A theory for the spinup of tropical depressions

D. J. Raymond,* S. L. Sessions and ˇZ. Fuchs
New Mexico Tech, Socorro, USA

ABSTRACT: Recent advances in our understanding of the physics of precipitation over tropical oceans is married with vorticity dynamics to produce a simple theory for the development or non-development of tropical depressions. This early phase of tropical cyclogenesis is perhaps the least understood part of the whole process. The theory predicts a finite amplitude threshold for intensification, with weaker disturbances decaying. Furthermore, only disturbances with radii larger than a critical value develop. Rough estimates of this critical radius place it near 1600 km, though better estimates of various parameters could change this value significantly. Copyright © 2007 Royal Meteorological Society

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1. Introduction

The empirically determined general conditions needed for the spinup of tropical cyclones have been known for many years, i.e. warm sea surface temperatures, low environmental shear and a pre-existing disturbance to provide low-level cyclonic vorticity (Gray, 1968; McBride and Zehr, 1981). However, in spite of the vast amount of numerical modelling work which has been done on tropical cyclones, major questions remain.


Somewhat less attention has been paid to the thermodynamic aspects of cyclogenesis. In a simple model, Emanuel (1989, 1995) explained the finite amplitude nature of cyclogenesis seen in the simulations of Rotunno and Emanuel (1987) as a consequence of the boundary layer divergence associated with convective downdrafts in weak disturbances. Only when the middle levels became moist enough to suppress downdraft production and increase precipitation efficiency was a weak vortex able to amplify. A notable facet of the Emanuel model is the abandonment of the direct linkage of Ekman pumping to convective mass fluxes seen in many early models of cyclones, e.g. Charney and Eliassen (1964) and Ooyama (1964, 1969). In a model similar to Emanuel’s (1989), Frisius (2006) emphasizes the importance of a negative radial gradient of precipitation efficiency in confining the convection to near the centre of the cyclone vortex, a point also touched upon by Emanuel.

Gray and Craig (1998) showed exponential growth of vortex intensity from small amplitude in a simple linearized model. However, they assumed that a saturated eyewall had already formed, so their results do not necessarily contradict those of Emanuel. Zehnder (2001) also explored the effect on tropical cyclogenesis of convective forcing based on the thermodynamic principle of boundary layer quasi-equilibrium (BLQ) of moist enthalpy or entropy. He found that quite different results were obtained using this formulation in comparison to those obtained from an Ekman pumping model of convective forcing. Craig and Gray (1996) also found that a cloud-resolving model of a tropical storm did not behave in a manner consistent with the direct forcing of convection by Ekman pumping.

Using results from the TExMEX (Tropical EXperiment in MExico) project, Bister and Emanuel (1997) showed that hurricane Guillermo (1991) formed out of a tropical depression with a mid-level vortex overlying a moist, cold core. They found that this structure resulted from cooling by the evaporation of precipitation and hypothesized that the eventual low-level vortex...
apparently developed downward from the mid-level circulation.

Raymond et al. (1998) also studied the genesis of cyclones observed in TExMEX. Using the form of the vorticity equation advocated by Haynes and McIntyre (1987), they noted that cyclone development occurs when the tendency of convergence to enhance the low-level circulation of a system defeats the tendency of surface friction to spin the system down. The downward development of circulation from the mid-level vortex, as envisioned by some investigators, is not supported by the Haynes–McIntyre formulation, which shows that the vertical component of vorticity undergoes only horizontal transport. The role of the mid-level vortex is more subtle in this view; it acts primarily through geostrophic (or nonlinear) balance to maintain the underlying cold core, which in turn has a thermodynamic effect on the convection occurring in this region. In particular, the cold core in conjunction with the moistening resulting from previous rainfall tends to produce a sounding profile with almost constant moist entropy (or equivalent potential temperature) above the boundary layer. An example of such a sounding taken during EPIC2001 (East Pacific Investigation of Climate, year 2001; Raymond et al. 2003, 2004) is shown in Figure 1.

Convection occurring in this environment is likely to have downdrafts suppressed due to the inability of subsequent rainfall to produce negative buoyancy by evaporation. The suppression of downdrafts means that the convergence at low levels associated with convective updrafts is not reduced by divergence associated with downdrafts. As a result, the low-level spinup of the system is enhanced.

We emphasize that to cause spinup of a cyclone, low-level convergence must be associated with heating, i.e. with the latent heat release from rainfall. Without the upward export of low-level air by moist convection, convergence results in dry-adiabatic ascent and the production of a cold anomaly. The consequence is hydrostatic enhancement of the surface pressure which opposes the low pressure in the centre of the vortex, resulting in spin-down (e.g. Gill 1982, p 353).

The importance of rainfall to the spinup of cyclones leads us to consider the mechanisms controlling its production. An approach based on thermodynamic principles was proposed by Raymond (2000). The consequences of the simple postulate that mean rainfall intensity is inversely proportional to the saturation deficit, i.e. the difference between the tropospheric saturated precipitable water and the actual precipitable water, were explored. In conjunction with Neelin and Held’s (1987) ideas about gross moist stability, this led to a model for the evolution of the humidity and precipitation rate in a tropospheric column.

Bretherton et al. (2004) performed a test of the hypothesis of Raymond (2000) by examining the relationship between precipitation and precipitable water over the tropical oceans, as determined by microwave measurements from Special Sensor Microwave/Imager (SSM/I) satellites. They discovered that a strong correlation exists between these quantities, but that the precise relationship varies between ocean basins. However, these differences vanished when the relationship was recast in terms of precipitation rate and saturation fraction, i.e. the ratio of precipitable water to saturated precipitable water. Bretherton et al. (2004) refer to this saturation fraction as the column relative humidity. Basically, all precipitation occurs for saturation fractions between 0.6 and 0.85, with a rapidly increasing precipitation rate as a function of saturation fraction within this range. Almost no cases were found with saturation fractions exceeding 0.85.

One might expect this type of relationship to be valid for long-time averages, but to break down on shorter time scales. However, it was found that the relation held even for daily time scales, suggesting that it reflects a fundamental physical process such as the deleterious effect of dry air entrainment on updrafts. The robust nature of this correlation is supported by Figure 2, which shows saturation fraction from individual soundings in the tropical east Pacific and the southwest Caribbean as a function of the coincident satellite infrared brightness temperature (considered as a proxy for precipitation) in a 50 km square centred on the sounding. In this figure, EPIC soundings were launched from the ship Ron Brown at 95°W, 10°N in the east Pacific intertropical convergence zone, whereas EPIC dropsondes were deployed from 470 hPa by the National Center for Atmospheric Research’s C-130 aircraft (Raymond et al., 2004). Only dropsondes deployed in the range 6.5°N to 12°N were used in order to limit measurements to regions of warm sea surface temperature. The ECAC (Experimental Climático en las Albercas de Agua Caliente de las Américas) soundings were launched from the Mexican research vessel Justo Sierra in the dry southwest Caribbean (Magana and Caetano, 2005).

![Figure 1. Dropsonde soundings taken in the tropical east Pacific on 29 September 2001. The thin lines represent undisturbed conditions at 95.01°W, 12.27°N while the thick lines show a sounding in the core of the developing tropical cyclone Lorena at 96.98°W, 9.97°N. In each case the left hand line is the specific moist entropy while the right hand line is the saturated specific moist entropy.](image-url)
The correlation between low infrared brightness temperature and humidity could simply be a consequence of higher humidity breeding more clouds. However, over warm ocean regions, low brightness temperature generally corresponds to high stratiform cloudiness with no necessary correlation with higher humidity at low to middle levels where most of the atmospheric moisture resides. On the other hand, area-averaged precipitation rates inferred from airborne radar data and satellite-derived infrared brightness temperature are highly correlated in the east Pacific intertropical convergence zone, with a correlation coefficient of $-0.90$ (Raymond et al. 2003). The scale on the right side of Figure 2 indicates rainfall values corresponding to brightness temperatures in this linear regression. We therefore argue that Figure 2 represents a valid, if somewhat noisy relationship between saturation fraction and precipitation rate.

More recent work by Back and Bretherton (2005) shows that some of the variance in precipitation rate at fixed saturation fraction can be explained by variations in surface wind speed, with higher wind speeds corresponding to higher precipitation rates. This result indicates that the original hypothesis of Raymond (2000) – that precipitation is a unique function of saturation fraction – is perhaps too simple. However, numerical simulations of convection, made in a context which we explain later, also support a strong relationship between rainfall rate and tropospheric humidity (Derbyshire et al., 2004; Raymond and Zeng, 2005). Furthermore, Derbyshire et al. (2004) show that many parametric treatments of convection in global models do not reproduce the sensitivity to moisture shown by the cumulus ensemble models.

The purpose of this paper is to marry the above ideas about precipitation and saturation fraction to the dynamics of tropical cyclones. The feedback loop which governs cyclone spinup or spindown can be described by the following three steps.

1. The existing circulation produces surface fluxes of water vapour, moist entropy and momentum which depend on the strength of the circulation and other environmental factors such as sea surface temperature. The moist entropy flux modifies the profile of moist entropy, and hence that of saturation fraction.
2. Convection and rainfall are considered to be governed by the saturation fraction in the convective environment inside the cyclone. Determining the dependence of convection on this environment is central to the theory of spinup.
3. The relative strengths of the spinup tendency due to low-level vorticity convergence induced by convective heating and the spindown tendency due to surface friction determine whether the cyclone vortex intensifies or decays.

We examine here only the case of weak systems such as tropical depressions in order to avoid the complications of nonlinearity. In this limit we do not expect to see the warm core which is the traditional signal that a tropical storm has formed. However, this phase of intensification is arguably less-well understood than the later phases of tropical cyclogenesis.

The paper is organized as follows: Section 2 updates the theory of Raymond (2000) to account for the results of Bretherton et al. (2004), while Section 3 links the updated thermodynamic results of Section 2 with the vorticity dynamics of the lower troposphere as discussed by Raymond et al. (1998). In Section 4 numerical results from the model of Raymond and Zeng (2005) are used to close the theory presented in Section 3 and the theory is applied to an axially symmetric vortex in Section 5. Conclusions are presented in Section 6.

2. Thermodynamic issues

We first examine the relevant theory governing water vapour and specific moist entropy. In this theory the gross moist stability (GMS) plays a central role. The concept of GMS depends on the existence of a thermodynamic variable conserved in moist processes. Unfortunately, there is no perfectly conserved variable in this context. Neelin and Held (1987) used the moist static energy in their pioneering work on GMS, whereas Raymond (2000) used equivalent potential temperature. In this work we use the specific moist entropy, as it is easier to deal with than equivalent potential temperature. It also has somewhat better-defined conservation properties than moist static energy, although given the simplified treatment here, this advantage is not realized.

An approximate form of the specific moist entropy suitable for the present work is

$$s = s_d + Lr/T_R. \quad (1)$$

Figure 2. Saturation fraction calculated from various soundings as a function of the infrared satellite brightness temperature coincident with the sounding in time and averaged over a 50 km square centred on the sounding position. The rainfall rate inferred from airborne radar measurements corresponding to the given brightness temperatures is given on the right hand scale. This correspondence should be considered approximate, especially at high brightness temperatures.
where \( s_e(T, p) \) is the specific entropy of dry air, \( L \) is the latent heat of condensation (assumed constant), \( r \) is the water vapour mixing ratio and \( T_R \) is a constant reference temperature. The specific moist entropy obeys the equation

\[
\frac{\partial s}{\partial t} + U \cdot \nabla s + \nabla \cdot (u_s) + \frac{\partial os}{\partial p} = G + \frac{\partial F_c}{\partial p},
\]

(2)

where \( g \) is the acceleration due to gravity, \( G \) is the irreversible generation of entropy per unit mass of air, and \( F_c \) is the vertical flux moist entropy due to diffusion, eddy transport and radiation. We have split the horizontal velocity \( u \) into a solenoidal barotropic part \( U \) (i.e. independent of height and divergence-free) and a baroclinic part \( u_s \), and have taken advantage of the fact that \( \nabla \cdot U = 0 \). We define \( U \) more precisely later. The quantity \( \omega \) is the pressure vertical velocity.

We now define the pressure integral operator

\[
Z = \frac{1}{p_s} \int_{p_1}^{p_s} Z \, dp,
\]

(3)

for any \( Z(p) \), where \( p_s \) and \( p_1 \) are respectively surface and tropopause pressures. Applying this operator to Equation (2) results in

\[
\frac{d\tilde{s}}{dt} + M\delta s = F_{es} - F_{et} + \overline{G}
\]

(4)

where \( F_{es} \) is the surface entropy flux due to latent and sensible heat flux plus radiation and \( F_{et} \) is the upward flux of entropy out of the top of the cylinder due to radiation. The total time derivative is defined

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + U \cdot \nabla.
\]

(5)

The second term in Equation (4) requires further explanation. Let us define the lateral mass outflow per unit time from a column of unit area as

\[
M = \frac{1}{g} \int s \cdot \hat{n} \, dp.
\]

(6)

where the plus and minus signs on the integrals indicate that only positive or negative values of the integrand are included in the integration. Due to the overall mass balance, the mass flowing into the column, given by the rightmost expression in Equation (6), equals the mass flowing out. The second term in Equation (4) can therefore be written

\[
\nabla \cdot (u_s) = \frac{1}{g} \int_{p_1}^{p_s} \nabla \cdot (u_s) \, dp
\]

\[
= \frac{1}{g} \int_{p_1}^{p_s} \int s \cdot \hat{n} \, d\ell \, dp = M\delta s
\]

(7)

where the line integral is around the periphery of the unit area defining the column, with \( \hat{n} \) being an outward unit normal. The quantity \( M \) is thus the mass current passing through the sides of the column in a reference frame moving with the solenoidal velocity \( U \), and \( \delta s = s_{out} - s_{in} \) is the difference between the mean outflow and inflow of moist entropy. The quantity \( \delta s \) is Neelin and Held’s (1987) GMS recast in terms of the moist entropy rather than the moist static energy. (Note that Neelin and Held (1987) approximate \( \overline{V} \cdot (u_s) \) by \( \overline{s \nabla u} \), thereby neglecting the entropy advection term \( \overline{u_s \nabla s} \) which is held to be insignificant.)

An equation similar to Equation (2) may be written for the water vapour mixing ratio \( r \):

\[
\frac{\partial r}{\partial t} + U \cdot \nabla r + \nabla \cdot (u_r) + \frac{\partial or}{\partial p} = - \frac{\partial}{\partial p} [(\omega + \omega_i)r_i] + \frac{\partial F_r}{\partial p}.
\]

(8)

The first term on the right side of this equation is the source of water vapour due to the formation and evaporation of precipitation, assuming that the horizontal and time scales are such that horizontal advection and storage of precipitation are negligible. The second term is the source due to the convergence of the vertical eddy and diffusional flux of water vapour \( F_r \). The role of condensed water in the form of cloud droplets or small ice crystals which advect with the air is neglected in this approximate analysis.

Applying Equation (3) to the moisture equation results in

\[
\frac{d\tilde{r}}{dt} - M\delta r = F_{rs} - R,
\]

(9)

where \( F_{rs} \) is the surface evaporation rate, \( R \) is the rainfall rate, and \( \delta r \) is defined as the mean inflow minus the mean outflow mixing ratio rather than the outflow minus the inflow.

We assume that \( R_s \), which is a function only of temperature, is approximately constant due to the near-invariance of the temperature profile in the tropics. It thus disappears in the time derivative in Equation (4), leaving the first term on the left hand side of this equation equal to \( (L/T_R)(d\overline{r}/dt) \). The mass flow \( M \) may then be eliminated between Equations (4) and (9), leaving us with

\[
(1 + \gamma) \frac{d\overline{r}}{dt} = \gamma(F_{rs} - R) + \frac{T_R(F_{es} - F_{et} + \overline{G})}{L}.
\]

(10)

Alternatively, the time derivative may be eliminated, resulting in an equation for the moisture convergence \( X \):

\[
X = M\delta r = \frac{L(R - F_{rs}) + T_R(F_{es} - F_{et} + \overline{G})}{(1 + \gamma)L}.
\]

(11)

The parameter

\[
\gamma = \frac{T_R\delta s}{L\delta r} = -\frac{T_R\overline{V} \cdot (u_s)}{\overline{V} \cdot (u_r)}
\]

(12)

is a dimensionless quantity closely related to the GMS. We shall refer to \( \gamma \) as the normalized gross moist stability.
(NGMS). The advantage over the GMS is that $\gamma$ is a dimensionless parameter which relates the GMS ($S_8$) to the moisture extracted ($\delta r$) and ultimately to the precipitation produced.

In a steady state, Equation (10) can be solved for the net rainfall rate $R - F_{re}$, which according to Equation (9) also equals the moisture convergence $X$ in this case:

$$
(R - F_{re})_{\text{steady}} = X_{\text{steady}} = \frac{T_R(F_{es} - F_{et} + \overline{G})}{\gamma L}. \quad (13)
$$

This equation expresses the results of Neelin and Held (1987), and illustrates how important the NGMS is to the large-scale forcing of precipitation. For a given moist entropy imbalance, represented by the numerator on the right side of Equation (13), the net rainfall, i.e. the rainfall minus evaporation, is inversely proportional to the NGMS in the steady state.

As it stands, Equation (13) is a purely diagnostic relationship. To give it prognostic value, two additional pieces of information are needed: a relationship between the rainfall rate $R$ and the precipitable water $P$ (or saturation fraction) and a way to estimate the NGMS. We now address these issues.

So far the analysis is similar to that of Raymond (2000) except that entropy is used in place of equivalent potential temperature. The solid line in Figure 3 shows graphically the humidity–precipitation relation assumed by Raymond (2000), expressed as a relationship between precipitation rate and saturation fraction. This assumption clearly does not fit the dependence on saturation deficit of the rainfall rate predicted by the cumulus ensemble model of Raymond and Zeng (2005), which is shown by the bullets in Figure 3. Nor does it agree with the form of the radiative-convective equilibrium rate, $\gamma(LF_{rs} + T_R(F_{es} - F_{et} + \overline{G}))$, shown by Raymond (2000), expressed as a relationship between the saturation fraction and a way to estimate the NGMS. The advantage over the GMS is that $\gamma$ is a dimensionless parameter which relates the GMS ($S_8$) to the moisture extracted ($\delta r$) and ultimately to the precipitation produced.

Eliminating $d\tau/dt$ between Equations (10) and (15) yields a predictive equation for the rainfall rate:

$$
\frac{dR}{dt} = \frac{R^2[\gamma L(F_{rs} - R) + T_R(F_{es} - F_{et} + \overline{G})]}{(1 + \gamma)LT_R(S_c - S_R)}. \quad (16)
$$

Non-dimensionalizing, we arrive at the same equation as Raymond (2000),

$$
\frac{d\alpha}{d\tau} = -\alpha^3 + \alpha^2 \Delta \phi, \quad (17)
$$

where $\alpha = R/R_R$ is the rainfall rate expressed in terms of the radiative-convective equilibrium rate,

$$
\Delta \phi = \frac{\gamma LF_{rs} + T_R(F_{es} - F_{et} + \overline{G})}{\gamma L R_R}. \quad (18)
$$

is the non-dimensional forcing and $\tau = t/t_0$ where the scaling time $t_0$ is defined

$$
t_0 = \frac{(1 + \gamma)\overline{F}_{rs}(S_c - S_R)}{\gamma R_R}. \quad (19)
$$

The quantity $\overline{F}_{rs}/R_R$ is simply the time scale to dry out a saturated atmosphere with a steady rain equal to the radiative-convective equilibrium rainfall rate. Numbers typical of the tropics yield 15 d for this ratio. (For a similar calculation, see Grabowski and Moncrieff (2004).) The factor $S_c - S_R \approx 0.06$, so the combination $\overline{F}_{rs}(S_c - S_R)/R_R \approx 1$ d. Depending on the value of the NGMS, $t_0$ will be somewhat greater than this value, perhaps 3 d if, as we find later, $\gamma \approx 0.5$.

In a steady state, the dimensionless precipitation rate equals the dimensionless forcing:

$$
\alpha = \alpha_0 = \Delta \phi \quad \text{(steady state)}. \quad (20)
$$

However, in the non-steady state case the rainfall rate relaxes to a steady state value as long as the NGMS...
3. Vorticity dynamics

The vorticity equation in isobaric coordinates as expressed by Haynes and McIntyre (1987) and used by Raymond et al. (1998) in the study of east Pacific tropical storms is

\[
\frac{\partial \zeta_a}{\partial t} + \nabla \cdot (\mathbf{u} \zeta_a + \mathbf{k} \times \mathbf{F}^*) = 0
\]

(22)

where the absolute vorticity is defined

\[
\zeta_a = f + \zeta = f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},
\]

(23)

\(f\) being the Coriolis parameter, and where

\[
\mathbf{F}^* = \mathbf{F} - \alpha \frac{\partial \mathbf{u}}{\partial p}.
\]

(24)

The viscous and turbulent force per unit mass acting on the air is \(\mathbf{F} = (F_x, F_y, 0)\). This force arises from the vertical eddy transport and diffusion of horizontal momentum and its vertical integral must be proportional to the surface stress \(\mathbf{T}\):

\[
\mathbf{F} = \mathbf{T}.
\]

(25)

We now divide the horizontal velocity into barotropic solenoidal and baroclinic parts as in Section 2, \(\mathbf{u} = \mathbf{U} + \mathbf{u}_t\), and use the vector identity \(\nabla \cdot (\mathbf{k} \times \mathbf{F}^*) = -\mathbf{k} \cdot (\nabla \times \mathbf{F}^*)\) to rewrite Equation (22) as

\[
\frac{d \zeta_a}{dt} + \nabla \cdot (\mathbf{u} \zeta_a) - \frac{\partial F_x^*}{\partial x} + \frac{\partial F_y^*}{\partial y} = 0.
\]

(26)

Within a weak disturbance such as a tropical depression, the absolute vorticity \(\zeta_a\) is unlikely to deviate much from the environmental value in the region \(\zeta_{se}\), which means that we can approximate the second term in the above equation by \(\nabla \cdot (\mathbf{u} \zeta_a) \approx \zeta_{se} \nabla \cdot \mathbf{u}_t\). In the absence of significant relative vorticity we would have \(\zeta_{se} \approx f\) where \(f\) is the Coriolis parameter, but we wish to retain the possibility that the disturbance is developing in a high vorticity region.

We wish to consider the behaviour of the vorticity at low levels in an incipient cyclone, as the development of a low-level circulation is the central feature of tropical cyclogenesis. We also wish to relate the low-level convergence to the rainfall rate. One way to meet both of those goals is to make a weighted average of Equation (26) in pressure with the weighting function being the ambient mixing ratio profile \(r_0(\zeta_a)\). Multiplying Equation (26) by \(r_0\), integrating over pressure, and dividing by \(g r_0\) results in

\[
\frac{d \zeta_a}{dt} = \frac{\zeta_{se} X}{r_0} + \frac{\partial \tilde{F}_x}{\partial x} - \frac{\partial \tilde{F}_y}{\partial y}
\]

(27)

where the moisture-weighted average

\[
\bar{Z} \equiv \int_{p_s}^{p_b} r_0 Z dp \int \int_{p_s}^{p_b} r_0 dp = \frac{1}{g r_0} \int_{p_s}^{p_b} r_0 Z dp
\]

(28)

for any variable \(Z\) and where the moisture convergence \(X\) is approximated by \(-\nabla \cdot (r_0 \mathbf{u}_t)\).

In computing the moisture convergence term we have assumed that \(\zeta_{se}\) does not vary much with height over the lower troposphere where the mixing ratio is large. For a vortex of horizontal size \(L\) this is a reasonable assumption if the vortex is in a balanced state and if the Rossby penetration depth \(L_f/N\) is greater than the scale height of the water vapour mixing ratio. \(N\) is the typical Brunt-Väisälä frequency.) In the tropics we typically have penetration depths of the order \(L/300\), so systems with a horizontal scale exceeding 900 km will have penetration depths exceeding the typical 3 km scale height of water vapour.

Neglecting the contribution of the term \(-\omega (\partial \mathbf{u} / \partial p)\) to \(\mathbf{F}^*\) in Equation (24) for now (see the end of Section 5), we consider the contribution due to the eddy and diffusive transport of momentum. We note that

\[
\tilde{F} = \frac{1}{g r_0} \int_{p_s}^{p_b} r_0 \mathbf{F} dp \equiv \frac{\tilde{r}_0}{g r_0} \int_{p_s}^{p_b} \mathbf{F} dp = \frac{\tilde{r}_0 \mathbf{T}}{r_0}
\]

(29)

where \(\tilde{r}_0\) is the pressure average of \(r_0\) weighted by the profile of the frictional force. Technically, \(\tilde{r}_0\) is a tensor, but as long as the vertical structure of \(F_x\) and \(F_y\) are similar, then this quantity can be represented as a scalar. If surface friction is deposited in the lowest layer of the atmosphere, then \(\tilde{r}_0\) equals the surface mixing ratio. For deeper distributions, \(\tilde{r}_0\) is correspondingly less and the spinwind tendency due to surface friction is weaker. Equation (27) thus simplifies to

\[
\frac{d \zeta_a}{dt} = \frac{\zeta_{se} X}{r_0} + \frac{\partial T_y}{\partial x} - \frac{\partial T_x}{\partial y}.
\]

(30)
We now identify $\mathbf{U}$ as the solenoidal part of $\mathbf{u}$. Since Equation (28) weights the vertical average strongly towards low levels where the mixing ratio is largest, we think of $\mathbf{U}$ as the non-divergent part of the low-level flow. We also note that

$$\tilde{\zeta}_a = f + \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}$$

(31)

where $\mathbf{U} = (U_x, U_y)$. Since $\mathbf{U}$ is divergence-free, we can define a stream function $\psi$ such that

$$U_x = \frac{\partial \psi}{\partial y}, \quad U_y = \frac{\partial \psi}{\partial x}.$$  

(32)

Combining this with Equation (31) results in an inversion relation

$$\nabla^2 \psi = \tilde{\zeta}_a - f$$

(33)

between the low-level vorticity and stream function.

If convection is active in the lowest few kilometres of the atmosphere, then there will be a tendency to homogenize vertically the horizontal velocity in this layer. In addition, relaxation towards a balanced state will have a similar effect on the vorticity if the lateral dimensions of the incipient cyclone are sufficiently large, as discussed above. To some degree of approximation, $\mathbf{U} = \mathbf{u}$ then becomes not only a weighted average of the low-level wind, but approximately equal to the actual wind throughout this layer, or at least to the non-divergent part of it. Thus, we can approximate bulk surface fluxes using $\mathbf{U} = |\mathbf{U}|$ for the near-surface wind. For surface evaporation and moist entropy flux we therefore assume

$$F_{rs} = \rho_s C \Delta r \mathbf{U}$$

(34)

$$F_{es} = \rho_s C \Delta s \mathbf{U}$$

(35)

where $\rho_s$ is the surface air density, $C \approx 0.001$ is an exchange coefficient, $\Delta r$ is the difference between the saturated mixing ratio at the sea surface temperature and pressure and the actual mixing ratio in the atmospheric boundary layer and $\Delta s$ is the equivalent difference for moist entropy. Similarly, the surface stress is

$$\mathbf{T} = -\rho_s C \mathbf{U} \mathbf{U}.$$  

(36)

In the present work we assume that the surface sensible heat flux is small over the oceans, which means that the evaporation rate and moist entropy flux are approximately related by $L F_{rs} \approx T_k F_{es}$ and therefore $L \Delta r \approx T_k \Delta s$. We also assume that $\Delta r$, $\Delta s$, and $C$ are independent of wind speed.

Equation (30) tells us that the vorticity tendency averaged over the layer of high moisture values is composed of two competing components, a part associated with moisture convergence, which tends to increase the vorticity, and a part associated with surface friction, which tends to decrease it. This by itself is no surprise. However, the analysis of the previous section provides a way for computing the moisture convergence $X$. In particular, Equation (11) gives us $X$ in terms of the rainfall rate and surface fluxes as long as the evolutionary time scale of the incipient cyclone is long compared to the moisture adjustment time scale, which we earlier estimated to be in the range $8 \text{ h} - 3 \text{ d}$. The rainfall rate is given by Equation (14) in terms of the precipitable water $\tilde{T}$, and the evolution of the precipitable water is governed by Equation (9) or Equation (10). The system is thus closed if the NGMS and the surface and tropopause moist entropy fluxes are known.

4. Determination of NGMS

In this section we use the cloud-resolving cumulus ensemble model of Raymond and Zeng (2005) (see also Derbyshire et al., 2004; Mapes 2004) to estimate the gross moist stability of deep convection. In this model, clouds interact with their surroundings according to the ideas of Sobel and Bretherton (2000). In particular, large-scale circulations are assumed to act to disperse buoyancy anomalies over large areas and thus keep the mean virtual temperature profile of the computational domain close to that of the surrounding environment. This action is approximated in the model by Newtonian relaxation of the virtual temperature profile toward that of a reference profile which represents the atmosphere surrounding the convection. The strength of the cooling needed to do this is diagnosed from the relaxation, and the mean vertical velocity required to produce this cooling by dry adiabatic lifting is then inferred from the cooling rate and the dry static stability. This imagined vertical velocity and the associated horizontal flow demanded by mass continuity do not actually exist in the model since cyclic lateral boundary conditions are imposed. However, their effects are included via the imposition of moist and moist entropy source terms consistent with this flow. Sobel et al. (2001) refer to the approximation implicit in this approach as the weak temperature gradient (WTG) approximation and we denote the above imaginary vertical velocity as the WTG vertical velocity. For shorthand, we refer to the cumulus ensemble model employing this approximation as the WTG model.

In a statistically-averaged sense, the characteristics of the convection and the precipitation rate in the WTG model are a function of the reference profiles of temperature, humidity, wind, the sea surface temperature and the imposed mean surface wind. Increasing either the sea surface temperature or the surface wind increase surface heat and moisture fluxes, and hence the precipitation rate.

We have yet to explore the response of the convection to a wide variety of reference profiles. In this paper we take as a reference profile the radiative-convective equilibrium profile resulting from an imposed surface wind of $5 \text{ m s}^{-1}$. However, much remains to be done in exploring the full range of possible driving conditions and the results reported here are preliminary.
In the formal derivation of the WTG model governing equations, we split the horizontal and vertical velocity fields into small-scale cyclic and large-scale non-cyclic parts, \( \mathbf{v} = \mathbf{v}_c + \mathbf{v}_nc \) and \( w = w_c + w_{nc} \), with the small-scale, cyclic flow computed by the cumulus ensemble model. The large-scale part represents the above-described interaction of the model with the surrounding environment. We further assume that \( w_{nc} \) depends only on \( z \) and \( t \). By mass continuity, \( \mathbf{v}_{nc} \) is therefore linear in \( x \) and \( y \).

The convective-scale equations for specific moist entropy \( s \) and total cloud water mixing ratio \( r \) are
\[
\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \mathbf{v}_c) + \frac{\partial \rho s w_c}{\partial z} = \rho S_c - \nabla \cdot (\rho s \mathbf{v}_{nc}) - \frac{\partial \rho s w_{nc}}{\partial z} = \rho (S_c - E_c) \tag{37}
\]
and
\[
\frac{\partial \rho r}{\partial t} + \nabla \cdot (\rho r \mathbf{v}_c) + \frac{\partial \rho r w_c}{\partial z} = \rho S_r - \nabla \cdot (\rho r \mathbf{v}_{nc}) - \frac{\partial \rho r w_{nc}}{\partial z} = \rho (S_r - E_r). \tag{38}
\]

The terms involving the non-cyclic, large-scale velocities are encapsulated into \( E_c \) and \( E_r \). These terms represent the rate at which the large-scale environment removes moist entropy and moisture from the convective domain, and as we now show, are precisely what is needed to compute the NGMS. The terms \( S_c \) and \( S_r \) represent the local sources of entropy and moisture associated with radiation, precipitation, subgrid-scale eddy fluxes, etc.

Assuming that the density \( \rho \) is a function only of height and applying the pressure integral operator Equation (3) to the definition of \( E_c \), we find
\[
E_c = \nabla \cdot (s \mathbf{v}_{nc}) + \frac{1}{\rho} \frac{\partial \rho s w_{nc}}{\partial z} = \nabla \cdot (s \mathbf{v}_{nc}) + \frac{\partial}{\partial \rho} \rho \frac{\partial s w_{nc}}{\partial \rho} = \frac{\partial}{\partial \rho} (\rho s w_{nc}) \tag{39}
\]
where we recall that normally \( E_r < 0 \). The model of Raymond and Zeng (2005) is written in terms of the equivalent potential temperature \( \theta_e \) rather than the moist entropy, but using a simplified relation between the two, \( s = C_p \ln(\theta_e/T_R) \), we easily find that \( E_{the} \) is the analog to \( E_r \) in the governing equation for equivalent potential temperature and where the angle brackets indicate a horizontal average over the domain of the cumulus ensemble model. Using arguments of Raymond and Zeng (2005) translated to the notation of this paper, we find that
\[
E_{the} = \frac{(\theta_e - \theta_{ex})}{\rho} \frac{\partial \rho w_{nc}}{\partial z} + w_{nc} \frac{\partial (\theta_e)}{\partial z}. \tag{41}
\]

where \( \theta_{ex} \) equals the reference profile of equivalent potential temperature in layers of net convergence (\( \partial w_{nc}/\partial z > 0 \)) and is equal to \( (\theta_e) \) in divergent layers. The first term in Equation (41) represents entrainment of environmental air into the convective region, while the second represents the effect of mean vertical advection. A similar expression exists for \( E_r \).

We have computed the NGMS for a number of simulations with our WTG model. As in the case of Raymond and Zeng (2005), we make two-dimensional computations on a small, 50 km domain. Expansion of the domain to 512 km changes the results very little. The radiation scheme of Raymond (2001) is used in place of fixed radiative cooling and particle fall speeds above the freezing level are reduced from 5 m s\(^{-1}\) to 1 m s\(^{-1}\) to emulate the behaviour of snow. Also, \( E_{the} \) is calculated directly via Equation (41) rather than indirectly, with more accurate results. Apart from these exceptions, the computations are identical to those of Raymond and Zeng (2005).

Figure 4 shows how the NGMS varies with wind speed and sea surface temperature (SST). Values of NGMS typically range from roughly 0.4 to 0.55 in these calculations. At fixed SST a minimum in NGMS occurs near a wind of 10 m s\(^{-1}\). At a fixed wind speed of 7 m s\(^{-1}\), the NGMS initially decreases with an increase in SST, though this decrease is partially reversed for SST increments greater than 1–2 K above the reference value.

As a check of our calculation of NGMS, we also solve Equation (13) for \( \gamma \) in the steady state
\[
\gamma_{steady} = \frac{T_R(F_{ex} - F_a + G)}{XL} = \frac{T_R(F_{ex} - F_a + G)}{(R - F_a)L}, \tag{42}
\]
and compare this result with that obtained using Equation (40) and find satisfactory agreement. This equation also provides a way to determine the NGMS from observations. In situations of intense precipitation, the steady-state assumption is not restrictive according to our theory as the humidity relaxation time is typically shorter than the dynamical time scale under these conditions.

The main conclusion from the above results is that radiative-convective equilibrium reference profiles yield NGMS values of the order 0.5. The simulations are of...
course very limited at this point, with a small, two-dimensional domain and no wind shear. Much needs to be done to refine and expand these calculations and to test them against observations. In particular, more general reference profiles differing from radiative-convective equilibrium need to be explored. Until more experience is gained with the WTG model, the above results must be regarded as tentative. However, as we shall see, the qualitative aspects of the theory do not depend on the precise value of the NGMS.

5. Results for an axially symmetric vortex

Insight as to how the present theory works may be obtained in a highly simplified context. We integrate Equation (30) over a circular area \( A \) of radius \( a \) and assume axial symmetry in all quantities. We also assume provisionally that the moisture convergence is well approximated by its steady-state value given by Equation (13), resulting in

\[
\frac{d\Gamma}{dt} = \frac{\pi a^2 \xi_e T_R (F_{\text{es}} - F_{\text{et}} + \bar{G})}{\gamma L \tau_0} - \frac{2\pi a \tau_0 |\mathbf{T}|}{\tau_0},
\]

where \( \Gamma \) is the relative circulation around the periphery of the area \( A \). The angle brackets here indicate an average over the area. In addition we assume that the flow is primarily tangential to the periphery, so that the tangential component of \( \mathbf{T} \) is approximately given by \( |\mathbf{T}| = \rho_s C U^2 \).

For the purposes of this analysis we ignore the irreversible generation of entropy and further assume that the radiative moist entropy sink is constant, i.e. independent of wind speed, which is tantamount to ignoring cloud–radiation interactions. The quantity inside the angle brackets in Equation (43) is therefore equal to the deviation of the surface flux from radiative-convective equilibrium conditions in which the imposed wind speed is \( U_0 \). Using Equation (35), we therefore have

\[
(F_{\text{es}} - F_{\text{et}} + \bar{G}) \approx \rho_s C \Delta S (U - U_0).
\]

The value of \( U_0 \) is uncertain but is likely to be approximately equal to the average surface wind speed in the tropics. As in Section 4, we assume \( U_0 = 5 \text{ m s}^{-1} \).

We interpret \( U \) as the surface wind speed on the periphery of the area \( A \), so that \( \Gamma = 2\pi a U \), and assume that the mean wind speed in the interior of the area is comparable to the wind speed on the periphery \( U \). This assumption is clearly a serious underestimate for a tropical storm, but perhaps is not so bad for a depression, which lacks a strong central vortex. Under these conditions, Equation (43) becomes

\[
\frac{dU}{dt} = \frac{U_c(U - U_0) - U^2}{D},
\]

where the critical velocity \( U_c \) is given by

\[
U_c = \frac{a \xi_e \Delta r}{2\gamma \tau_0},
\]

and the parameter \( D \) is

\[
D = \frac{\tau_0}{\rho_s C \tau_0}.
\]

Though the NGMS was shown to change somewhat with wind speed, let us assume here that it is constant. The numerical simulations in the previous section suggest a value near \( \gamma = 0.5 \). Then, for any intensification to be possible, we must have \( U_c > 4U_0 \). In this case, \( dU/dt = 0 \) for two particular velocities,

\[
U_{1,2} = [U_c \pm (U_c^2 - 4U_c U_0)^{1/2}] / 2,
\]

(47)

where \( U_1 < U_2 \) and \( dU/dt > 0 \) only for \( U_1 < U < U_2 \), as illustrated in Figure 5. If \( U_c < 4U_0 \), decay occurs for all values of \( U \).

The condition \( U_c > 4U_0 \) can be recast as a condition on the system radius \( a \):

\[
a > \frac{8\gamma \tau_0 U_0}{\xi_e \Delta r} \equiv a_c.
\]

(48)

The intensification of a vortex in this model thus depends on the vortex’s initial size and intensity. If the radius \( a < a_c \), then decay is inevitable. If \( a > a_c \), then for initial peripheral tangential velocities \( U < U_1 \) there is decay, while intensification up to \( U = U_2 \) occurs for \( U > U_1 \). For \( U > U_2 \), the vortex decays to \( U = U_2 \).

If \( \gamma = 0.5 \), \( U_0 = 5 \text{ m s}^{-1} \), \( \rho_s = 1.2 \text{ kg m}^{-3} \), \( \xi_e = f = 3 \times 10^{-5} \text{ s}^{-1} \), \( \tau_0 = 12 \text{ g kg}^{-1} \), \( \tau_0 = 50 \text{ kg m}^{-2} \), and \( \Delta r = 5 \text{ g kg}^{-1} \), then \( a_c \approx 1600 \text{ km} \) and \( D \approx 3500 \text{ km} \). If, as an example, we take \( U_c = 6U_0 = 30 \text{ m s}^{-1} \), then \( U_{1,2} = (3 \pm 3^{1/2})U_0 \), or 6.3 m s\(^{-1}\) and 23.7 m s\(^{-1}\). The specified value of \( \Delta r \) is that for 30°C.
Figure 5. Schematic illustration of growth rate for an axially symmetric system. Point $U_1$ is a point of unstable equilibrium, while $U_2$ exhibits stable equilibrium. The heavy arrows indicate the direction of evolution of $U$ with time.

Sea surface temperature, 29°C air temperature and 80% relative humidity.

The time constant for intensification can now be determined. Equation (44) can be rewritten in terms of $U_1$ and $U_2$ as

$$dU/dt = (U - U_1)(U_2 - U)/D.$$  (49)

The maximum growth rate occurs when $U = (U_1 + U_2)/2$, and for this value the time constant for growth is

$$\tau = \left(\frac{1}{U} \frac{dU}{dt}\right)^{-1} = 2D(U_1 + U_2)/(U_2 - U_1)^2.$$  (50)

Using the above determined values, we find that $\tau \approx 8$ days. This growth time is longer than the range of possible moisture relaxation time constants, which implies that the steady state assumption for moisture convergence used in Equation (43) is justified.

Returning to the question of neglecting $-\omega(\partial u/\partial p)$ in $F^*$ in Equation (24), by virtue of Stokes’ theorem, this term contributes only to the periphery of the area $A$. If this area is large enough to bound the region of convection, then the periphery of $A$ will see little or no convection, and we can assume that $\omega \approx 0$ there. On the other hand, if the periphery is embedded in the convection, then this term must be considered. Tropical depressions tend to be cold-core at low levels, which from thermal wind considerations tells us that $\partial u_t/\partial p < 0$ where $u_t$ is the component of the velocity in the direction of a cyclonic traverse around the area $A$. The contribution of this term will therefore be to retard spinup in a region of upward motion ($\omega < 0$).

6. Conclusions

In this paper a theory is developed for the early stages of the spinup of tropical cyclones. It is limited to the early development stage in order to avoid complications associated with feedback of the developing vortex on the vortex environment, i.e. nonlinear effects. The theory incorporates recent advances in our understanding of the thermodynamic forcing of tropical convection and precipitation. Beyond that, it bears a strong conceptual resemblance to the simple cyclone model of Emanuel (1989, 1995), although the scales are very different.

The key assumptions of the theory are: (1) cyclogenesis results when the spinup tendency of low-level convergence forced by latent heat release associated with precipitation production exceeds the spindown tendency of surface friction, and (2) precipitation is governed by the surface moist entropy flux with a small lag (several hours to a few days) and is modulated in intensity by the normalized gross moist stability.

Though numerous uncertainties exist in the details of this theory, its qualitative basis is reasonably robust. The first of the above points follows directly from the vorticity equation in flux form (Haynes and McIntyre, 1987). Emanuel’s (1989) model makes the second of the above assumptions and the numerical results of Craig and Gray (1996) support the hypothesis that convection is forced by surface fluxes rather than Ekman pumping.

Our focus on the role of precipitation rather than convection in cyclogenesis is appropriate because of the closer relationship of the former to low-level convergence. It is possible to have strong convection without significant low-level convergence if the convection produces strong downdrafts, as noted by Emanuel (1989, 1995). Such convection does not produce low-level spinup. However, heavy precipitation without strong low-level convergence is considerably less likely, as most of the moisture for intensely precipitating systems is drawn in laterally from the moisture-rich lower troposphere.

The lag mentioned above in the second key assumption is the time required for the convective environment to moisten or dry in response to a change in the surface moist entropy flux. Ample evidence exists to suggest that the production of precipitation is exceptionally sensitive to the environmental relative humidity (Bretherton et al., 2004; Derbyshire et al., 2004; Raymond and Zeng 2005). This considerably shortens the lag compared to simple estimates which do not take this factor into account; the exact form of rainfall rate versus saturation fraction shown in Figure 3 is not essential to the theory.

The idea of relating the convective mass flux and precipitation to the surface entropy flux via the gross moist stability is well established (Neelin and Held, 1987; Raymond, 2000), although the means for estimating the NGMS in different circumstances are relatively new and untested. Recent work not presented here suggests that moistening and stabilization of the environment act to decrease the NGMS by a factor of 2 to 3 relative to the values reported here.

More observations are needed in developing (and non-developing) cyclone precursor disturbances to test the present theory. In particular, measurements of NGMS under a variety of circumstances are needed, as are observations of the vertical distribution of momentum.
fluxes associated with surface drag. We believe that a well-tested cumulus ensemble model run in WTG mode will eventually constitute a reliable tool for estimating the NGMS.

For an axially symmetric vortex, the theory predicts that spinup can only occur if the radius of the vortex is greater than some critical value. Smaller vortices decay according to the theory. For parameter values typical of the tropical environment, the minimum radius is estimated to be 1600 km. Smaller NGMS values would decrease the critical radius. For radii greater than this value, a finite threshold for further growth exists in the tangential velocity. Vortices weaker than this threshold decay, while those stronger than the threshold intensify further.

The critical radius governs spinup as, with other things being equal, the integrated precipitation rate and hence the spinup tendency over the area being considered scale with the area. The frictional spinup tendency scales with the circumference, and hence the square root of the area. Increasing the radius therefore increases the spinup-spindown ratio; the critical radius is that for which this ratio equals unity.

The threshold for growth exists because the ratio of spinup and spindown tendencies $U_0(U - U_0)/U^2$ increases with surface wind speed, exceeding unity at the threshold. Recall that $U_0$ is the wind speed associated with radiative-convective equilibrium and $U_c$ is a constant proportional to disturbance radius. The spinup tendency scales as $U - U_0$ since rainfall, which varies with the surface moist entropy flux and hence the wind speed, must exceed the radiative-convective equilibrium value for moisture convergence to occur.

The surface friction increases as the square of the wind speed due to the form of the bulk formula for surface stress. Because of this factor, a limit is eventually reached where the frictional spinup tendency exceeds the convergence-driven spinup tendency, and the cyclone cannot intensify beyond this limit. However, this maximum in wind speed is well into the nonlinear regime where the detailed assumptions of the theory are invalid.

One factor not explored in our axisymmetric vortex model is the tendency discovered by Emanuel (1989) and examined further by Frisius (2006) for convection to expand away from the centre of the cyclone in a concentric ring, thus halting cyclogenesis. This phenomenon is not addressed in our axisymmetric control volume example, but it could be accounted for in a spatially resolved version of our model by a radial gradient in the NGMS, in analogy to the precipitation efficiency gradient in the articles cited above.

Relaxation of some of the assumptions behind the theory allows one to explore qualitatively certain behaviours not strictly encompassed by the theory as it stands. If the atmosphere is sheared, one might expect that differentially advected dry air could make its way into the region of convection, with resulting suppression of precipitation. This would manifest itself as an increase in the NGMS and a corresponding decrease in the potential for intensification. Alternatively, if the system is embedded in a region of cyclonic relative vorticity, then ingested air would have absolute vorticity larger than $f$ and the spinup would be enhanced. The theory is thus in essential agreement with the ideas of Challa and Pfeffer (1980) and Pfeffer and Challa (1981).

Also interesting would be to apply the theory to non-axisymmetric situations. Equations (10) for humidity and (30) for vorticity along with associated diagnostic relations form a closed system assuming that the NGMS can be determined. These equations could form the basis for studying the intensification of easterly waves and related disturbances into tropical depressions. Such an effort would be highly worthwhile, given that this may be the least-understood phase of tropical cyclogenesis.

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