The Electromagnetic Radiation from a Finite Antenna

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Textbooks rarely give time-domain solutions to antenna problems. For the case of a finite linear antenna along which a fixed current waveform propagates, we present analytical time-domain solutions for the electric and magnetic radiation (far) fields. We also give computer solutions for the total (near and far) fields. The current waveform used as an example in the computer calculations approximates that of a lightning return-stroke, a common geophysical example of the type of radiation source under consideration.

INTRODUCTION

Most textbooks on classical electromagnetic theory^{1–6} do not consider the problem of radiation from finite antennas in the time-domain. The conventional treatment of radiation phenomena in free space begins with a discussion of the time-dependent Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$
 (1)

and their general solutions in terms of retarded scalar and vector potentials

$$\mathbf{E} = -\nabla \phi - \partial \mathbf{A}/\partial t \tag{2}$$

$$\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}.\tag{3}$$

where

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}', t - R/c)}{R} dV' \qquad (4)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{VI} \frac{\mathbf{J}(\mathbf{r}', t - R/c)}{R} dV' \qquad (5)$$

and

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \tag{6}$$

for the geometry sketched in Fig. 1. In addition, a polarization potential or Hertz vector is usually defined and shown to satisfy a vector wave equation. From this Hertz vector, electric and magnetic fields can be obtained by taking appropriate derivatives. The reader is assumed to be familiar with this development.

Next, as an application of this formalism, most texts find the fields from an infinitesimal dipole source oscillating at some particular angular frequency. Finite antennas are treated by a spatial integration of the dipole solutions over the antenna. From this frequency domain analysis, time-domain solutions may be obtained using the appropriate Fourier integrals, but this approach is quite cumbersome and rarely used in textbooks. A further disadvantage of this technique is that most physical insight is obscured.

The primary purpose of this paper is to describe a finite antenna problem which can be solved analytically in the time domain. For the case of a finite linear antenna along which a fixed current

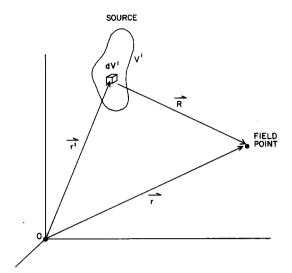


Fig. 1. The geometry for general solutions of the time-dependent Maxwell equations.

waveform propagates, we present analytical solutions for the electric and magnetic radiation (far) fields. Further, we give computer solutions for the total (near and far) fields for the special case of a triangular current waveform.

THE PROBLEM

Consider a straight vertical antenna of height Habove a perfectly conducting ground plane, as shown schematically in Fig. 2. Boundary conditions at the plane are satisfied by adding the image antenna shown dashed in Fig. 2. The radius of the antenna cross-section is assumed to be very small compared to the wavelength of any radiation under consideration. The current at any height will be assumed to be some arbitrary continuous function, i(z, t), which is zero everywhere at t=0. With the geometry of Fig. 2, $\mathbf{r}'=z\hat{a}_z$, and the differential current source $\mathbf{J}(\mathbf{r}', t-R/c)dV'$ becomes $i(z, t-R/c)dz \hat{a}_z$. Now, the vector potential can be computed using Eq. (5) and B by using (3). The result for a point on the ground plane a distance D from the antenna base is

$$B_{\phi}(D, t) = \frac{\mu_0}{2\pi} \int_0^H \frac{\sin\theta}{R^2} i(z, t - R/c) dz$$

$$+ \frac{\mu_0}{2\pi} \int_0^H \frac{\sin\theta}{cR} \frac{\partial i(z, t - R/c)}{\partial t} dz. \quad (7)$$

Details of this computation are given in the Appendix. The first term on the right of Eq. (7) is the induction field and the second term the radiation field.

The electric field can be found from Eq. (2) starting with A and using Eq. (6) in the form

$$\phi(\mathbf{R}, t) = -c^2 \int_0^t \nabla \cdot \mathbf{A} d\tau. \tag{8}$$

The result for a point on the plane a distance D from the antenna is

$$E_{z}(D, t) = \frac{1}{2\pi\epsilon_{0}} \left[\int_{0}^{H} \frac{(2-3\sin^{2}\theta)}{R^{3}} \right]$$

$$\times \int_{0}^{t} i(z, \tau - R/c) d\tau dz$$

$$+ \int_{0}^{H} \frac{(2-3\sin^{2}\theta)}{cR^{2}} i(z, t - R/c) dz$$

$$- \int_{0}^{H} \frac{\sin^{2}\theta}{c^{2}R} \frac{\partial i(z, t - R/c)}{\partial t} dz \right]. \quad (9)$$

Details of this computation are also given in the Appendix. The first term on the right of Eq. (9) is the electrostatic field, the second term the induction or intermediate field, and the third term the radiation field.

Figure 3 shows computer solutions to Eqs. (7) and (9) for the special case where a current pulse propagates up the antenna as it would a transmission line, i.e.,

$$i(z, t) = i(t-z/v),$$
 (10)

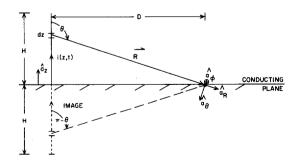


Fig. 2. A straight vertical antenna of height H above a perfectly conducting ground plane.

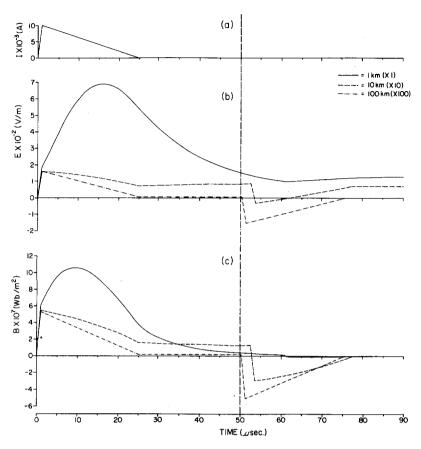


Fig. 3. Computer solutions to Eqs. (7) and (9) for the case of the current pulse shown in (a) propagating up a 4-km antenna at a constant speed of 8×10^7 m/sec. The electric field intensities which would be observed at distances of 1, 10, and 100 km are shown in (b) and the magnetic fields in (c). The time axis for (b) and (c) is t - D/c, and the dashed vertical line shows the time the current pulse reaches the top of the antenna. Positively plotted electric field is in the negative z-direction.

where v is the velocity of the current pulse and is assumed to be constant. These calculations assume that the current terminates at H and that no current is reflected downward from the top of the antenna. The height, H=4 km, speed of propagation, $v=8\times 10^7$ m/sec, and input current waveform (a) in Fig. 3 have been chosen to resemble a return stroke in a lightning discharge to ground, which is a common geophysical example of this type of radiation source. Further discussions of the theory and the application to lightning discharges are given in a series of papers by Uman and co-workers.⁸⁻¹³

In Fig. 3, the radiation field terms dominate the E and B waveforms at the initial times for all distances because the initial time derivative of the current is large. At 1 km, the near field terms produce a large hump in E_z which represents

the integrals of the current being modulated by the geometrical factors as the current propagates upward. The magnetic field at 1 km is dominated by the induction field term. As the distance D increases, the R^{-1} dependence causes the radiation fields to dominate the R^{-2} and R^{-3} terms. The final value of the magnetic field is zero at all distances because the final current and time-rate-of-change of current have been chosen to be zero. The electric field, on the other hand, does have a finite final value because the current waveform results in the effective transfer of a point charge

$$Q = \int_0^{\infty} idt$$

from ground to the top of the antenna which creates a dipole field given by the well-known

relation

$$E_z(D, \infty) = -\; \frac{1}{2\pi\epsilon_0} \frac{QH}{(H^2 + D^2)^{3/2}} \eqno(11)$$

It should be noted in Fig. 3 that the shape of the electric and magnetic field pulses at large distances is the same as the shape of the current pulse for $t \le H/v + D/c$. This is a general, and perhaps unexpected, result which we now will prove analytically. Consider the case where $D\gg H$. Then $\theta \simeq \pi/2$, $R\simeq D$, the radiation field terms dominate, and Eqs. (7) and (9) become

$$B_{\phi} \simeq B_{\rm RAD}(D, t) = \frac{\mu_0}{2\pi} \frac{1}{cD} \int_0^H \frac{\partial i(t - z/v - D/c)}{\partial t} dz, \tag{12}$$

$$E_{s} \simeq E_{RAD}(D, t) = -\frac{1}{2\pi\epsilon_{0}} \frac{1}{c^{2}D}$$

$$\times \int_{0}^{H} \frac{\partial i(t - z/v - D/c)}{\partial t} dz. \quad (13)$$

Now, since v is constant,

$$\frac{\partial i(t-z/v)}{\partial t} = -v \frac{\partial i(t-z/v)}{\partial z} \tag{14}$$

and Eqs. (12) and (13) can be written

$$B_{\rm RAD}(D,\,t) = -\,\frac{\mu_0 v}{2\pi c D} \int_0^H \frac{\partial i (t-z/v-D/c)}{\partial z} \,dz \eqno(15)$$

$$E_{\rm RAD}(D,t) \equiv + \frac{\mu_0 v}{2\pi D} \int_0^H \frac{\partial i (t-z/v-D/c)}{\partial z} \, dz, \eqno(16)$$

Eqs. (15) and (16) can now be integrated directly to provide

$$B_{\text{RAD}}(D, t) = (u_0 v / 2\pi c D)$$

$$\times \left[i(t - D/c) - i(t - H/v - D/c) \right] \quad (17)$$

$$E_{\rm RAD}(D, t) = -(\mu_0 v/2\pi D)$$

$$\times [i(t-D/c)-i(t-H/v-D/c)].$$
 (18)

Since
$$i(\tau) = 0$$
 for all $\tau \le 0$, Eqs. (17) and (18)

become simply

$$B_{\rm RAD}(D, t) = (\mu_0 v / 2\pi c D) i (t - D/c)$$
 (19)

$$E_{\rm RAD}(D, t) = -(\mu_0 v/2\pi D) i(t - D/c)$$
 (20)

for t < H/v + D/c.

Equations (19) and (20) can be derived in a slightly different manner by noting that for $t \le H/v + D/c$ the upper limit to the integrals in Eqs. (15) and (16) can be replaced by the maximum height from which radiation can be seen at distance D at time t, $z_{\max} = v(t - D/c)$. Above z_{\max} , the current is zero, since the front of the current pulse has not passed z_{\max} at time t - D/c. Therefore, Eq. (16) becomes

$$E_{\rm RAD}(D,t) = \frac{\mu_0 v}{2\pi D} \int_{z=0}^{z_{\rm max}=v(t-D/c)} \frac{\partial i(t-z/v-D/c)}{\partial z} \, dz$$

 \mathbf{or}

$$E_{ extbf{RAD}}(D, t) = -\left(\mu_0 v/2\pi D\right) i(t-D/c)$$
 ;
$$t \leq H/v + D/c$$

which is the same as (20).

As is evident from Eqs. (19) and (20), the magnetic and electric radiation fields have the same shape as the current pulse as long as the pulse has not reached the top of the antenna. The minus sign in Eq. (20) refers only to the fact that the electric radiation field in Eq. (9) points in a direction opposite the current flow. In a practical situation, if the radiation fields are measured together with D and v, then the current producing these fields can be easily determined.

For the case t>H/v+D/g, both terms in Eqs. (17) and (18) must be retained. The second terms are proportional to -i(t-H/v-D/c) and lead to the "mirror image" of the initial field peak seen at large distances in Fig. 3 for times after the current pulse reaches the top of the antenna. The initial radiation field peak arises because a propagating current wave has been turned on at the bottom of the antenna, and the mirror image arises because the same current wave has been turned off at the top. We have treated the antenna as a vertical transmission line terminated in its characteristic impedance at H; that is, we have allowed no current to be reflected downward. It

is a relatively simple matter to include a reflected current in the analysis, if that is desired.

APPENDIX

Consider an infinitesimal vertical dipole of length dz having a current i(z, t) a distance z above a perfectly conducting plane, as shown in Fig. 2. The plane can be replaced by an image dipole a distance z below the plane. The electric and magnetic fields at an observation point on the plane a distance D from the antenna base are the sums of the fields from the real and image dipoles.

The differential magnetic field $d\mathbf{B}$ of the dipole can be determined from the vector potential $d\mathbf{A}$ since

$$d\mathbf{B} = \nabla \times d\mathbf{A}.\tag{A1}$$

The current has only a z-component, and the resulting vector potential has only a z-component, which is given by

$$dA_z = \frac{\mu_0}{4\pi} \frac{i(z, t - R/c)}{R} dz,$$
 (A2)

where R is the distance from the dipole to the point of observation. If we use a spherical coordinate system with an origin at the dipole, then

$$d\mathbf{A} = \frac{\mu_0}{4\pi} \left[i(z, t - R/c) \frac{\cos\theta}{R} \, \hat{a}_R \right]$$

$$-i(z, t-R/c) \frac{\sin\theta}{R} \hat{a}_{\theta} dz$$
, (A3)

where d_R , d_{θ} , and d_{ϕ} are the unit vectors sketched in Fig. 2. Evaluating the curl gives

$$\nabla \times d\mathbf{A} = \frac{\mu_0 dz}{4\pi} \left[-\frac{\sin\theta}{R} \frac{\partial i(z, t - R/c)}{\partial R} \right]$$

$$+\frac{\sin\theta}{R^2}i(z,t-R/c)$$
 δ_{ϕ} , (A4)

and using the identity

$$\frac{\partial i(z, t - R/c)}{\partial R} = -\frac{1}{c} \frac{\partial i(z, t - R/c)}{\partial t}, \quad (A5)$$

we have

$$\begin{split} d\mathbf{B} &= \frac{\mu_0 dz}{4\pi} \sin\!\theta \left[\frac{i(z,\, t\!-\!R/c)}{R^2} \right. \\ &\left. + \frac{1}{cR} \frac{\partial i(z,\, t\!-\!R/c)}{\partial t} \right] \hat{a}_{\phi}. \quad (\mathbf{A6}) \end{split}$$

The expression for the magnetic field from the image dipole will be Eq. (A6) with θ replaced by $\pi-\theta$. Now, the total magnetic field at the observation point is in the ϕ direction (parallel to the plane) and is given by

$$\begin{split} dB_{\phi}(R,\,\theta,\,t) &= \frac{\mu_0 dz}{2\pi} \sin\!\theta \left[\frac{i(z,\,t\!-\!R/c)}{R^2} \right. \\ &\left. + \frac{1}{cR} \frac{\partial i(z,\,t\!-\!R/c)}{\partial t} \right]. \quad (A7) \end{split}$$

The differential electric field due to the source dipole can be determined from Eq. (2) using the $d\mathbf{A}$ from Eq. (A3) and the scalar potential $d\phi$ found by substituting Eq. (A3) in Eq. (8); that is.

$$\begin{split} d\phi(R,\,\theta,\,t) &= \frac{dz\,\cos\!\theta}{4\pi\epsilon_0} \bigg[\frac{1}{R^2} \int_0^t i(z,\,\tau\!-\!R/c)\,d\tau \\ &\quad + \frac{i(z,\,t\!-\!R/c)}{cR}\bigg]\,. \quad (\text{A8}) \end{split}$$

After using Eq. (A5) to convert the spatial derivatives to time derivatives, we obtain the following expression for the differential electric field due to the infinitesimal source dipole,

$$d\mathbf{E}(R,\,\theta,\,t) = \frac{dz}{4\pi\epsilon_0} \left\{ \cos\theta \left[\frac{2}{R^3} \int_0^t i(z,\,\tau - R/c) d\tau \right. \right.$$

$$\left. + \frac{2}{cR^2} i(z,\,t - R/c) \right] \hat{a}_R$$

$$\left. + \sin\theta \left[\frac{1}{R^3} \int_0^t i(z,\,\tau - R/c) d\tau \right. \right.$$

$$\left. + \frac{1}{cR^2} i(z,\,t - R/c) \right.$$

$$\left. + \frac{1}{c^2R} \frac{\partial i(z,\,t - R/c)}{\partial t} \right] \hat{a}_\theta \right\}. \quad (A9)$$

The electric field from the infinitesimal image dipole is given by a similar expression with θ replaced by $\pi-\theta$, except that the unit vectors in the R and θ directions are different for the real and image dipoles. The total electric field will be the vector sum of the fields from the real and the image dipoles and will be in the vertical direction. The horizontal electric field will be zero, since no tangential electric field can exist at the surface of a perfect conductor. The following expressions relate the unit vectors \hat{a}_R , \hat{a}_θ (Fig. 2) and \hat{a}_R^i , \hat{a}_θ^i , the comparable image quantities, to \hat{a}_z , \hat{a}_H , which are unit vectors perpendicular and parallel to the ground plane, respectively:

$$\hat{a}_{R} = -\hat{a}_{z}\cos(\pi - \theta) + \hat{a}_{H}\cos[\theta - (\pi/2)]$$

$$= \hat{a}_{z}\cos\theta + \hat{a}_{H}\sin\theta; \qquad (A10)$$

$$\hat{a}_{\theta} = -\hat{a}_{z}\cos[\theta - (\pi/2)] + \hat{a}_{H}\cos\theta$$

$$= -\hat{a}_{z}\sin\theta + \hat{a}_{H}\cos\theta; \qquad (A11)$$

$$\hat{a}_{R}^{i} = \hat{a}_{z}\cos(\pi - \theta) + \hat{a}_{H}\cos[\theta - (\pi/2)]$$

$$= -\hat{a}_{z}\cos\theta + \hat{a}_{H}\sin\theta; \qquad (A12)$$

$$\hat{a}_{\theta}^{i} = -\hat{a}_{z} \cos[\theta - (\pi/2)] + \hat{a}_{H} \cos(\pi - \theta)$$

$$= -\hat{a}_{z} \sin\theta - \hat{a}_{H} \cos\theta. \tag{A13}$$

Using these identities, we find the sum of the

source and image dipole solutions is

$$\begin{split} dE_z(R,\,\theta,\,t)\,\hat{a}_z \\ &= \frac{dz}{2\pi\epsilon_0} \bigg[\frac{(2\!-\!3\,\sin^2\!\theta)}{R^3} \int_0^t i(z,\,\tau\!-\!R/c)\,d\tau \\ &\quad + \frac{(2\!-\!3\,\sin^2\!\theta)}{cR^2}\,i(z,\,t\!-\!R/c) \\ &\quad - \frac{\sin^2\!\theta}{c^2\!R} \frac{\partial i(z,\,t\!-\!R/c)}{\partial t} \bigg]\,\hat{a}_z \quad (\text{A}14) \end{split}$$

for a point on the ground plane.

The total electric and magnetic fields from the complete vertical antenna of height H are obtained by integrating the infinitesimal dipole solutions Eqs. (A7) and (A14) over the antenna

$$B_{\phi}(D, t) = \int_{0}^{H} dB_{\phi}(R, \theta, t) dz \qquad (A15)$$

$$E_z(D, t) = \int_0^H dE_z(R, \theta, t) dz$$
 (A16)

which prove Eqs. (7) and (9), respectively.

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