Physics 535 - Physics of Lightning
New Mexico Tech

This is one derivation
split over multiple lectures

pages 1-4 Lecture 15
5-8 Lecture 17
9-11 Lecture 20
12-14 Lecture 21

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INTRO TO UMAN 1975

$$\left.\begin{array}{ll} (1) & \nabla\cdot\vec{B} = 0 \leftarrow \rho_m \\ (2) & \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \leftarrow J_m \end{array}\right] \text{"Source Free"}$$

$$\left.\begin{array}{ll} (1) & \nabla\cdot\vec{B} = 0 \\ (2) & \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \\ (3) & \nabla\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \\ (4) & \nabla\cdot\vec{E} = \rho/\epsilon_0 \end{array}\right\} [U1]$$

Uman states:

[U3] $\vec{B} = \nabla\times\vec{A}$

Because $\nabla\cdot(\nabla\times\vec{f}) = 0 \quad \forall \vec{f}$,

making definition U3 automatically gives us (1)

U3 and (2) imply

[U2] $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$

Justification

$$\nabla\times\vec{E} + \frac{\partial}{\partial t}(\nabla\times\vec{A}) = 0 \quad \text{(from (2) and U3)}$$

$$\therefore \nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \quad \text{(interchange } \tfrac{\partial}{\partial t} \text{ \& } \nabla\text{)}$$

For any scalar function χ, $\nabla\times(\nabla\chi) \equiv 0$

$\therefore$ Define χ s.t. $\nabla\chi = \vec{E} + \frac{\partial\vec{A}}{\partial t}$,

$$\frac{\partial\chi}{\partial x} = E_x + \frac{\partial A_x}{\partial t} \qquad \frac{\partial\chi}{\partial y} = E_y + \frac{\partial A_y}{\partial t} \quad \text{etc.}$$

Redefine $\phi = -\chi$

So $-\nabla\phi = \vec{E} + \frac{\partial\vec{A}}{\partial t}$

$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$. This is U2. We have not yet shown that ϕ is electrostatic potential

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{r}\,',t)$$

is solved by

$$\psi(\vec{r},t) = \int \frac{[f(\vec{r}\,',t')]_{ret}}{|\vec{r}-\vec{r}\,'|}\,dV'$$

$$[f(\vec{r}\,',t')]_{ret} = f(\vec{r}\,',t - |\vec{r}-\vec{r}\,'|/c)$$

If we can show

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Then $-4\pi f = -\mu_0 J_x \Rightarrow f = \frac{\mu_0}{4\pi}J_x$

$$\therefore\ A_x(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{J_{x\,ret}}{|\vec{r}-\vec{r}\,'|}\,dV' \quad [U5]$$

Also if

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{-\rho}{\epsilon_0}$$

Then $-4\pi f = \frac{-\rho}{\epsilon_0} \Rightarrow f = \frac{\rho}{4\pi\epsilon_0}$

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_{ret}}{|\vec{r}-\vec{r}\,'|}\,dV' \quad [U4]$$

We will prove this, and along the way require [U6] is true (and that one can require U6)

Begin with 4

$$\nabla \cdot E = \rho/\epsilon_0$$

[U2] $\vec{E} = -\nabla\phi - \dfrac{\partial \vec{A}}{\partial t}$ $\therefore$

$$\nabla \cdot \left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right) = \rho/\epsilon_0$$

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\vec{A} = \rho/\epsilon_0$$

Let us arbitrarily require $\nabla\cdot\vec{A} = -\dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t}$

$\nabla\cdot A$ can be anything since $\nabla\times\nabla\cdot\vec{A} = 0$ ← Check this

Thus $\nabla\cdot A = 0$ (Coulomb condition)

$\nabla\cdot\vec{A} = -\dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t}$ (Lorenz condition)

$$\nabla^2\phi + \frac{\partial}{\partial t}\nabla\cdot A = -\rho/\epsilon_0$$

$$\nabla^2\phi + \frac{\partial}{\partial t}\left(-\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\rho/\epsilon_0$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\rho/\epsilon_0 \quad \text{Q.E.D.}$$

Now begin with 3

$$\nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \qquad (B = \nabla\times\vec{A})$$

$$\nabla\times\nabla\times A - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0\vec{J}$$

$$\nabla\times\nabla\times A = \nabla(\nabla\cdot\vec{A}) - \nabla^2 A$$

$$-\nabla^2 A + \nabla(\nabla\cdot A) - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

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U-04

Since $\nabla \times (\nabla \cdot \vec{A}) = 0$ $\nabla \cdot \vec{A}$ can be anything

Now differentiate U2

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left[E = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right]$$

$$-\frac{1}{c^2}\frac{\partial E}{\partial t} = \frac{1}{c^2}\nabla\frac{\partial \phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}$$

So

$$-\nabla^2 A + \nabla(\nabla \cdot A) + \frac{1}{c^2}\nabla\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \mu_0 \vec{J}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla\underbrace{\left(\nabla \cdot A + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right)}_{=0 \text{ Lorenz condition}} = \mu_0 J$$

$$\boxed{\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \mu_0 J}$$

Q.E.D.

Lecture 17 | 2016 U-05

We established $E = -\nabla\phi - \frac{\partial A}{\partial t}$

$B = \nabla \times A$ ↙ $R = |\vec{r} - \vec{r}\,'|$

$$\phi(r, t) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho(\vec{r}\,', t - R/c)}{R}\, dV'$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\quad)}{R}\, dV'$$

(6) $\nabla \cdot A + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$ ← Lorenz Condition

Show Lorenz condition holds (by assuming it doesn't)

We had

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\rho/\epsilon_0$$ [Before asserting (6)]

We also had

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla\left(\nabla\cdot A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \mu_0 J$$

Let's say that for $\vec{A}$ & ϕ there is no Lorenz condition. Can we pick A' & ϕ' so it <u>does</u> work?

$A' = A + \nabla\chi$ ← for a χ we don't yet know

$B' = \nabla\times A' = \nabla\times A = B$ (Why?) -- So we are allowed to do this

Also $\phi' = \phi - \frac{\partial\chi}{\partial t}$. This doesn't change B.

Does it change E?

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

$$E' = \left(-\nabla\phi + \cancel{\frac{\partial}{\partial t}\nabla\chi}\right) - \frac{\partial A}{\partial t} - \cancel{\frac{\partial}{\partial t}\nabla\chi}$$

$$= -\nabla\phi - \frac{\partial A}{\partial t} = E$$

So it does not change E either.

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U-06

17-2

We now plug $\vec{A}'$ & ϕ' into $\nabla \cdot A + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$

$$\nabla \cdot A' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = \nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \left(\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}\right)$$

If we knew $-f(\vec{r},t)$ not $=0$

Then $\chi(r,t) = \frac{1}{4\pi}\int \frac{f(r',t-\frac{R'}{c})}{R'}\,dV'$

Let $+f(r,t) = \nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$ (Which we said $\neq 0$)

Then we find χ and

$$\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = -f(r,t) = -\nabla \cdot \vec{A} - \frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

So RHS of equation $=0$

$$\nabla \cdot \vec{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0.$$

Thus $\exists$ A, ϕ s.t. Lorenz Condition is true and thus we calculate those with equation 4, 5

No reason to call them A' & ϕ'

$\vec{J} = nQ\vec{v}$

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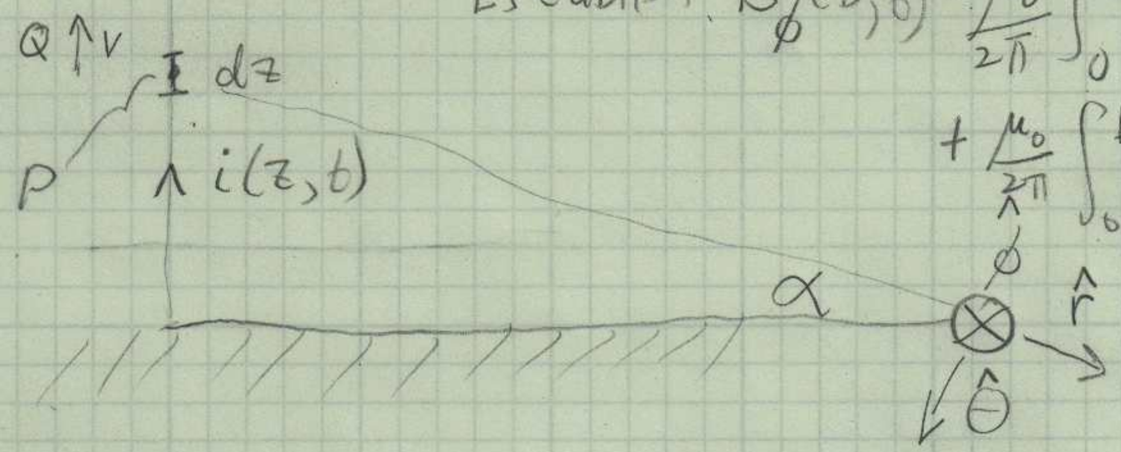
U-07

Establish $B_\phi(D,t) = \frac{\mu_0}{2\pi}\int_0^H \frac{\sin\theta}{R^2}\, i\left(z, t-\frac{R}{c}\right)dz + \frac{\mu_0}{2\pi}\int_0^H \frac{\sin\theta}{cR}\frac{\partial i(\;)}{\partial t}\,dz$

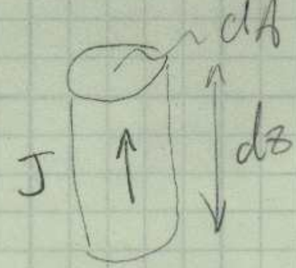


Q', v' ↓ $\quad Q' = -Q \quad v' = -v$ What direction is image current?

i_{image} has same sign and direction as i

$\boxed{d\vec{B} = \nabla \times d\vec{A}}$ Equation 1

Need to evaluation $\vec{J}(\vec{r}', t - \frac{R}{c})\, dV'$

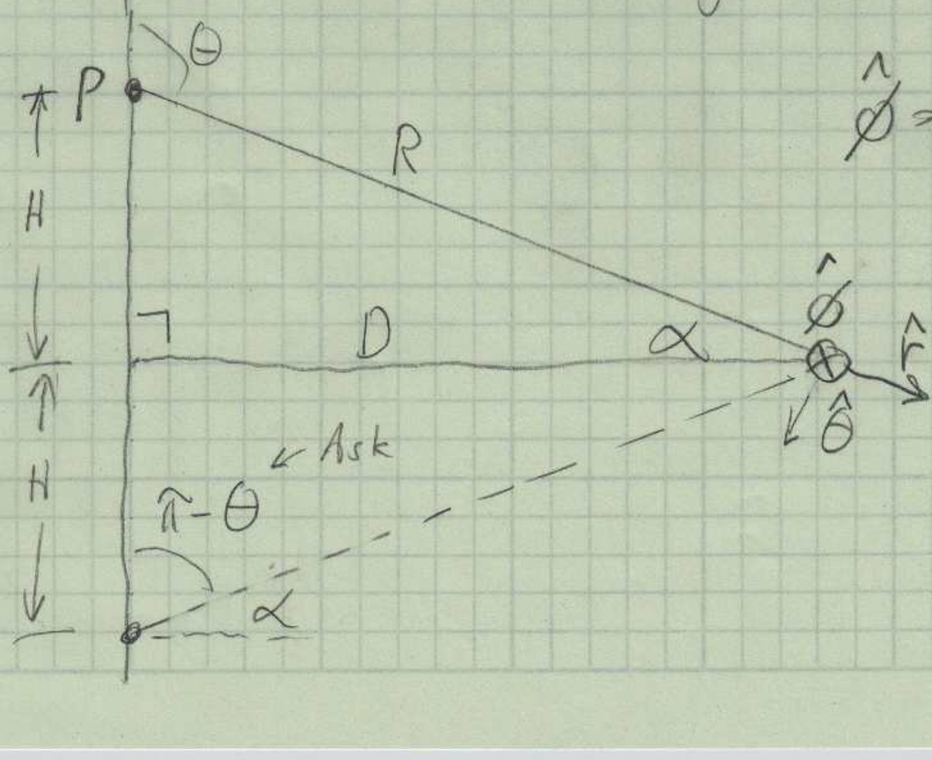


$dV' = dA\,dz$

$J\,dV' = J_z\,dA\,dz\,\hat{z} = i\,dz\,\hat{z}$

$\vec{J}\,dV' = i(z', t - \frac{R}{c})\,dz\,\hat{z}$

$|R_{image}| = |R|$. Thus $\vec{J}_{image}\,dV'$



$\hat{\phi} = \hat{r} \times \hat{\theta}$

[Spherical Coords with origin at P

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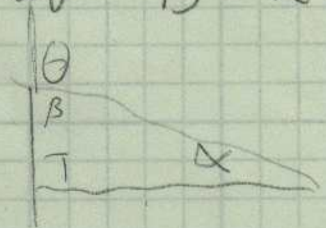
17-4 U-08

It is obvious that $D = R\cos\alpha$

How about θ?



$\pi - \theta = \beta$

$\beta = \pi - \frac{\pi}{2} - \alpha$

$\quad = \frac{\pi}{2} - \alpha$

$\pi - \theta = \frac{\pi}{2} - \alpha$

$\frac{\pi}{2} - \theta = -\alpha$

So $\theta = \frac{\pi}{2} - \alpha$

or $\alpha = \frac{\pi}{2} - \theta$

So $D = R\cos\alpha$

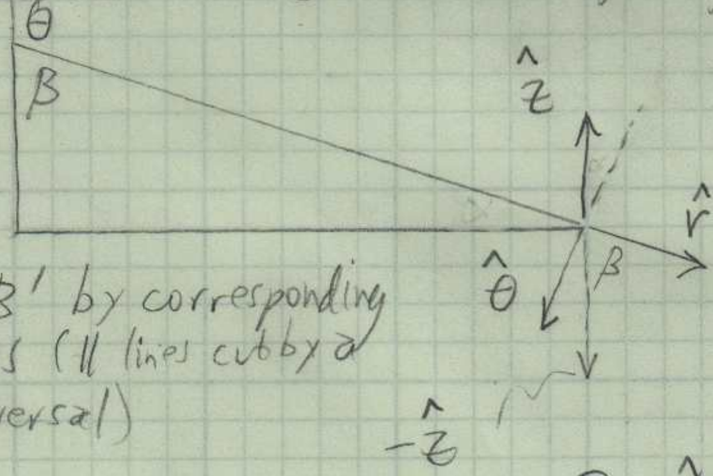
$\quad = R\sin\theta$

(Complimentary angles or use sum formula)

$\cos\alpha = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta = \sin\theta$

$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{J}}{R}dV = \frac{\mu_0}{4\pi}\frac{i\,dz\,\hat{z}}{R}$$ Equation A2

Need to write $\hat{z}$ in terms of $\hat{r}, \hat{\theta}, \hat{\phi}$ (Clearly no $\hat{\phi}$)



$\beta = \beta'$ by corresponding angles (∥ lines cut by a transversal)

Clearly $\beta = \pi - \theta$

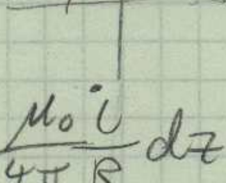
$\theta = \pi - \beta$

Also

$-\hat{z} = \hat{r}\cos\beta + \hat{\theta}\sin\beta$

$\cos(\pi - \theta) = -\cos\theta$

$\sin(\pi - \theta) = \sin\theta$



So $\hat{z} = -\hat{r}\cos\beta - \hat{\theta}\sin\beta$

$\hat{z} = +\hat{r}\cos\theta - \hat{\theta}\sin\theta$

$$d\vec{A} = \frac{\mu_0 i}{4\pi R}dz(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$ ← Equation A3

Lecture 20

2016 U-09

20-1

$$d\vec{A} = \frac{\mu_0}{4\pi} i \left[\frac{\cos\theta}{R}\hat{r} - \frac{\sin\theta}{R}\hat{\theta} \right] dz \qquad [A_3]$$

General Curl in Spherical Coords

$$\nabla \times d\vec{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, dA_\phi) - \frac{\partial}{\partial\phi} dA_\theta\right]$$

$$+ \hat{\theta}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\phi} dA_r - \frac{1}{r}\frac{\partial}{\partial r}(r\, dA_\phi)\right]$$

$$+ \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

By cyl. symettry $[dA]_\phi = 0$

Also $[dA]_\theta$ is indep. of ϕ

$[dA]_r$ is indep. of ϕ

Remaining term is $[\nabla \times d\hat{A}]_\phi$

$$\nabla \times d\vec{A} = -\hat{\phi}\frac{\mu_0}{4\pi} dz \left\{ \frac{1}{r}\frac{\partial}{\partial r}(r \underbrace{i\sin\theta/R}_{A_\theta}) + \frac{1}{r}\frac{\partial}{\partial\theta}\overbrace{\frac{i\cos\theta}{R}}^{A_r} \right\}$$

Because of how we defined problem (Figure 2) the general "r" from spherical coords and "R" are the same. Thus replace r by R and $\frac{\partial}{\partial r}$ by $\frac{\partial}{\partial R}$.

$$-(dB)_\phi = \frac{\mu_0}{4\pi} dz \left\{ \frac{1}{R}\sin\theta \frac{\partial i(t - R/c)}{\partial R} - \frac{1}{R^2}\sin\theta\, i + \frac{1}{R^2}\cos\theta \frac{\partial i}{\partial\theta} \right\}$$

We think $\frac{\partial i}{\partial\theta} = 0$ (not entirely happy w/ this)

$$\text{Also } \frac{\partial}{\partial R} = \frac{\partial}{\partial(t - R/c)}\frac{\partial}{\partial R}(t - R/c) = -\frac{1}{c}\frac{\partial}{\partial t} \quad \leftarrow [A5]$$

$$-(dB)_\phi = \frac{\mu_0}{4\pi} dz \sin\theta \left\{ \frac{1}{R}\left(\frac{-1}{c}\frac{\partial i}{\partial t}\right) - \frac{i}{R^2} \right\}$$

$$\partial B_\phi = \frac{\mu_0}{4\pi} dz \sin\theta \left\{ \frac{i}{R^2} + \frac{1}{Rc}\frac{\partial i}{\partial t} \right\} \qquad [A6]$$

More rigorous derivation of [A5]

Define $u = t - \frac{R}{c}$ obviously $\frac{\partial u}{\partial R} = -\frac{1}{c}$

$$\frac{\partial}{\partial R} i(z, t - R/c) = \frac{\partial}{\partial u} i(z,u) \frac{\partial u}{\partial R} = G \frac{\partial u}{\partial R} \quad \left(\text{Where } G \equiv \frac{\partial}{\partial u} i(z,u)\right)$$

$$\frac{\partial}{\partial t} i(z, t - \tfrac{R}{c}) = \frac{\partial}{\partial u} i(z,u) \frac{\partial u}{\partial t} = G \frac{\partial u}{\partial t}$$

$$G \frac{\partial u}{\partial R} = -\frac{G}{c} \qquad G \frac{\partial u}{\partial t} = G$$

$$\therefore \frac{\partial}{\partial R} i = -\frac{1}{c}\frac{\partial}{\partial t} i$$

$$\sin(\pi - \theta) = \sin\theta$$

∂B_ϕ image is same as ∂B_ϕ but $\theta \to \pi - \theta$
recall current was in same direction. All that happens is A6 doubles

$$[A7] \quad dB_\phi = \frac{\mu_0 dz}{2\pi} \sin\theta \left[\frac{i}{R^2} + \frac{1}{Rc}\frac{\partial i}{\partial t} \right]$$

Given (6)

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

$$-c^2 \int \nabla \cdot \vec{A} \, d\tau = \phi + \text{const} \quad \text{and } c^2 = \frac{1}{\mu_0 \epsilon_0} \quad \frac{\mu_0}{4\pi} c^2 = \frac{1}{4\pi\epsilon_0}$$

$$\nabla \cdot A = \frac{1}{R^2}\frac{\partial}{\partial R} R^2 A_R + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta} \sin\theta A_\theta$$

$$\boxed{\frac{\partial}{\partial\theta} \sin^2\theta \, i = 2\sin\theta\cos\theta \, i + \sin^2\frac{\partial i}{\partial\theta}}$$

Using A3

$$\phi = -\frac{dz}{4\pi\epsilon_0} \int \left[\frac{1}{R^2}\frac{\partial}{\partial R} R^2 \frac{\cos\theta \, i}{R} - \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta} \frac{\sin\theta \sin\theta \, i}{R} \right] d\tau$$

$$= \frac{-dz}{4\pi\epsilon_0} \int d\tau \left[\overbrace{\frac{1}{R^2}\cos\theta \, i + \frac{R}{R^2}\cos\theta \frac{\partial i}{\partial R}} - \frac{1}{R^2\sin\theta} 2\sin\theta\cos\theta \, i \right]$$

Combine these two terms

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U-11

$$-\phi = \frac{dz}{4\pi\epsilon_0}\int d\hat{\tau}\left[\frac{1}{R^2}\cos\theta\, i - \frac{\cos\theta}{Rc}\frac{\partial i}{\partial t} - \frac{2\cos\theta\, i}{R^2}\right]$$

$$\phi = \frac{dz\cos\theta}{4\pi\epsilon_0}\left[\int \frac{1}{R^2}\, i\, d\tau + \frac{\dot{i}}{Rc}\right] \quad [A_8]$$

Lecture 21

2016 U-12

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \quad (2) \qquad \frac{1}{c^2} = \mu_0\epsilon_0 \rightarrow \mu_0 = \frac{1}{\epsilon_0 c^2}$$

$$\nabla\varphi = \hat{r}\frac{\partial\varphi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\cancel{\frac{\partial\varphi}{\partial\phi}} \rightarrow 0$$

<u>$\hat{r}$-component</u>

$$dA_r = \frac{\mu_0 i}{4\pi}\frac{\cos\theta}{R}dz = \frac{i}{4\pi\epsilon_0}\frac{\cos\theta}{Rc^2}dz$$

$$-\frac{\partial}{\partial t}dA_r = \frac{-dz}{4\pi\epsilon_0}\frac{\cos\theta}{Rc^2}\frac{\partial i}{\partial t}$$

$$-\frac{\partial\varphi}{\partial r} = \frac{-dz}{4\pi\epsilon_0}\cos\theta\left[\frac{\partial}{\partial r}\frac{1}{r^2}\int_0^t i\,d\tau + \frac{\partial}{\partial r}\frac{i}{rc}\right]$$

$$= \frac{-dz}{4\pi\epsilon_0}\cos\theta\left[\frac{-2}{r^3}\int_0^t i\,d\tau + \frac{1}{r^2}\int_0^t\frac{\partial i}{\partial r}d\tau - \frac{i}{r^2c} + \frac{1}{rc}\frac{\partial i}{\partial r}\right]$$

Again $\frac{\partial i}{\partial R} = \frac{-1}{c}\frac{\partial i}{\partial t}$ (and distribute − sign)

$$= \frac{dz}{4\pi\epsilon_0}\cos\theta\left[\frac{2}{r^3}\int_0^t i\,d\tau + \frac{1}{cr^2}\int_0^t\frac{\partial i}{\partial t}d\tau + \frac{i}{r^2c} + \frac{1}{rc^2}\frac{\partial i}{\partial t}\right]$$

The $\frac{1}{rc^2}\frac{\partial i}{\partial t}$ terms from A + ϕ cancel

$$E_r = \frac{dz}{4\pi\epsilon_0}\cos\theta\left[\frac{2}{r^3}\int_0^t i\,d\tau + \frac{2}{r^2c}i\right] \quad \text{A9 } \hat{r}$$

$\hat{\theta}$-component

$$dA_\theta = -\frac{\mu_0 i}{4\pi}\frac{\sin\theta}{r}dz = \frac{-i}{4\pi\epsilon_0}\frac{\sin\theta}{rc^2}dz$$

$$-\frac{\partial}{\partial t}dA_\theta = \frac{dz}{4\pi\epsilon_0}\frac{\sin\theta}{rc^2}\frac{\partial i}{\partial t}$$

$$-\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{-1}{r}\frac{dz}{4\pi\epsilon_0}\left[\frac{1}{r^2}\int_0^t i\,d\tau\,\frac{\partial}{\partial\theta}\cos\theta + \frac{i}{rc}\frac{\partial}{\partial\theta}\cos\theta\right]$$

$$= \frac{dz}{4\pi\epsilon_0}\sin\theta\left[\frac{1}{r^3}\int_0^t i\,d\tau + \frac{i}{r^2c}\right]$$

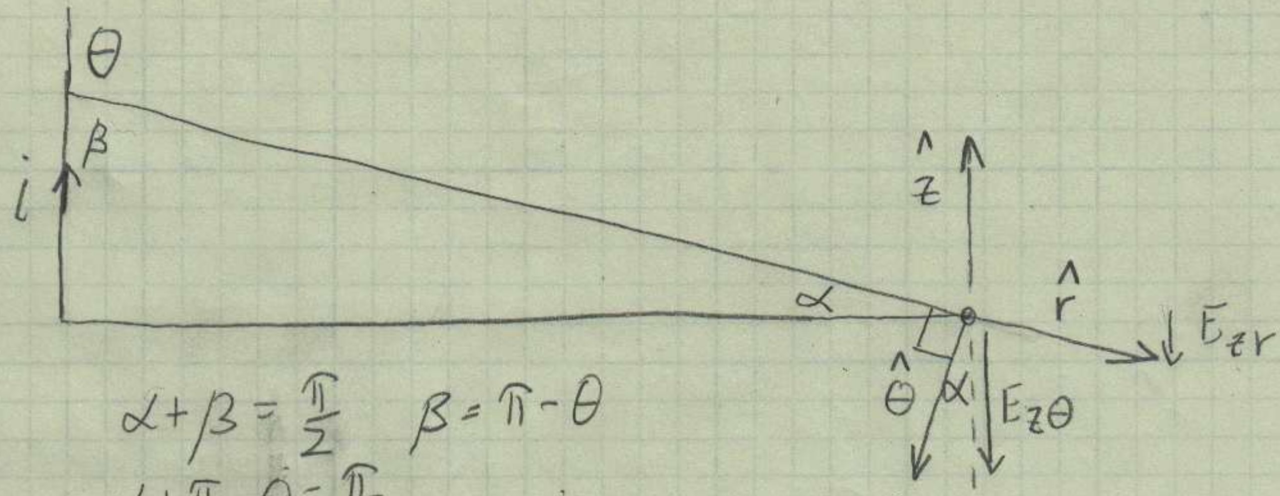
$$dE_\theta = \frac{dz}{4\pi\epsilon_0}\sin\theta\left[\frac{1}{r^3}\int_0^t i\,d\tau + \frac{i}{r^2c} + \frac{1}{rc^2}\frac{\partial i}{\partial t}\right] \quad \text{A9 } \hat{\theta}$$

$$dE_z = 2dE_r\cos\theta - 2dE_\theta\sin\theta$$ [See p. U-14]

$$= \frac{dz}{2\pi\epsilon_0}\cos^2\theta\left[\frac{2}{r^3}\int_0^t i\,d\tau + \frac{2}{r^2c}i\right] \quad \{\cos^2\theta = 1-\sin^2\theta\}$$

$$-\frac{dz}{2\pi\epsilon_0}\sin^2\theta\left[\frac{1}{r^3}\int_0^t i\,d\tau + \frac{i}{r^2c} + \frac{1}{rc^2}\frac{\partial i}{\partial t}\right]$$

$$dE_z = \frac{dz}{2\pi\epsilon_0}\left[\frac{1}{r^3}(2-3\sin^2\theta)\int_0^t i\,d\tau + \frac{1}{r^2c}(2-3\sin^2\theta)i - \frac{\sin^2\theta}{rc^2}\frac{\partial i}{\partial t}\right] \quad \text{A14}$$



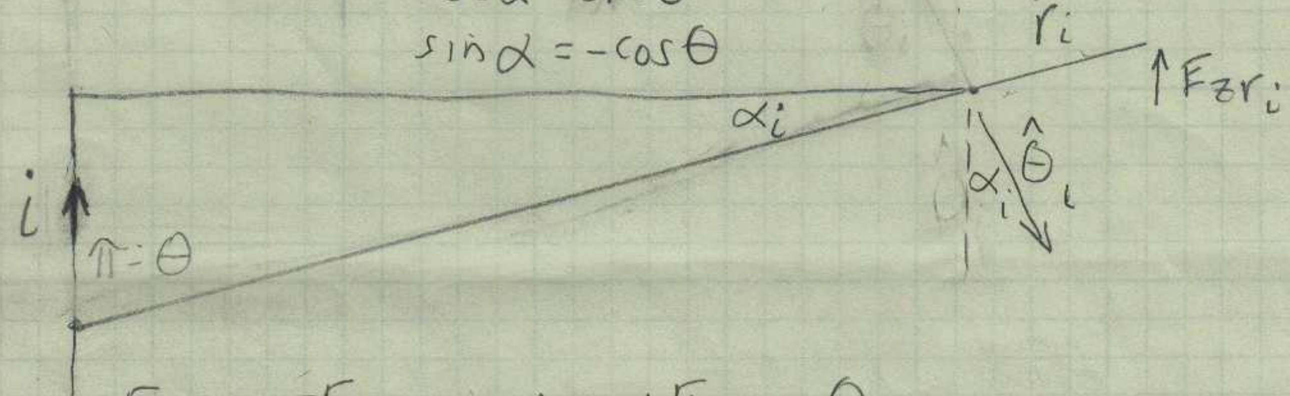
$\alpha + \beta = \frac{\pi}{2}$ $\quad \beta = \pi - \theta$

$\alpha + \pi - \theta = \frac{\pi}{2}$

$\alpha + \frac{\pi}{2} = \theta$ $\quad \alpha = \theta - \frac{\pi}{2}$

$\cos\alpha = \sin\theta$

$\sin\alpha = -\cos\theta$



$E_{zr} = -E_r \sin\alpha = +E_r \cos\theta$

$E_{z\theta} = -E_\theta \cos\alpha = -E_\theta \sin\theta$

Include image currents -- Double E_z

Physics 535 - Physics of Lightning
New Mexico Tech

This is one derivation
split over multiple lectures

pages 1-4 Lecture 15
5-8 Lecture 17
9-11 Lecture 20
12-14 Lecture 21

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