $4 < d < 6$ . Therefore,

$$[m]_{\text{eff}} = \begin{cases} 2 & \text{for } 2 < d < 4 \\ 4/(d-2) & \text{for } 4 < d < 6 \\ 1 & \text{for } d > 6, \end{cases} \quad (4.10b)$$

which leads to

$$\beta = \begin{cases} 2/(d-2) & \text{for } 2 < d < 6 \\ 1/2 & \text{for } d > 6. \end{cases} \quad (4.10c)$$

Finally, the effective scale dimension of the external magnetic field  $h$  is

$$[h]_{\text{eff}} = [t] + [m]_{\text{eff}} = \begin{cases} d & \text{for } 2 < d < 4 \\ 2d/(d-2) & \text{for } 4 < d < 6 \\ 3 & \text{for } d > 6, \end{cases} \quad (4.11a)$$

which implies

$$\delta = \begin{cases} d/2 & \text{for } 2 < d < 6 \\ 3 & \text{for } d > 6. \end{cases} \quad (4.11b)$$

An inspection of Eq. (3.5) shows that all corrections to the Gaussian action are irrelevant with respect to the Gaussian fixed point, except that  $c_2$  is marginal in the event that the factor  $\sqrt{T}$  in this coupling carries the diffusive time scale. It was shown in I and II that this can indeed happen, and that this makes  $c_2$  marginally relevant with respect to the Gaussian fixed point. The actual critical fixed point therefore contains the effects of  $c_2$ . This is a result of the existence of two time scales in the problem. Furthermore, the terms of  $O(q^4)$ , which are irrelevant by power counting, turn out to be effectively marginal as well. This was also shown in I and II, and the logarithmic corrections to scaling that result from these marginal operators were explicitly determined.

There is no need to repeat the discussion of the logarithmic corrections to scaling, since it is now clear how this solution carries over to the present case. Above we have shown explicitly that scaling works in the ferromagnetic phase, with the magnetization having a scale dimension  $[m] = 2$ . This means, in particular, that the results obtained in II from scaling arguments for the free energy were correct. For instance, the exponent  $\beta$  is given by the result of the generalized Landau theory, Eq. (2.10), with logarithmic corrections as given in Eq. (3.6e) of II.

## V. DISCUSSION AND CONCLUSION

We conclude with a summary of our results and some additional remarks.

### A. Summary

In summary, we have constructed an effective theory for the instability of the ferromagnetic phase of a disordered itinerant Heisenberg ferromagnet at the quantum critical point. We have shown that the presence of ferromagnetic Goldstone modes, or spin waves, does not change the critical behavior compared to that obtained by supplementing results from the paramagnetic phase with scaling arguments. We have also given a very simple generalized Landau theory for this problem, which takes into account the effects of soft fermionic modes independent of the order parameter, and which yields the correct critical behavior in all dimensions  $d > 2$  apart from logarithmic corrections to power-law scaling. The results of the Landau theory are the same as those originally obtained from a nonlocal order parameter theory in Ref. 9.

### B. Hertz's fixed point

We briefly discuss how Hertz's fixed point<sup>15</sup> relates to the above discussion. Suppose one ignored the mode-mode coupling effects that are represented by the coefficients  $a_{\ell,d-2}$  and  $a_{t,d-2}$  in the Gaussian action, Eqs. (3.1), and by  $v$  in the equation of state, Eq. (2.9b). Then one has, for all  $d > 2$ ,

$$[t] = [m] = 2, [u] = -2, \quad (5.1a)$$

which leads to

$$[m]_{\text{eff}}=1, [h]_{\text{eff}}=3. \quad (5.1b)$$

The dynamical critical exponent is then

$$z_c=4, \quad (5.2a)$$

and all static exponents have mean-field values

$$\eta=0, \nu=\beta=1/2, \delta=3. \quad (5.2b)$$

Of course, for  $2 < d < 4$ , this fixed point is unstable against the mode-mode coupling effects as was discussed in detail in I. For  $4 < d < 6$ , it is actually stable, and the only reason why Hertz's theory does not yield the correct critical behavior is that it misses the leading dangerous irrelevant variable for the magnetization, which is  $v$  rather than  $u$ . For  $d > 6$ , the exact critical behavior is mean-field like.

### C. General remarks

We finally come back to some of the points mentioned in the Introduction. We have shown, by explicitly considering the ordered phase, that the scaling arguments used in I and II to extract the critical behavior of the magnetization were correct, and can be justified by a renormalization-group (RG) analysis. In particular, the presence of the magnetic Goldstone modes does not invalidate these arguments since the transverse fermionic modes and the magnetic Goldstone modes essentially just switch roles as one goes into the ferromagnetic phase. We mention, however, that we have not addressed the question of the size of the quantum scaling region for any particular observable. This question can only be answered by an explicit crossover calculation for the observable of interest, which follows the behavior of the observable through the entire critical region at  $T > 0$ .

Another point of interest is the relation between the theory developed here and Ref. 9. Like Hertz's theory, the latter was a pure Landau-Ginzburg-Wilson theory, i.e., a field theory formulated entirely in terms of the order parameter field; the fermionic degrees of freedom had been integrated out. This integrating out of soft modes resulted in nonlocal vertices that made the theory unsuitable for explicit calculations, but the critical behavior could be extracted by a combination of power counting and scaling arguments. It is interesting to see that the current series of papers has completely vindicated this treatment. In particular, the equation of state derived in Ref. 9 was Eq. (2.9a) expanded in powers of  $m^2$ , and the critical behavior determined in the nonlocal theory was indeed exact, except for the logarithmic corrections to scaling in  $2 < d < 4$ , which were discussed in II. It is also interesting to see that the much simpler generalized Landau theory of Sec. II B, which is based on a Gaussian approximation, reproduces the results of Ref. 9, and thus is also exact except for the logarithms. The reason is that, as we have seen, only two classes of non-Gaussian terms ultimately contribute to the critical fixed point, namely, the terms of  $O(Mq^2)$  and those of  $O(q^4)$ , and both of these lead only to logarithmic corrections to the Gaussian critical behavior. Section II therefore provides a very simple way to

obtain the essentially exact critical behavior, but of course this is clear only *a posteriori* once the RG analysis has been performed.

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### APPENDIX A: DERIVATION OF THE EFFECTIVE ACTION

In this appendix we give the exact effective action at the Gaussian level, as it emerges from a derivation from the microscopic theory. A schematic version of this result, which is sufficient for power counting, was given in Eqs. (3.1), and motivated by general arguments in Sec. III. We start with the general field theory, Eqs. (2.10) of I. The first step is to find a saddle-point solution that is appropriate for a ferromagnetic phase. One then separates soft and massive modes, and integrates out the latter in tree approximation. The result is the desired effective field theory for the soft modes.

#### 1. Stoner saddle point

Equations (2.10) of I allow for a homogeneous saddle-point solution that reproduces the Stoner theory. This is the saddle-point considered in Ref. 10, supplemented with a saddle-point value for the order parameter field  $M$ ,

$${}^i_r Q_{12}(\mathbf{x})|_{\text{sp}} = \delta_{12}[\delta_{r0}\delta_{i0}Q_{n_1}^0 + \delta_{r3}\delta_{i3}Q_{n_1}^3], \quad (A1a)$$

$${}^i_r \tilde{\Lambda}_{12}(\mathbf{x})|_{\text{sp}} = \delta_{12}[-\delta_{r0}\delta_{i0}i\Sigma_{n_1} + \delta_{r3}\delta_{i3}i\Delta_{n_1}], \quad (A1b)$$

$${}^i M_1(\mathbf{x})|_{\text{sp}} = \delta_{n_1 0} \delta_{i3} m / \sqrt{T}. \quad (A1c)$$

Here we use the same notation as in I, with  $l \equiv (n_1, \alpha_1)$ , etc., comprising both frequency and replica indices. By substituting Eqs. (A1) into the effective action  $\mathcal{A}$  [Eqs. (2.10) in paper I], and using the saddle-point conditions  $\delta\mathcal{A}/\delta Q = \delta\mathcal{A}/\delta\tilde{\Lambda} = \delta\mathcal{A}/\delta M = 0$ , one obtains the saddle-point equations

$$m = 4i \left(\frac{\Gamma_t}{2}\right)^{1/2} T \sum_m Q_m^3 e^{i\omega_m 0}, \quad (A2a)$$

$$Q_n^0 = \frac{i}{2V} \sum_k G_n^0(\mathbf{k}), \quad (A2b)$$

$$Q_n^3 = \frac{i}{2V} \sum_k G_n^3(\mathbf{k}), \quad (A2c)$$

$$\Sigma_n = \frac{-i}{\pi N_F \tau_{\text{el}}} Q_n^0 - 4i T \Gamma_s \sum_m Q_m^0 e^{i\omega_m 0}, \quad (A2d)$$

$$\Delta_n = \frac{i}{\pi N_F \tau_{\text{el}}} Q_n^3 + \Delta, \quad (\text{A2e})$$

with

$$\Delta = -(2\Gamma_t)^{1/2} m. \quad (\text{A2f})$$

Here

$$G_n^0(\mathbf{k}) = \frac{1}{2} [\mathcal{G}_n^+(\mathbf{k}) + \mathcal{G}_n^-(\mathbf{k})], \quad (\text{A3a})$$

$$G_n^3(\mathbf{k}) = \frac{1}{2} [\mathcal{G}_n^+(\mathbf{k}) - \mathcal{G}_n^-(\mathbf{k})], \quad (\text{A3b})$$

are Green's functions given in terms of

$$G_n^\pm(\mathbf{k}) = \frac{1}{i\omega_n - \xi_{\mathbf{k}} \pm \Delta_n - \Sigma_n}, \quad (\text{A3c})$$

with  $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \mu$ .  $\Gamma_{s,t}$  are the interaction amplitudes proportional to  $K_{s,t}$  that were defined in I. Upon substituting Eqs. (A2a) and (A2f) into (A2e), one recovers the saddle-point equations of Ref. 10, and hence the Stoner theory.

It is useful to define various transport and thermodynamic quantities in terms of these Green's functions. We will need

$$\begin{aligned} \sigma_0^\pm &= \frac{1}{\pi m V} \lim_{n,m \rightarrow 0} \sum_{\mathbf{k}} \left[ \frac{1}{2} [\mathcal{G}_n^\pm(\mathbf{k}) + \mathcal{G}_m^\pm(\mathbf{k})] \right. \\ &\quad \left. + \frac{1}{dm} \mathbf{k}^2 \mathcal{G}_n^\pm(\mathbf{k}) \mathcal{G}_m^\pm(\mathbf{k}) \right], \end{aligned} \quad (\text{A4a})$$

and

$$\begin{aligned} \tilde{\sigma}_0^\pm &= \frac{1}{\pi m V} \lim_{n,m \rightarrow 0} \sum_{\mathbf{k}} \left[ \frac{1}{2} [\mathcal{G}_n^\pm(\mathbf{k}) + \mathcal{G}_m^\mp(\mathbf{k})] \right. \\ &\quad \left. + \frac{1}{dm} \mathbf{k}^2 \mathcal{G}_n^\pm(\mathbf{k}) \mathcal{G}_m^\mp(\mathbf{k}) \right], \end{aligned} \quad (\text{A4b})$$

as well as

$$N_F^\pm = \frac{i}{\pi V} \lim_{n \rightarrow 0^+} \sum_{\mathbf{k}} \mathcal{G}_n^\pm(\mathbf{k}). \quad (\text{A4c})$$

These quantities represent the Born approximation for various conductivities and densities of states in the split-band system of the Stoner theory. They are generalizations of the analogous quantities defined in Ref. 4. For a physical interpretation of these quantities, see Ref. 11.

## 2. Gaussian soft-mode theory

The separation of soft and massive modes works in analogy to I, although the procedure is more cumbersome in the presence of a nonzero magnetization. The massive modes are integrated out in tree approximation to arrive at an effective soft-mode action, and the soft fermion modes are expanded in powers of  $q$  [cf. Eq. (2.5d)], again in analogy to I. The complete procedure and result can be found in Ref. 11. Here we list the Gaussian (i.e., bilinear in  $M$  and  $q$ ) contribution to the effective action. The higher-order terms are obtained analogously.

The Gaussian action has the general form

$$\mathcal{A}^G[q, M] = \mathcal{A}_{\text{NL}\sigma\text{M}}[q] + \mathcal{A}_{\text{LGW}}[M] + \mathcal{A}_c[q, M]. \quad (\text{A5})$$

We expand the  $q$  matrices in a spin-quaternion basis, see Eq. (2.5b). The fermionic part of the action, which is the Gaussian part of a generalized nonlinear  $\sigma$  model, is given by

$$\begin{aligned} \mathcal{A}_{\text{NL}\sigma\text{M}}[q] &= -4 \sum_{\alpha\beta} \sum_{n_1, n_2, n_3, n_4} \sum_{\mathbf{k}} \sum_{r,s=0,3} \sum_{i,j=0}^3 (\delta_{n_1 n_3} \delta_{n_2 n_4} [\kappa_{ij,rs}^0 (\delta_{i0} + \delta_{i3}) (\mathbf{k}^2/G^s + H^s \Omega_{n_1-n_2}) \\ &\quad + \kappa_{ij,rs}^0 (\delta_{i1} + \delta_{i2}) (\mathbf{k}^2/\tilde{G}^s + H^s \Omega_{n_1-n_2}) + \kappa_{ij,rs}^\ell (\mathbf{k}^2/G^a + H^a \Omega_{n_1-n_2}) + \kappa_{ij,rs}^t (\mathbf{k}^2/\tilde{G}^a + 2iH^s \Delta)] \\ &\quad - \delta_{n_1-n_2, n_3-n_4} \delta_{\alpha\beta} T \Gamma_s \pi^2 \{ \kappa_{ij,rs}^0 [(N_F^s)^2 \delta_{i0} + (N_F^a)^2 \delta_{i3}] + \kappa_{ij,rs}^\ell N_F^s N_F^a \}_{r, q_{n_1 n_2}^{\alpha\beta}(\mathbf{k})_s^j q_{n_3 n_4}^{\alpha\beta}(-\mathbf{k})}. \end{aligned} \quad (\text{A6})$$

Here  $\kappa$  represents trace in spin-quaternion space,

$$\kappa_{ij,rs}^0 = \frac{1}{4} \text{tr}(\tau_r \tau_s^\dagger) \text{tr}(s_i s_j^\dagger),$$

$$\kappa_{ij,rs}^\ell = \frac{1}{8} \text{tr}(\tau_3 \tau_r \tau_s^\dagger) [\text{tr}(s_3 s_i s_j^\dagger) + \text{tr}(s_i s_3 s_j^\dagger)], \quad (\text{A7a})$$

$$\kappa_{ij,rs}^t = \frac{1}{8} \text{tr}(\tau_3 \tau_r \tau_s^\dagger) [\text{tr}(s_3 s_i s_j^\dagger) - \text{tr}(s_i s_3 s_j^\dagger)],$$

$$\kappa_{ij,rs}^3 = -\frac{1}{4} \text{tr}(\tau_r \tau_s^\dagger) \text{tr}(s_3 s_i s_3 s_j^\dagger). \quad (\text{A7b})$$

The coupling constants  $G$  and  $H$  in Eq. (A6), as well as the densities of states  $N_F$ , are magnetization dependent generalizations of the analogous quantities in I. They are given by

$$G^s = 16/\pi(\sigma_0^+ + \sigma_0^-), \quad G^a = 16/\pi(\sigma_0^+ - \sigma_0^-),$$

$$\tilde{G}^s = 16/\pi(\tilde{\sigma}_0^+ + \tilde{\sigma}_0^-), \quad \tilde{G}^a = 16/\pi(\tilde{\sigma}_0^+ - \tilde{\sigma}_0^-), \quad (\text{A8a})$$

$$N_F^s = (N_F^+ + N_F^-)/2, \quad N_F^a = (N_F^+ - N_F^-)/2, \quad (\text{A8b})$$

$$H^{s,a} = \pi N_F^{s,a}/4. \quad (\text{A8c})$$

As can be seen by a direct comparison, the schematic action given by Eq. (3.1) reflects all qualitative features of the complete result. Notice that the mass  $H\Delta$  in the transverse vertex

comes from the coupling between  $M$  and  $q$ , cf. Eq. (2.4), but has been included in  $\mathcal{A}_{\text{NL}\sigma M}$ , since it contains only fermionic fluctuations.

The LGW part of the action that results from integrating out the massive modes is

$$\begin{aligned} \mathcal{A}_{\text{LGW}}[M] = & - \sum_{\alpha} \sum_{n \geq 0} (2 - \delta_{n0}) \sum_{\mathbf{k}} \sum_{i,j=1}^3 \left\{ \delta_{ij} + (\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} - i\kappa'_{12,ij}) \Gamma_t T \sum'_m [\mathcal{E}_{m+n,m}^+ + (-)^{i+j} \mathcal{E}_{m+n,m}^-] \right. \\ & \left. + \delta_{i3}\delta_{j3} \Gamma_t T \sum'_m (\mathcal{D}_{m+n,m}^+ + \mathcal{D}_{m+n,m}^-) + \delta_{i3}\delta_{j3} \frac{\Gamma_t \Gamma_s \left[ T \sum'_m (\mathcal{D}_{m+n,m}^+ - \mathcal{D}_{m+n,m}^-) \right]^2}{1 - \Gamma_s T \sum'_m (\mathcal{D}_{m+n,m}^+ + \mathcal{D}_{m+n,m}^-)} \right\} i M_n^{\alpha}(\mathbf{k})^j M_{-n}^{\alpha}(-\mathbf{k}). \end{aligned} \quad (\text{A9})$$

Here, we have adopted the notation of Ref. 10;  $\mathcal{D}$  and  $\mathcal{E}$  correspond to modified ‘‘diffusons’’ in the longitudinal and transverse spin channels, respectively. They are given by

$$\mathcal{D}_{nm}^{\pm}(\mathbf{k}) = \phi_{nm}^{\pm}(\mathbf{k}) / [1 - \phi_{nm}^{\pm}(\mathbf{k}) / 2\pi N_F \tau], \quad (\text{A10a})$$

$$\mathcal{E}_{nm}^{\pm}(\mathbf{k}) = \eta_{nm}^{\pm}(\mathbf{k}) / [1 - \eta_{nm}^{\pm}(\mathbf{k}) / 2\pi N_F \tau], \quad (\text{A10b})$$

with

$$\phi_{nm}^{\pm} = (\varphi_{nm}^{00} + \varphi_{nm}^{33}) \pm (\varphi_{nm}^{03} + \varphi_{nm}^{30}), \quad (\text{A10c})$$

$$\eta_{nm}^{\pm} = (\varphi_{nm}^{00} - \varphi_{nm}^{33}) \pm (\varphi_{nm}^{03} - \varphi_{nm}^{30}), \quad (\text{A10d})$$

and

$$\varphi_{nm}^{uv} = \frac{1}{V} \sum_{\mathbf{p}} G_n^u(\mathbf{p}) G_m^v(\mathbf{p} + \mathbf{k}) \quad (u, v = 0, 3), \quad (\text{A10e})$$

with  $G^{0,3}$  given by Eqs. (A3). We have also introduced the notation

$$T \sum'_m [\dots] \equiv T \sum_m \Theta[m(m+n)] [\dots]. \quad (\text{A11})$$

The LGW vertex simplifies in the limit of long wavelengths and small frequencies. By using<sup>10</sup>

$$T \sum_m \mathcal{D}_{mm}^{\pm}(\mathbf{k}=0) = -N_F^{\pm}, \quad (\text{A12a})$$

and

$$T \sum_m \mathcal{E}_{mm}^{\pm}(\mathbf{k}=0) = -1/2\Gamma_t, \quad (\text{A12b})$$

one recovers the functional form given in Eqs. (3.1), except for the nonanalytic terms that emerge only at one-loop order and were added in Sec. III.

Finally, the coupling between  $M$  and  $q$  to bilinear order is

$$\begin{aligned} \mathcal{A}_c[M, q] = & \sum_{12} \sum_{\mathbf{k}} \sum_{r,s=0,3} \sum_{i=1}^3 \sum_{j=0}^3 4\pi \sqrt{2\Gamma_t} [\kappa_{ij,rs}^0 (N_F)^2 \\ & + \kappa_{ij,rs}^{\ell} (N_F^a)^2] {}_r b_{12}(\mathbf{k}) {}_s q_{12}(-\mathbf{k}). \end{aligned} \quad (\text{A13a})$$

Here,  $b$  is a field with components

$$\begin{aligned} {}_r b_{12}(\mathbf{k}) = & \delta_{\alpha_1 \alpha_2} (-)^{r/2} \sum_n \delta_{n, n_1 - n_2} \\ & \times [{}^i M_n^{\alpha_1}(\mathbf{k}) + (-)^{r+1} {}^i M_{-n}^{\alpha_1}(\mathbf{k})]. \end{aligned} \quad (\text{A13b})$$

Again, this has the same functional form as the schematic representation in Eqs. (3.1).

### 3. Gaussian propagators

The Gaussian propagators can be obtained by inverting the quadratic form given by the Gaussian action in the preceding subsection. We do not list the complete propagators here, which can be found in Ref. 11, but give only the hydrodynamic parts of the diffusons, which are needed in the calculation. For  $nm < 0$ , one finds

$$\mathcal{D}_{nm}^{\pm}(\mathbf{k}) = \frac{2\pi N_F^{\pm}}{D^{\pm} \mathbf{k}^2 + |\Omega_{n-m}|}. \quad (\text{A14a})$$

Here,  $D^{\pm} = 1/G^{\pm} H^{\pm}$  with

$$1/G^{\pm} = 1/G^s \pm 1/G^a, \quad (\text{A14b})$$

$$H^{\pm} = H^s \pm H^a \quad (\text{A14c})$$

can be interpreted as the Boltzmann diffusivities in the upper and lower Stoner band, respectively.

Similarly, one finds for  $n > 0$  and  $m < 0$ ,

$$\mathcal{E}_{nm}^{\pm}(\mathbf{k}) = \frac{2\pi N_F}{[\Omega_{n-m} + \tilde{D}^{\pm} \mathbf{k}^2] \frac{1}{1 \pm 2i\Delta\tau} \pm 2i\Delta}. \quad (\text{A15a})$$

Here,  $\tilde{D}^{\pm} = D/(1 \pm 2i\Delta\tau) = 1/\tilde{G}^{\pm}H$ , with

$$1/\tilde{G}^{\pm} = 1/\tilde{G}^s \pm 1/\tilde{G}^a. \quad (\text{A15b})$$

Notice that Eq. (A15a) holds only for  $n > 0$  and  $m < 0$ . The result for  $n < 0$ ,  $m > 0$  is given by the relation<sup>19</sup>

$$\mathcal{E}_{nm}^{\pm}(\mathbf{k}) = \mathcal{E}_{mn}^{\mp}(\mathbf{k}). \quad (\text{A15c})$$

By means of these expressions, one readily finds that the propagators have functional forms as given in Sec. III. Again, the nonanalytic dependences on  $\mathbf{k}$  and  $\Delta$  are not included in the Gaussian theory, and have been added in Sec. III.

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- <sup>1</sup>D. Belitz, T.R. Kirkpatrick, M.T. Mercaldo, and S.L. Sessions, Phys. Rev. B **63**, 174427 (2001).  
<sup>2</sup>D. Belitz, T.R. Kirkpatrick, M.T. Mercaldo, and S.L. Sessions, Phys. Rev. B **63**, 174428 (2001).  
<sup>3</sup>K.B. Efetov, A.I. Larkin, and D.E. Khmel'nitskii, Zh. Éksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].  
<sup>4</sup>D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **56**, 6513 (1997).  
<sup>5</sup>J.W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).  
<sup>6</sup>F. Wegner, Z. Phys. B **35**, 207 (1979).  
<sup>7</sup>A.M. Finkelstein, Zh. Éksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].  
<sup>8</sup>D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).  
<sup>9</sup>T.R. Kirkpatrick and D. Belitz, Phys. Rev. B **53**, 14364 (1996).  
<sup>10</sup>T.R. Kirkpatrick and D. Belitz, Phys. Rev. B **62**, 952 (2000).  
<sup>11</sup>S.L. Sessions, Ph.D. thesis, University of Oregon, 2002.  
<sup>12</sup>T. Moriya, *Spin Fluctuations in Itinerant Electron Magnetism* (Springer, Berlin, 1985).

- <sup>13</sup>D. Belitz, T.R. Kirkpatrick, A.J. Millis, and T. Vojta, Phys. Rev. B **58**, 14155 (1998).  
<sup>14</sup>S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA, 1976).  
<sup>15</sup>J. Hertz, Phys. Rev. B **14**, 1165 (1976).  
<sup>16</sup>We use the notation,  $a \approx b$  for “ $a$  approximately equals  $b$ ,”  $a \cong b$  for “ $a$  is isomorphic to  $b$ ,” and  $a \sim b$  for “ $a$  scales like  $b$ .”  
<sup>17</sup> $z$  is defined by the scaling relation between frequency and wave number,  $\Omega \sim |\mathbf{k}|^z$ .  
<sup>18</sup>We note that it is permissible at this point to use the generalized mean-field equation of state, whose validity we want to establish. The point is that the equation of state serves only to obtain the values of the exponents  $\beta$  and  $\delta$  from  $[m]$ , while the fact that  $[m]=2$  suffices to justify the scaling arguments used in I, which, in turn, establishes the validity of the equation of state up to logarithmic corrections. See also the last paragraph of this section.  
<sup>19</sup>This distinction between the cases  $n > 0$ ,  $m < 0$  and  $n < 0$ ,  $m > 0$  was not made in Ref. 10.