To demonstrate that this goes to $\delta^3(\mathbf{r})$ as $\epsilon \to 0$:

(a) Show that $D(r, \epsilon) = (3\epsilon^2/4\pi)(r^2 + \epsilon^2)^{-5/2}$.

(b) Check that $D(0, \epsilon) \to \infty$, as $\epsilon \to 0$.

(c) Check that $D(r, \epsilon) \to 0$, as $\epsilon \to 0$, for all $r \neq 0$.

(d) Check that the integral of $D(r, \epsilon)$ over all space is 1.

CHAPTER 2 Electrostatics

2.1 ■ THE ELECTRIC FIELD

2.1.1 ■ Introduction

The fundamental problem electrodynamics hopes to solve is this (Fig. 2.1): We have some electric charges, $q_1, q_2, q_3, \ldots$ (call them **source charges**); what force do they exert on another charge, Q (call it the **test charge**)? The positions of the source charges are *given* (as functions of time); the trajectory of the test particle is *to be calculated.* In general, both the source charges and the test charge are in motion.

The solution to this problem is facilitated by the **principle of superposition**, which states that the interaction between any two charges is completely unaffected by the presence of others. This means that to determine the force on Q, we can first compute the force $\mathbf{F}_1$, due to q_1 alone (ignoring all the others); then we compute the force $\mathbf{F}_2$, due to q_2 alone; and so on. Finally, we take the vector sum of all these individual forces: $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$ Thus, if we can find the force on Q due to a *single* source charge q, we are, in principle, done (the rest is just a question of repeating the same operation over and over, and adding it all up).[1]

Well, at first sight this looks very easy: Why don't I just write down the formula for the force on Q due to q, and be done with it? I *could,* and in Chapter 10 I shall, but you would be shocked to see it at this stage, for not only does the force on Q depend on the separation distance $\mathscr{r}$ between the charges (Fig. 2.2), it also

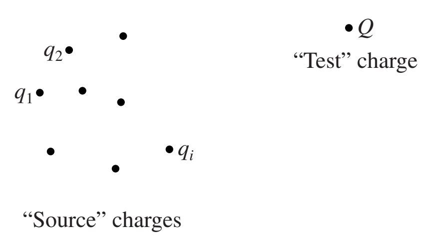


FIGURE 2.1

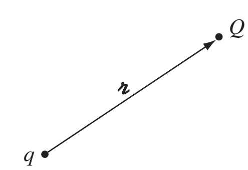


FIGURE 2.2

[1]The principle of superposition may seem "obvious" to you, but it did not have to be so simple: if the electromagnetic force were proportional to the *square* of the total source charge, for instance, the principle of superposition would not hold, since $(q_1 + q_2)^2 \neq q_1^2 + q_2^2$ (there would be "cross terms" to consider). Superposition is not a logical necessity, but an experimental fact.

depends on *both* their velocities and on the *acceleration* of q. Moreover, it is not the position, velocity, and acceleration of q *right now* that matter: electromagnetic "news" travels at the speed of light, so what concerns Q is the position, velocity, and acceleration q *had* at some earlier time, when the message left.

Therefore, in spite of the fact that the basic question ("What is the force on Q due to q?") is easy to state, it does not pay to confront it head on; rather, we shall go at it by stages. In the meantime, the theory we develop will allow for the solution of more subtle electromagnetic problems that do not present themselves in quite this simple format. To begin with, we shall consider the special case of **electrostatics** in which all the *source charges are stationary* (though the test charge may be moving).

2.1.2 ■ Coulomb's Law

What is the force on a test charge Q due to a single point charge q, that is at *rest* a distance $\mathscr{r}$ away? The answer (based on experiments) is given by **Coulomb's law**:

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.} \tag{2.1}$$

The constant ϵ_0 is called (ludicrously) the **permittivity of free space.** In SI units, where force is in newtons (N), distance in meters (m), and charge in coulombs (C),

$$\epsilon_0 = 8.85 \times 10^{-12}\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}.$$

In words, the force is proportional to the product of the charges and inversely proportional to the square of the separation distance. As always (Sect. 1.1.4), $\boldsymbol{\mathscr{r}}$ is the separation vector from $\mathbf{r}'$ (the location of q) to $\mathbf{r}$ (the location of Q):

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'; \tag{2.2}$$

$\mathscr{r}$ is its magnitude, and $\hat{\boldsymbol{\mathscr{r}}}$ is its direction. The force points along the line from q to Q; it is repulsive if q and Q have the same sign, and attractive if their signs are opposite.

Coulomb's law and the principle of superposition constitute the physical input for electrostatics—the rest, except for some special properties of matter, is mathematical elaboration of these fundamental rules.

Problem 2.1

(a) Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon (for instance, one on each numeral of a clock face). What is the net force on a test charge Q at the center?

(b) Suppose *one* of the 12 q's is removed (the one at "6 o'clock"). What is the force on Q? Explain your reasoning carefully.

(c) Now 13 equal charges, q, are placed at the corners of a regular 13-sided polygon. What is the force on a test charge Q at the center?

(d) If one of the 13 q's is removed, what is the force on Q? Explain your reasoning.

2.1.3 ■ The Electric Field

If we have *several* point charges $q_1, q_2, \ldots, q_n$, at distances $\mathscr{r}_1, \mathscr{r}_2, \ldots, \mathscr{r}_n$ from Q, the total force on Q is evidently

$$\begin{aligned}\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots &= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right)\\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \frac{q_3}{\mathscr{r}_3^2}\hat{\boldsymbol{\mathscr{r}}}_3 + \ldots\right),\end{aligned}$$

or

$$\boxed{\mathbf{F} = Q\mathbf{E},} \tag{2.3}$$

where

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i. \tag{2.4}$$

$\mathbf{E}$ is called the **electric field** of the source charges. Notice that it is a function of position ($\mathbf{r}$), because the separation vectors $\boldsymbol{\mathscr{r}}_i$ depend on the location of the **field point** P (Fig. 2.3). But it makes no reference to the test charge Q. The electric field is a vector quantity that varies from point to point and is determined by the configuration of source charges; physically, $\mathbf{E}(\mathbf{r})$ is the force per unit charge that would be exerted on a test charge, if you were to place one at P.

What exactly *is* an electric field? I have deliberately begun with what you might call the "minimal" interpretation of $\mathbf{E}$, as an intermediate step in the calculation of electric forces. But I encourage you to think of the field as a "real" physical

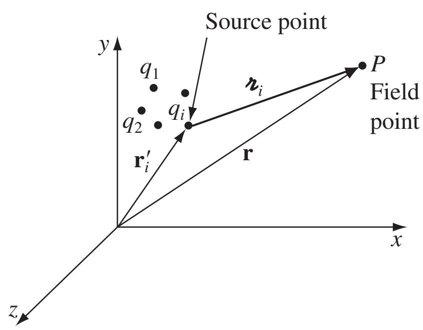


FIGURE 2.3

entity, filling the space around electric charges. Maxwell himself came to believe that electric and magnetic fields are stresses and strains in an invisible primordial jellylike "ether." Special relativity has forced us to abandon the notion of ether, and with it Maxwell's mechanical interpretation of electromagnetic fields. (It is even possible, though cumbersome, to formulate classical electrodynamics as an "action-at-a-distance" theory, and dispense with the field concept altogether.) I can't tell you, then, what a field *is*—only how to calculate it and what it can do for you once you've got it.

Example 2.1. Find the electric field a distance z above the midpoint between two equal charges (q), a distance d apart (Fig. 2.4a).

Solution

Let $\mathbf{E}_1$ be the field of the left charge alone, and $\mathbf{E}_2$ that of the right charge alone (Fig. 2.4b). Adding them (vectorially), the horizontal components cancel and the vertical components conspire:

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta.$$

Here $\mathscr{r} = \sqrt{z^2 + (d/2)^2}$ and $\cos\theta = z/\mathscr{r}$, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + (d/2)^2\right]^{3/2}}\,\hat{\mathbf{z}}.$$

Check: When $z \gg d$ you're so far away that it just looks like a single charge $2q$, so the field should reduce to $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\,\hat{\mathbf{z}}$. And it *does* (just set $d \to 0$ in the formula).

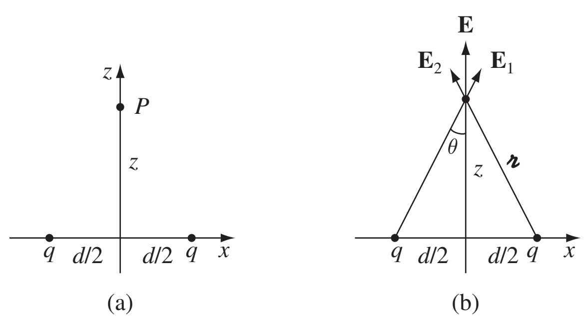
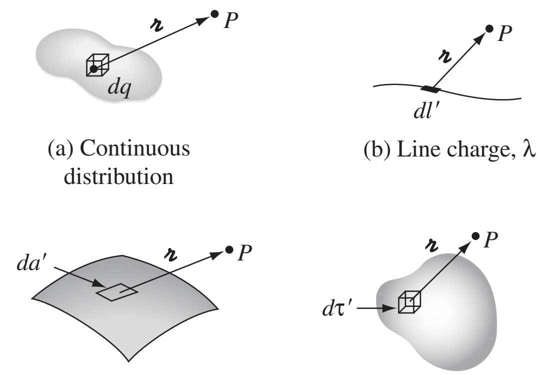


FIGURE 2.4

Problem 2.2 Find the electric field (magnitude and direction) a distance z above the midpoint between equal and opposite charges ($\pm q$), a distance d apart (same as Example 2.1, except that the charge at $x = +d/2$ is $-q$).



(a) Continuous distribution

(b) Line charge, λ

(c) Surface charge, σ

(d) Volume charge, ρ

FIGURE 2.5

2.1.4 ■ Continuous Charge Distributions

Our definition of the electric field (Eq. 2.4) assumes that the source of the field is a collection of discrete point charges q_i. If, instead, the charge is distributed continuously over some region, the sum becomes an integral (Fig. 2.5a):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dq. \tag{2.5}$$

If the charge is spread out along a *line* (Fig. 2.5b), with charge-per-unit-length λ, then $dq = \lambda\,dl'$ (where dl' is an element of length along the line); if the charge is smeared out over a *surface* (Fig. 2.5c), with charge-per-unit-area σ, then $dq = \sigma\,da'$ (where da' is an element of area on the surface); and if the charge fills a *volume* (Fig. 2.5d), with charge-per-unit-volume ρ, then $dq = \rho\,d\tau'$ (where $d\tau'$ is an element of volume):

$$dq \rightarrow \lambda\,dl' \sim \sigma\,da' \sim \rho\,d\tau'.$$

Thus the electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dl'; \tag{2.6}$$

for a surface charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,da'; \tag{2.7}$$

and for a volume charge,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'.} \tag{2.8}$$

Equation 2.8 itself is often referred to as "Coulomb's law," because it is such a short step from the original (2.1), and because a volume charge is in a sense the most general and realistic case. Please note carefully the meaning of $\boldsymbol{\mathscr{r}}$ in these formulas. Originally, in Eq. 2.4, $\boldsymbol{\mathscr{r}}_i$ stood for the vector from the source charge q_i to the field point $\mathbf{r}$. Correspondingly, in Eqs. 2.5–2.8, $\boldsymbol{\mathscr{r}}$ is the vector from dq (therefore from dl', da', or $d\tau'$) to the field point $\mathbf{r}$.[2]

Example 2.2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ (Fig. 2.6).

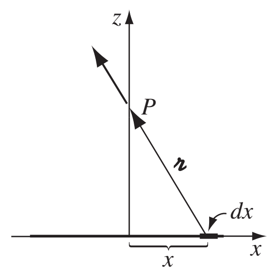


FIGURE 2.6

Solution

The simplest method is to chop the line into symmetrically placed pairs (at $\pm x$), quote the result of Ex. 2.1 (with $d/2 \to x$, $q \to \lambda\, dx$), and integrate ($x : 0 \to L$). But here's a more general approach:[3]

$$\mathbf{r} = z\,\hat{\mathbf{z}}, \quad \mathbf{r}' = x\,\hat{\mathbf{x}}, \quad dl' = dx;$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}, \quad \mathscr{r} = \sqrt{z^2 + x^2}, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}.$$

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda}{z^2 + x^2} \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}\, dx \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \int_{-L}^{L} \frac{1}{(z^2 + x^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_{-L}^{L} \frac{x}{(z^2 + x^2)^{3/2}}\, dx \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \left(\frac{x}{z^2\sqrt{z^2 + x^2}} \right)\Bigg|_{-L}^{L} - \hat{\mathbf{x}} \left(-\frac{1}{\sqrt{z^2 + x^2}} \right)\Bigg|_{-L}^{L} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}\,\hat{\mathbf{z}}.
\end{aligned}$$

[2] *Warning:* The unit vector $\hat{\boldsymbol{\mathscr{r}}}$ is *not* constant; its *direction* depends on the source point $\mathbf{r}'$, and hence it *cannot be taken outside the integrals* (Eqs. 2.5–2.8). In practice, *you must work with Cartesian components* ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ *are* constant, and *do* come out), even if you use curvilinear coordinates to perform the integration.

[3] Ordinarily I'll put a prime on the source coordinates, but where no confusion can arise I'll remove the prime to simplify the notation.

For points far from the line ($z \gg L$),

$$E \cong \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2}.$$

This makes sense: From far away the line looks like a point charge $q = 2\lambda L$. In the limit $L \to \infty$, on the other hand, we obtain the field of an infinite straight wire:

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}. \tag{2.9}$$

Problem 2.3 Find the electric field a distance z above one end of a straight line segment of length L (Fig. 2.7) that carries a uniform line charge λ. Check that your formula is consistent with what you would expect for the case $z \gg L$.

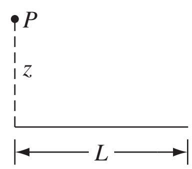


FIGURE 2.7

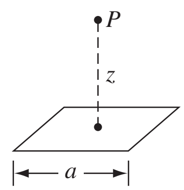


FIGURE 2.8

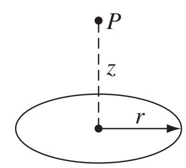


FIGURE 2.9

Problem 2.4 Find the electric field a distance z above the center of a square loop (side a) carrying uniform line charge λ (Fig. 2.8). [*Hint:* Use the result of Ex. 2.2.]

Problem 2.5 Find the electric field a distance z above the center of a circular loop of radius r (Fig. 2.9) that carries a uniform line charge λ.

Problem 2.6 Find the electric field a distance z above the center of a flat circular disk of radius R (Fig. 2.10) that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

! **Problem 2.7** Find the electric field a distance z from the center of a spherical surface of radius R (Fig. 2.11) that carries a uniform charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere. [*Hint:* Use the law of cosines to write $\mathscr{r}$ in terms of R and θ. Be sure to take the *positive* square root: $\sqrt{R^2 + z^2 - 2Rz} = (R - z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]

Problem 2.8 Use your result in Prob. 2.7 to find the field inside and outside a solid sphere of radius R that carries a uniform volume charge density ρ. Express your answers in terms of the total charge of the sphere, q. Draw a graph of $|\mathbf{E}|$ as a function of the distance from the center.

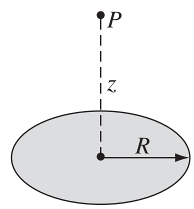


FIGURE 2.10

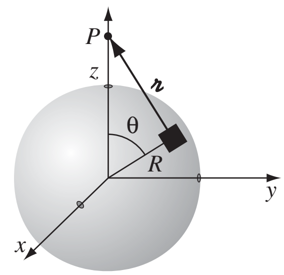


FIGURE 2.11

2.2 ■ DIVERGENCE AND CURL OF ELECTROSTATIC FIELDS

2.2.1 ■ Field Lines, Flux, and Gauss's Law

In principle, we are *done* with the subject of electrostatics. Equation 2.8 tells us how to compute the field of a charge distribution, and Eq. 2.3 tells us what the force on a charge Q placed in this field will be. Unfortunately, as you may have discovered in working Prob. 2.7, the integrals involved in computing $\mathbf{E}$ can be formidable, even for reasonably simple charge distributions. Much of the rest of electrostatics is devoted to assembling a bag of tools and tricks for avoiding these integrals. It all begins with the divergence and curl of $\mathbf{E}$. I shall calculate the divergence of $\mathbf{E}$ directly from Eq. 2.8, in Sect. 2.2.2, but first I want to show you a more qualitative, and perhaps more illuminating, intuitive approach.

Let's begin with the simplest possible case: a single point charge q, situated at the origin:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.10}$$

To get a "feel" for this field, I might sketch a few representative vectors, as in Fig. 2.12a. Because the field falls off like $1/r^2$, the vectors get shorter as you go farther away from the origin; they always point radially outward. But there is a

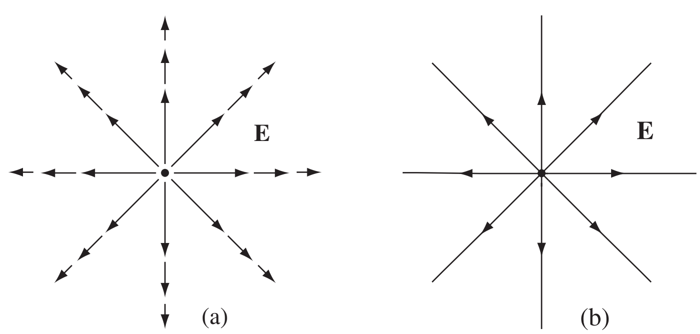


FIGURE 2.12

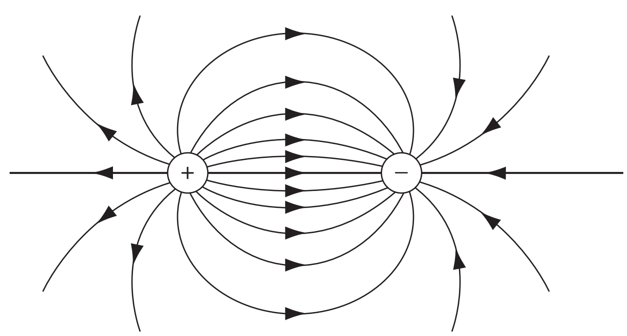
nicer way to represent this field, and that's to connect up the arrows, to form **field lines** (Fig. 2.12b). You might think that I have thereby thrown away information about the *strength* of the field, which was contained in the length of the arrows. But actually I have not. The magnitude of the field is indicated by the *density* of the field lines: it's strong near the center where the field lines are close together, and weak farther out, where they are relatively far apart.

In truth, the field-line diagram is deceptive, when I draw it on a two-dimensional surface, for the density of lines passing through a circle of radius r is the total number divided by the circumference $(n/2\pi r)$, which goes like $(1/r)$, not $(1/r^2)$. But if you imagine the model in three dimensions (a pincushion with needles sticking out in all directions), then the density of lines is the total number divided by the area of the sphere $(n/4\pi r^2)$, which *does* go like $(1/r^2)$.

Such diagrams are also convenient for representing more complicated fields. Of course, the number of lines you draw depends on how lazy you are (and how sharp your pencil is), though you ought to include enough to get an accurate sense of the field, and you must be consistent: If q gets 8 lines, then $2q$ deserves 16. And you must space them fairly—they emanate from a point charge symmetrically in all directions. Field lines begin on positive charges and end on negative ones; they cannot simply terminate in midair,[4] though they may extend out to infinity. Moreover, field lines can never cross—at the intersection, the field would have two different directions at once! With all this in mind, it is easy to sketch the field of any simple configuration of point charges: Begin by drawing the lines in the neighborhood of each charge, and then connect them up or extend them to infinity (Figs. 2.13 and 2.14).

In this model, the *flux* of **E** through a surface $\mathcal{S}$,

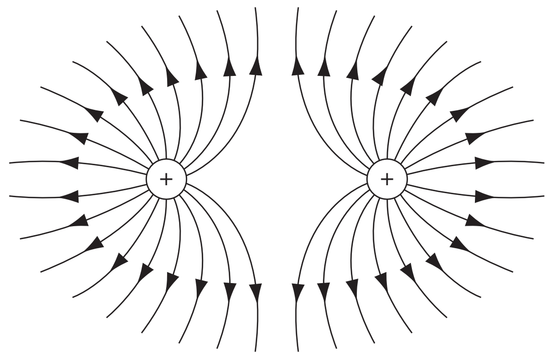
$$\Phi_E \equiv \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a}, \tag{2.11}$$



Opposite charges

FIGURE 2.13

[4]If they *did*, the divergence of **E** would not be zero, and (as we shall soon see) that cannot happen in empty space.



Equal charges

FIGURE 2.14

is a measure of the "number of field lines" passing through $\mathcal{S}$. I put this in quotes because of course we can only draw a representative *sample* of the field lines—the *total* number would be infinite. But *for a given sampling rate* the flux is *proportional* to the number of lines drawn, because the field strength, remember, is proportional to the density of field lines (the number per unit area), and hence $\mathbf{E} \cdot d\mathbf{a}$ is proportional to the number of lines passing through the infinitesimal area $d\mathbf{a}$. (The dot product picks out the component of $d\mathbf{a}$ along the direction of $\mathbf{E}$, as indicated in Fig. 2.15. It is the area *in the plane perpendicular to* $\mathbf{E}$ that we have in mind when we say that the density of field lines is the number per unit area.)

This suggests that the flux through any *closed* surface is a measure of the total charge inside. For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside (Fig. 2.16a). On the other hand, a charge *outside* the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other (Fig. 2.16b). This is the *essence* of **Gauss's law**. Now let's make it quantitative.

In the case of a point charge q at the origin, the flux of $\mathbf{E}$ through a spherical surface of radius r is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q. \tag{2.12}$$

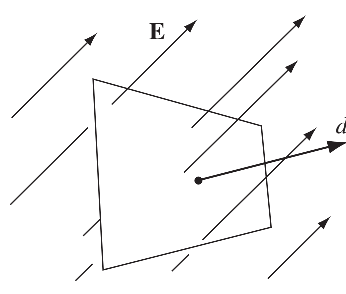


FIGURE 2.15

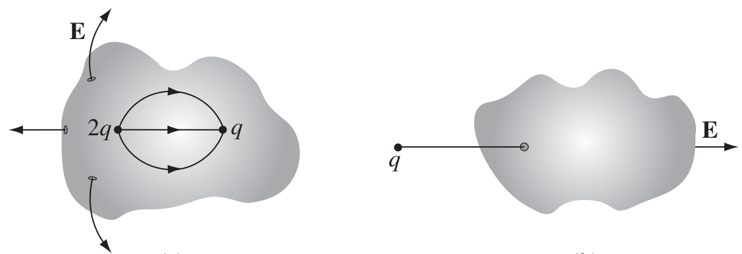


FIGURE 2.16

Notice that the radius of the sphere cancels out, for while the surface area goes *up* as r^2, the field goes *down* as $1/r^2$, so the product is constant. In terms of the field-line picture, this makes good sense, since the same number of field lines pass through any sphere centered at the origin, regardless of its size. In fact, it didn't have to be a sphere—*any* closed surface, whatever its shape, would be pierced by the same number of field lines. Evidently *the flux through any surface enclosing the charge is* q/ϵ_0.

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i .$$

The flux through a surface that encloses them all is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i \right)$$

For any closed surface, then,

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},} \tag{2.13}$$

where Q_{enc} is the total charge enclosed within the surface. This is the quantitative statement of Gauss's law. Although it contains no information that was not already present in Coulomb's law plus the principle of superposition, it is of almost magical power, as you will see in Sect. 2.2.3. Notice that it all hinges on the $1/r^2$ character of Coulomb's law; without that the crucial cancellation of the r's in Eq. 2.12 would not take place, and the total flux of $\mathbf{E}$ would depend on the surface chosen, not merely on the total charge enclosed. Other $1/r^2$ forces (I am thinking particularly of Newton's law of universal gravitation) will obey "Gauss's laws" of their own, and the applications we develop here carry over directly.

As it stands, Gauss's law is an *integral* equation, but we can easily turn it into a *differential* one, by applying the divergence theorem:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau.$$

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.} \tag{2.14}$$

Equation 2.14 carries the same message as Eq. 2.13; it is **Gauss's law in differential form.** The differential version is tidier, but the integral form has the advantage in that it accommodates point, line, and surface charges more naturally.

Problem 2.9 Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).

(a) Find the charge density ρ.

(b) Find the total charge contained in a sphere of radius R, centered at the origin. (Do it two different ways.)

Problem 2.10 A charge q sits at the back corner of a cube, as shown in Fig. 2.17. What is the flux of $\mathbf{E}$ through the shaded side?

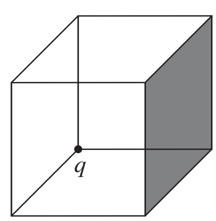


FIGURE 2.17

2.2.2 ■ The Divergence of E

Let's go back, now, and calculate the divergence of **E** directly from Eq. 2.8:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'. \tag{2.15}$$

(Originally the integration was over the volume occupied by the charge, but I may as well extend it to all space, since $\rho = 0$ in the exterior region anyway.) Noting that the **r**-dependence is contained in $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$, we have

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) \rho(\mathbf{r}')\,d\tau'.$$

This is precisely the divergence we calculated in Eq. 1.100:

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}).$$

Thus

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')\,d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r}), \tag{2.16}$$

which is Gauss's law in differential form (Eq. 2.14). To recover the integral form (Eq. 2.13), we run the previous argument in reverse—integrate over a volume and apply the divergence theorem:

$$\int\limits_{\mathcal{V}} \nabla \cdot \mathbf{E}\,d\tau = \oint\limits_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int\limits_{\mathcal{V}} \rho\,d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

2.2.3 ■ Applications of Gauss's Law

I must interrupt the theoretical development at this point to show you the extraordinary power of Gauss's law, in integral form. When symmetry permits, it affords *by far* the quickest and easiest way of computing electric fields. I'll illustrate the method with a series of examples.

Example 2.3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Solution

Imagine a spherical surface at radius $r > R$ (Fig. 2.18); this is called a **Gaussian surface** in the trade. Gauss's law says that

$$\oint\limits_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},$$

and in this case $Q_{\text{enc}} = q$. At first glance this doesn't seem to get us very far, because the quantity we want (**E**) is buried inside the surface integral. Luckily, symmetry allows us to extract **E** from under the integral sign: **E** certainly points radially outward,[5] as does $d\mathbf{a}$, so we can drop the dot product,

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{S}} |\mathbf{E}|\, da,$$

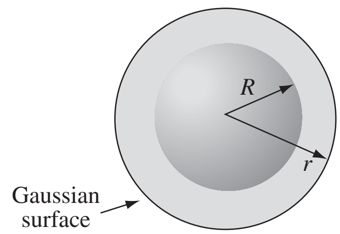


FIGURE 2.18

and the *magnitude* of **E** is constant over the Gaussian surface, so it comes outside the integral:

$$\int_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2.$$

Thus

$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

or

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

Notice a remarkable feature of this result: The field outside the sphere is exactly *the same as it would have been if all the charge had been concentrated at the center.*

Gauss's law is always *true*, but it is not always *useful.* If ρ had not been uniform (or, at any rate, not spherically symmetrical), or if I had chosen some other shape for my Gaussian surface, it would still have been true that the flux of **E** is q/ϵ_0, but **E** would not have pointed in the same direction as $d\mathbf{a}$, and its magnitude would not have been constant over the surface, and without that I cannot get $|\mathbf{E}|$ outside

[5]If you doubt that **E** is radial, consider the alternative. Suppose, say, that it points due east, at the "equator." But the orientation of the equator is perfectly arbitrary—nothing is spinning here, so there is no natural "north-south" axis—any argument purporting to show that **E** points east could just as well be used to show it points west, or north, or any other direction. The only *unique* direction on a sphere is radial.

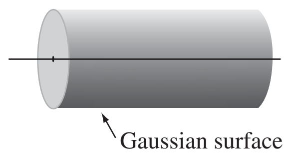


FIGURE 2.19

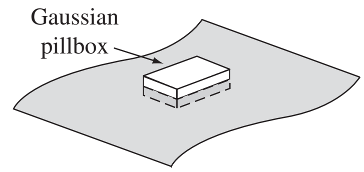


FIGURE 2.20

of the integral. *Symmetry is crucial* to this application of Gauss's law. As far as I know, there are only three kinds of symmetry that work:

1. *Spherical symmetry.* Make your Gaussian surface a concentric sphere.
2. *Cylindrical symmetry.* Make your Gaussian surface a coaxial cylinder (Fig. 2.19).
3. *Plane symmetry.* Use a Gaussian "pillbox" that straddles the surface (Fig. 2.20).

Although (2) and (3) technically require infinitely long cylinders, and planes extending to infinity, we shall often use them to get approximate answers for "long" cylinders or "large" planes, at points far from the edges.

Example 2.4. A long cylinder (Fig. 2.21) carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

Solution

Draw a Gaussian cylinder of length l and radius s. For this surface, Gauss's law states:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

The enclosed charge is

$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \tfrac{2}{3}\pi k l s^3.$$

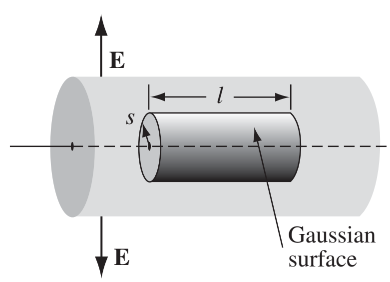


FIGURE 2.21

(I used the volume element appropriate to cylindrical coordinates, Eq. 1.78, and integrated ϕ from 0 to 2π, dz from 0 to l. I put a prime on the integration variable s', to distinguish it from the radius s of the Gaussian surface.)

Now, symmetry dictates that **E** must point radially outward, so for the curved portion of the Gaussian cylinder we have:

$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}|\, da = |\mathbf{E}| \int da = |\mathbf{E}|\, 2\pi s l,$$

while the two ends contribute nothing (here **E** is perpendicular to $d\mathbf{a}$). Thus,

$$|\mathbf{E}|\, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3,$$

or, finally,

$$\mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$$

Example 2.5. An infinite plane carries a uniform surface charge σ. Find its electric field.

Solution

Draw a "Gaussian pillbox," extending equal distances above and below the plane (Fig. 2.22). Apply Gauss's law to this surface:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

In this case, $Q_{\text{enc}} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, **E** points away from the plane (upward for points above, downward for points below). So the top and bottom surfaces yield

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$$

whereas the sides contribute nothing. Thus

$$2A\, |\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A,$$

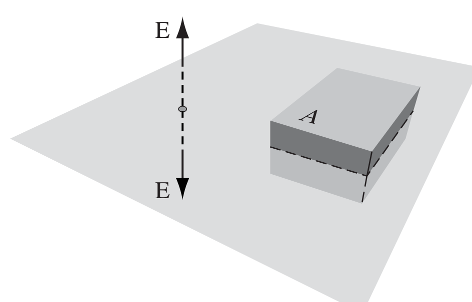


FIGURE 2.22

or

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}, \tag{2.17}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing away from the surface. In Prob. 2.6, you obtained this same result by a much more laborious method.

It seems surprising, at first, that the field of an infinite plane is *independent of how far away you are.* What about the $1/r^2$ in Coulomb's law? The point is that as you move farther and farther away from the plane, more and more charge comes into your "field of view" (a cone shape extending out from your eye), and this compensates for the diminishing influence of any particular piece. The electric field of a sphere falls off like $1/r^2$; the electric field of an infinite line falls off like $1/r$; and the electric field of an infinite plane does not fall off at all (you cannot escape from an infinite plane).

Although the direct use of Gauss's law to compute electric fields is limited to cases of spherical, cylindrical, and planar symmetry, we can put together *combinations* of objects possessing such symmetry, even though the arrangement as a whole is not symmetrical. For example, invoking the principle of superposition, we could find the field in the vicinity of two uniformly charged parallel cylinders, or a sphere near an infinite charged plane.

Example 2.6. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 2.23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

Solution

The left plate produces a field $(1/2\epsilon_0)\sigma$, which points away from it (Fig. 2.24)—to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\epsilon_0)\sigma$, which points *toward* it—to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). *Conclusion:* The field between the plates is σ/ϵ_0, and points to the right; elsewhere it is zero.

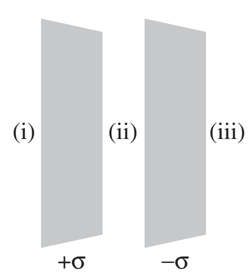


FIGURE 2.23

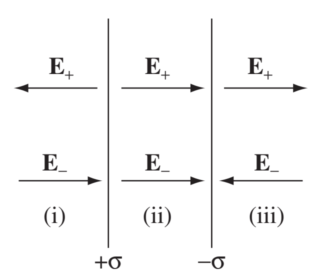


FIGURE 2.24

Problem 2.11 Use Gauss's law to find the electric field inside and outside a spherical shell of radius R that carries a uniform surface charge density σ. Compare your answer to Prob. 2.7.

Problem 2.12 Use Gauss's law to find the electric field inside a uniformly charged solid sphere (charge density ρ). Compare your answer to Prob. 2.8.

Problem 2.13 Find the electric field a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compare Eq. 2.9.

Problem 2.14 Find the electric field inside a sphere that carries a charge density proportional to the distance from the origin, $\rho = kr$, for some constant k. [*Hint:* This charge density is *not* uniform, and you must *integrate* to get the enclosed charge.]

Problem 2.15 A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \quad (a \leq r \leq b)$$

(Fig. 2.25). Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

Problem 2.16 A long coaxial cable (Fig. 2.26) carries a uniform *volume* charge density ρ on the inner cylinder (radius a), and a uniform *surface* charge density on the outer cylindrical shell (radius b). This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field in each of the three regions: (i) inside the inner cylinder ($s < a$), (ii) between the cylinders ($a < s < b$), (iii) outside the cable ($s > b$). Plot $|\mathbf{E}|$ as a function of s.

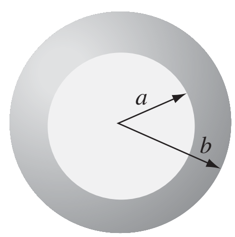


FIGURE 2.25

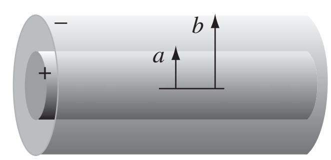


FIGURE 2.26

Problem 2.17 An infinite plane slab, of thickness $2d$, carries a uniform volume charge density ρ (Fig. 2.27). Find the electric field, as a function of y, where $y = 0$ at the center. Plot E versus y, calling E positive when it points in the $+y$ direction and negative when it points in the $-y$ direction.

• **Problem 2.18** Two spheres, each of radius R and carrying uniform volume charge densities $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap (Fig. 2.28). Call the vector from the positive center to the negative center $\mathbf{d}$. Show that the field in the region of overlap is constant, and find its value. [*Hint:* Use the answer to Prob. 2.12.]

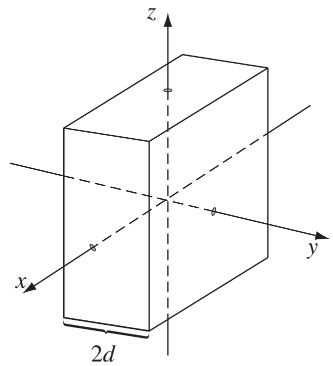


FIGURE 2.27

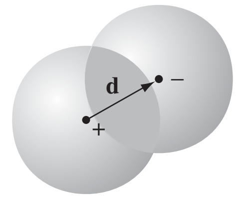


FIGURE 2.28

2.2.4 ■ The Curl of E

I'll calculate the curl of **E**, as I did the divergence in Sect. 2.2.1, by studying first the simplest possible configuration: a point charge at the origin. In this case

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$

Now, a glance at Fig. 2.12 should convince you that the curl of this field has to be zero, but I suppose we ought to come up with something a little more rigorous than that. What if we calculate the line integral of this field from some point **a** to some other point **b** (Fig. 2.29):

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l}.$$

In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, so

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr.$$

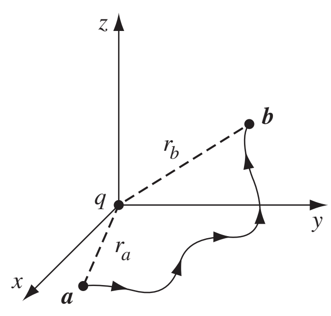


FIGURE 2.29

Therefore,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2} dr = \frac{-1}{4\pi\epsilon_0} \frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right), \tag{2.18}$$

where r_a is the distance from the origin to the point **a** and r_b is the distance to **b**. The integral around a *closed* path is evidently zero (for then $r_a = r_b$):

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = 0,} \tag{2.19}$$

and hence, applying Stokes' theorem,

$$\boxed{\nabla \times \mathbf{E} = \mathbf{0}.} \tag{2.20}$$

Now, I proved Eqs. 2.19 and 2.20 only for the field of a single point charge at the origin, but these results make no reference to what is, after all, a perfectly arbitrary choice of coordinates; they hold no matter *where* the charge is located. Moreover, if we have many charges, the principle of superposition states that the total field is a vector sum of their individual fields:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots,$$

so

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = \mathbf{0}.$$

Thus, Eqs. 2.19 and 2.20 hold for *any static charge distribution whatever.*

Problem 2.19 Calculate $\nabla \times \mathbf{E}$ directly from Eq. 2.8, by the method of Sect. 2.2.2. Refer to Prob. 1.63 if you get stuck.

2.3 ■ ELECTRIC POTENTIAL

2.3.1 ■ Introduction to Potential

The electric field **E** is not just *any* old vector function. It is a very special *kind* of vector function: one whose curl is zero. $\mathbf{E} = y\hat{\mathbf{x}}$, for example, could not possibly be an electrostatic field; *no* set of charges, regardless of their sizes and positions, could ever produce such a field. We're going to exploit this special property of electric fields to reduce a *vector* problem (finding **E**) to a much simpler *scalar* problem. The first theorem in Sect. 1.6.2 asserts that any vector whose curl is zero is equal to the gradient of some scalar. What I'm going to do now amounts to a proof of that claim, in the context of electrostatics.