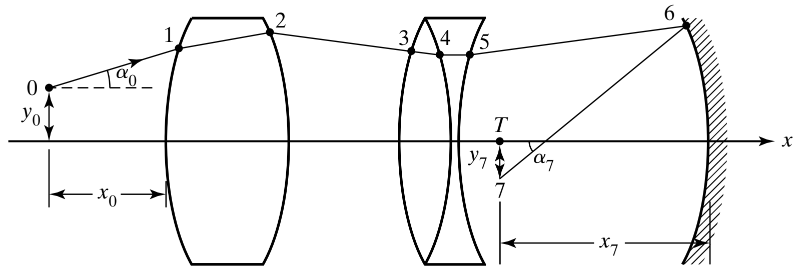


18 *Matrix Methods in Paraxial Optics*

INTRODUCTION

This chapter deals with methods of analyzing optical systems when they become complex, involving a number of refracting and/or reflecting elements in trainlike fashion. Beginning with a description of a single *thick lens* in terms of its *cardinal points*, the discussion proceeds to an analysis of a train of optical elements by means of multiplication of 2×2 matrices representing the elementary refractions or reflections involved in the train. In this way, a *system matrix* for the entire optical system can be found that is related to the same cardinal points characterizing the thick lens. Finally, computer ray-tracing methods for tracing a given ray of light through an optical system are briefly described.

1 THE THICK LENS

Consider a spherical *thick lens*, that is, a lens whose thickness along its optical axis cannot be ignored without leading to serious errors in analysis. Just when a lens moves from the category of *thin to thick* clearly depends on the accuracy required. The thick lens can be treated by methods you should already be familiar with. The glass medium is bounded by two spherical refracting surfaces. The image of a given object, formed by refraction at the first surface, becomes the object for refraction at the second surface. The object distance for the second surface takes into account the thickness of the lens. The image formed by the second surface is then the final image due to the action of the composite thick lens.

The thick lens can also be described in a way that allows graphical determination of images corresponding to arbitrary objects, much like the ray rules for a thin lens. This description, in terms of the so-called *cardinal points* of the lens, is useful also because it can be applied to more complex optical

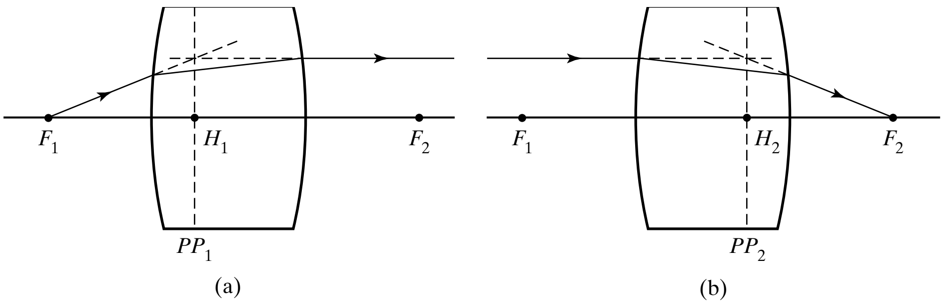


Figure 1 Illustration of the (a) first (PP_1) and (b) second (PP_2) principal planes of an optical system. The principal points H_1 and H_2 are also shown.

systems, as will become evident in this chapter. Thus, even though we are at present interested in a single thick lens, the following description is applicable to an arbitrary optical system that we can imagine is contained within the outlines of the thick lens.

There are six cardinal points on the axis of a thick lens, from which its imaging properties can be deduced. Planes[1] normal to the axis at these points are called the *cardinal planes*. The six cardinal points (see Figures 1 and 2) consist of the first and second *system focal points* (F_1 and F_2), which are already familiar; the first and second *principal points* (H_1 and H_2); and the first and second *nodal points* (N_1 and N_2).

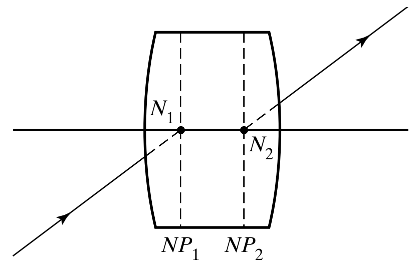


Figure 2 Illustration of the nodal points (N_1 and N_2) and nodal planes (NP_1 and NP_2) of an optical system.

A ray from the first focal point, F_1, is rendered parallel to the axis (Figure 1a), and a ray parallel to the axis is refracted by the lens through the second focal point, F_2 (Figure 1b). The extensions of the incident and resultant rays in each case intersect, by definition, in the *principal planes*, and these cross the axis at the principal points, H_1 and H_2. If the thick lens were a single thin lens, the two principal planes would coincide at the vertical line that is usually drawn to represent the lens. Principal planes in general do not coincide and may even be located outside the optical system itself. Once the locations of the principal planes are known, accurate ray diagrams can be drawn. The usual rays, determined by the focal points, change direction at their intersections with the principal planes, as in Figure 1. The third ray usually drawn for thin-lens diagrams is one through the lens center, undeviated and negligibly displaced. The nodal points of a thick lens, or of any optical system, permit the correction to this ray, as shown in Figure 2. Any ray directed toward the first nodal point, N_1, emerges from the optical system parallel to the incident ray, but displaced so that it appears to come from the second nodal point on the axis, N_2.

The positions of all six cardinal points are indicated in Figure 3. Distances are *directed*, positive or negative, by a sign convention that makes distances directed to the left negative and distances to the right positive. Notice that for the thick lens, the distances r and s determine the positions of the

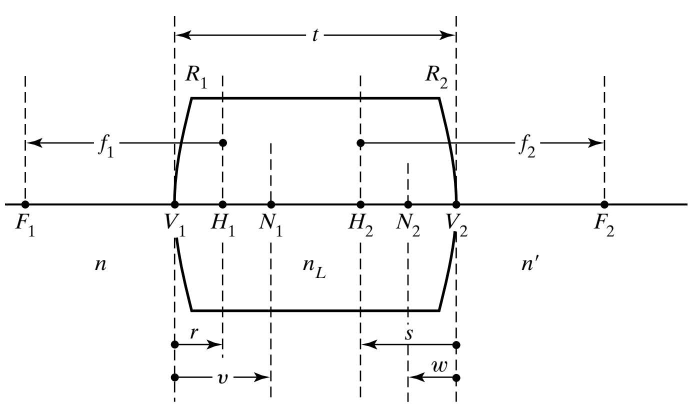


Figure 3 Symbols used to signify the cardinal points and locations for a thick lens. Axial points include focal points (F), vertices (V), principal points (H), and nodal points (N). Directed distances separating their corresponding planes are defined in the drawing.

[1]These "planes" are actually slightly curved surfaces that can be considered plane in the paraxial approximation.

principal points relative to the vertices V_1 and V_2, while f_1 and f_2 determine focal point positions relative to the principal points H_1 and H_2, respectively. Note carefully that these focal points are *not* measured from the vertices of the lens.

We summarize the basic equations for the thick lens without proof. Although the derivations involve simple algebra and geometry, they are rather arduous. We shall be content to await the matrix approach later in this chapter as a simpler way to justify these equations, and even then some of the work is relegated to the problems.

Utilizing the symbols defined in Figure 3, the focal length f_1 is given by

$$\frac{1}{f_1} = \frac{n_L - n'}{nR_2} - \frac{n_L - n}{nR_1} - \frac{(n_L - n)(n_L - n')}{nn_L}\frac{t}{R_1 R_2} \tag{1}$$

and the focal length f_2 is conveniently expressed in terms of f_1 by

$$f_2 = -\frac{n'}{n} f_1 \tag{2}$$

where n, n', and n_L are the refractive indices of the three regions indicated in Figure 3.

Notice that the two-focal lengths have the same magnitude if the lens is surrounded by a single refractive medium, so that $n = n'$. The principal planes can be located next using

$$r = \frac{n_L - n'}{n_L R_2} f_1 t \quad \text{and} \quad s = -\frac{n_L - n}{n_L R_1} f_2 t \tag{3}$$

The positions of the nodal points are given by

$$v = \left(1 - \frac{n'}{n} + \frac{n_L - n'}{n_L R_2} t\right) f_1 \quad \text{and} \quad w = \left(1 - \frac{n}{n'} - \frac{n_L - n}{n_L R_1} t\right) f_2 \tag{4}$$

Image and object distances and lateral magnification are related by

$$-\frac{f_1}{s_o} + \frac{f_2}{s_i} = 1 \quad \text{and} \quad m = -\frac{n s_i}{n' s_o} \tag{5}$$

as long as the distances s_o and s_i, as well as focal lengths, are measured relative to corresponding principal planes. The signs for s_o and s_i follow the usual sign convention. In the ordinary case of a lens in air, with $n = n' = 1$, notice that $r = v$ and $s = w$: First and second principal points are superimposed over corresponding nodal points. Also, first and second focal lengths are equal in magnitude, and the usual thin lens equations,

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \quad \text{and} \quad m = -\frac{s_i}{s_o} \tag{6}$$

are valid. Here we have noted that $f = f_2 = -f_1$.

Example 1

Determine the focal lengths and the principal points for a 4-cm thick, biconvex lens with refractive index of 1.52 and radii of curvature of 25 cm, when the lens caps the end of a long cylinder filled with water ($n = 1.33$).

Solution

Use the equations for the thick lens in the order given:

$$\frac{1}{f_1} = \frac{1.52 - 1.33}{1(-25)} - \frac{1.52 - 1}{1(+25)} - \frac{(1.52 - 1)(1.52 - 1.33)}{1(1.52)} \frac{4}{(+25)(-25)}$$

or $f_1 = -35.74$ cm to the *left* of the first principal plane. Then

$$f_2 = -\left(\frac{1.33}{1}\right)(-35.74) = 47.53 \text{ cm}$$

to the *right* of the second principal plane, and

$$r = \frac{1.52 - 1.33}{(1.52)(-25)}(-35.74)(4) = 0.715 \text{ cm}$$

$$s = -\frac{1.52 - 1}{(1.52)(+25)}(47.53)(4) = -2.60 \text{ cm}$$

Thus the principal point H_1 is situated 0.715 cm to the *right* of the left vertex of the lens, and H_2 is situated 2.60 cm to the *left* of the right vertex V_2.

2 THE MATRIX METHOD

When the optical system consists of several elements—for example, the four or five lenses that constitute a photographic lens—we need a systematic approach that facilitates analysis. As long as we restrict our analysis to *paraxial rays*, this systematic approach is well handled by the matrix method. We now present a treatment of image formation that employs matrices to describe changes in the height and angle of a ray as it makes its way by successive reflections and refractions through an optical system. We show that, in the paraxial approximation, changes in height and direction of a ray can be expressed by linear equations that make this matrix approach possible. By combining matrices that represent individual refractions, reflections, and translations, a given optical system may be represented by a single matrix, from which the essential properties of the composite optical system may be deduced. The method lends itself to computer techniques for tracing a ray through an optical system of arbitrary complexity.

Figure 4 shows the progress of a single ray through an arbitrary optical system. The ray is described at distance x_0 from the first refracting surface in terms of its height y_0 and slope angle α_0 relative to the optical axis. Changes in angle occur at each *refraction*, such as at points 1 through 5, and at each *reflection*, such as at point 6. The height of the ray changes during *translations* between these points. We look for a procedure that will allow us to calculate the height and slope angle of the ray at any point in the optical system, for example, at point T, a distance x_7 from the mirror. In other words,

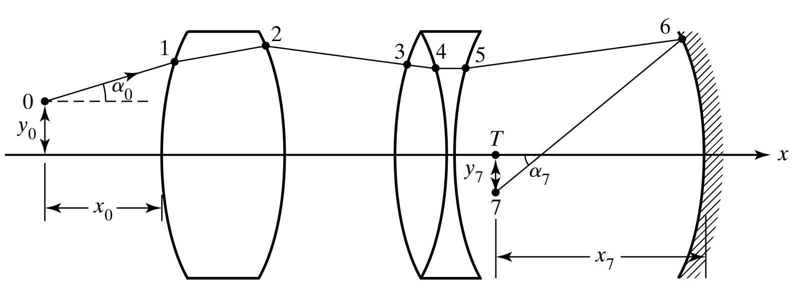


Figure 4 Steps in tracing a ray through an optical system. Progress of a ray can be described by changes in its elevation and direction.

given the input data (y_0, α_0) at point 0, we wish to predict values of (y_7, α_7) at point 7 as output data.

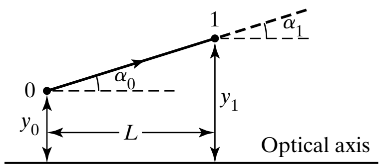


Figure 5 Simple translation of a ray.

3 THE TRANSLATION MATRIX

Consider a simple translation of the ray in a homogeneous medium, as in Figure 5. Let the axial progress of the ray be L, as shown, such that at point 1, the elevation and direction of the ray are given by "coordinates" y_1 and α_1, respectively. Evidently,

$$\alpha_1 = \alpha_0 \quad \text{and} \quad y_1 = y_0 + L \tan \alpha_0$$

These equations may be put into an ordered form,

$$\begin{aligned} y_1 &= (1)y_0 + (L)\alpha_0 \\ \alpha_1 &= (0)y_0 + (1)\alpha_0 \end{aligned} \tag{7}$$

where the paraxial approximation $\tan \alpha_0 \cong \alpha_0$ has been used. In matrix notation, the two equations are written

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \tag{8}$$

The 2×2 *ray-transfer matrix* represents the effect of the translation on a ray. The input data (y_0, α_0) is modified by the ray-transfer matrix to yield the correct output data (y_1, α_1).

4 THE REFRACTION MATRIX

Consider next the refraction of a ray at a spherical interface separating media of refractive indices n and n', as shown in Figure 6. We need to relate the ray coordinates (y', α') after refraction to those before refraction, (y, α). Since refraction occurs at a point, there is no change in elevation, and $y = y'$.

The angle α', on the other hand, is, by inspection of Figure 6 and the use of small angle approximations,

$$\alpha' = \theta' - \phi = \theta' - \frac{y}{R} \quad \text{and} \quad \alpha = \theta - \phi = \theta - \frac{y}{R}$$

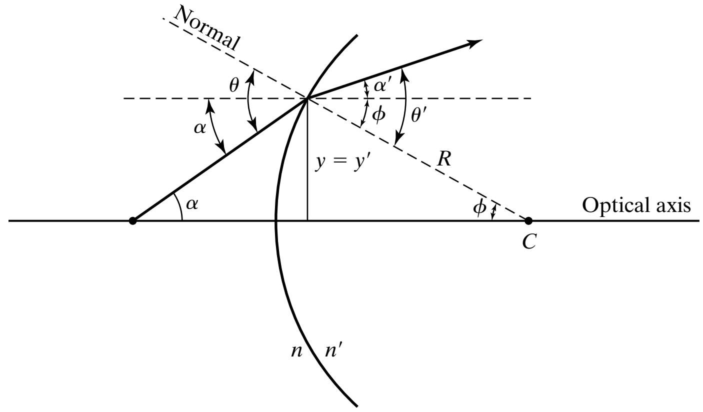


Figure 6 Refraction of a ray at a spherical interface.

Incorporating the paraxial form of Snell's law,

$$n\theta = n'\theta'$$

we have

$$\alpha' = \left(\frac{n}{n'}\right)\theta - \frac{y}{R} = \left(\frac{n}{n'}\right)\left(\alpha + \frac{y}{R}\right) - \frac{y}{R}$$

or

$$\alpha' = \left(\frac{1}{R}\right)\left(\frac{n}{n'} - 1\right)y + \left(\frac{n}{n'}\right)\alpha$$

The appropriate linear equations are then

$$\begin{aligned} y' &= (1)y + (0)\alpha \\ \alpha' &= \left[\left(\frac{1}{R}\right)\left(\frac{n}{n'} - 1\right)\right]y + \left(\frac{n}{n'}\right)\alpha \end{aligned} \tag{9}$$

or, in matrix form,

$$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{R}\left(\dfrac{n}{n'} - 1\right) & \dfrac{n}{n'} \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix} \tag{10}$$

Here, we use a sign convention for R that should be familiar to you. If the surface is instead concave, R is negative. Furthermore, allowing $R \rightarrow \infty$ yields the appropriate refraction matrix for a plane interface.

5 THE REFLECTION MATRIX

Finally, consider reflection at a spherical surface, illustrated in Figure 7. In the case considered, a concave mirror, R, is negative. We need to add a sign convention for the angles that describe the ray directions. Angles are considered positive for all rays pointing upward, either before or after a reflection; angles for rays pointing downward are considered negative. The sign convention is summarized in the inset of Figure 7.

From the geometry of Figure 7, with both α and α' positive,

$$\alpha = \theta + \phi = \theta + \frac{y}{-R} \quad \text{and} \quad \alpha' = \theta' - \phi = \theta' - \frac{y}{-R}$$

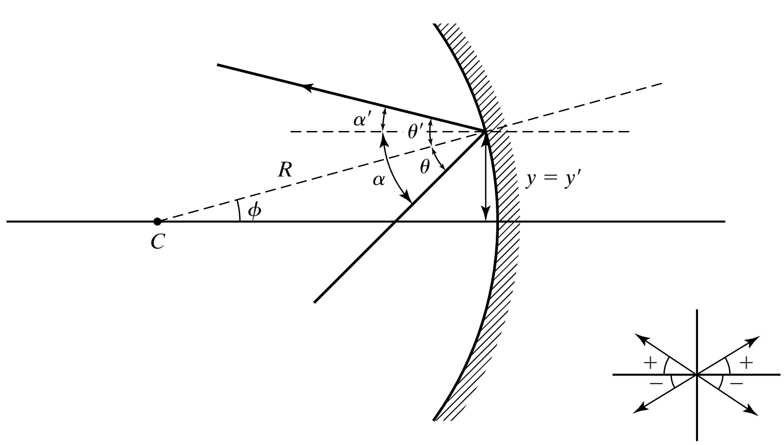


Figure 7 Reflection of a ray at a spherical surface. The inset illustrates the sign convention for ray angles.

where we have made the usual small angle approximations. Using these relations together with the law of reflection, $\theta = \theta'$,

$$\alpha' = \theta' + \frac{y}{R} = \theta + \frac{y}{R} = \alpha + \frac{2y}{R}$$

and so the two desired linear equations are

$$\begin{aligned} y' &= (1)y + (0)\alpha \\ \alpha' &= \left(\frac{2}{R}\right)y + (1)\alpha \end{aligned} \tag{11}$$

In matrix form,

$$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix} \tag{12}$$

6 THICK-LENS AND THIN-LENS MATRICES

We construct now a matrix that represents the action of a thick lens on a ray of light. For generality, we assume different media on opposite sides of the lens, having refractive indices n and n', as shown in Figure 8. In traversing the lens, the ray undergoes two refractions and one translation, steps for which we have already derived matrices. Referring to Figure 8, where we have chosen for simplicity a lens with positive radii of curvature, we may write, symbolically,

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \quad \text{for the first refraction}$$

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = M_2 \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} \quad \text{for the translation}$$

and

$$\begin{bmatrix} y_3 \\ \alpha_3 \end{bmatrix} = M_3 \begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} \quad \text{for the second refraction}$$

Telescoping these matrix equations results in

$$\begin{bmatrix} y_3 \\ \alpha_3 \end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

Evidently the entire thick lens can be represented by a matrix $M = M_3 M_2 M_1$. Recalling that the multiplication of matrices is associative but not commutative, the descending order must be maintained. The individual matrices operate on the light ray in the same order in which the corresponding optical actions influence the light ray as it traverses the system. Generalizing, the matrix equation representing any number N of translations, reflections, and refractions is given by

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \tag{13}$$

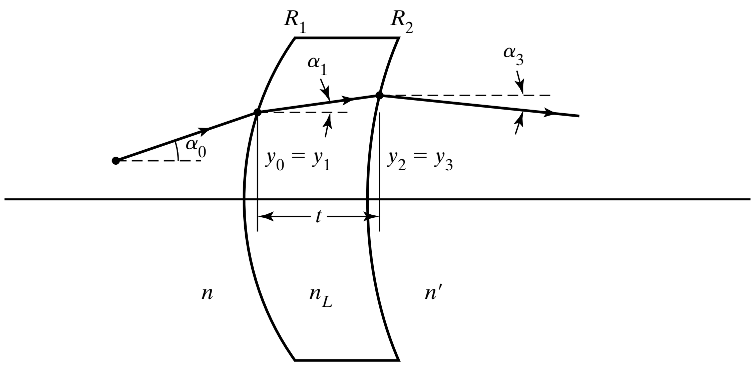


Figure 8 Progress of a ray through a thick lens.

and the ray-transfer matrix representing the entire optical system is

$$M = M_N M_{N-1} \cdots M_2 M_1 \tag{14}$$

We apply this result first to the thick lens of Figure 8, whose index is n_L and whose thickness for paraxial rays is t. The correct approximation for a thin lens is then made by allowing $t \rightarrow 0$. Letting $\mathfrak{R}$ represent a refraction matrix and $\mathfrak{T}$ represent a translation matrix, the matrix for the thick lens is, by Eq. (14), the composite matrix

$$M = \mathfrak{R}_2 \mathfrak{T} \mathfrak{R}_1$$

or

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n_L - n'}{n' R_2} & \dfrac{n_L}{n'} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{n - n_L}{n_L R_1} & \dfrac{n}{n_L} \end{bmatrix} \tag{15}$$

For the case where t is negligible $(t = 0)$ and where the lens is surrounded by the same medium on either side $(n = n')$,

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n_L - n}{n R_2} & \dfrac{n_L}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{n - n_L}{n_L R_1} & \dfrac{n}{n_L} \end{bmatrix} \tag{16}$$

Simplifying Eq. (16),

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n_L - n}{n}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right) & 1 \end{bmatrix} \tag{17}$$

The matrix element in the first column, second row, may be expressed in terms of the focal length of the lens, by the lensmaker's formula,

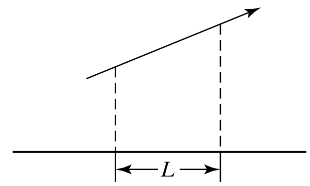
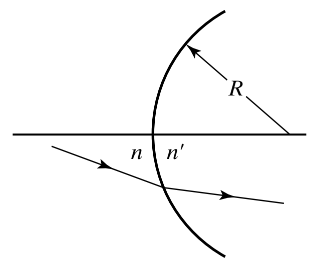
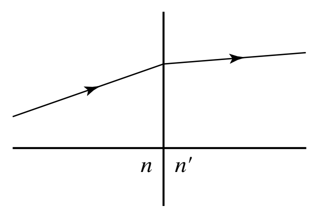
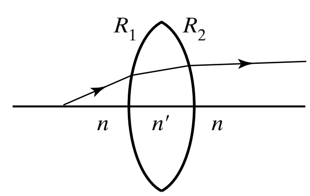
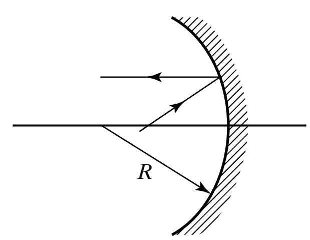
$$\frac{1}{f} = \frac{n_L - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

so that the thin-lens ray-transfer matrix is simply

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \tag{18}$$

As usual, f is taken as positive for a convex lens and negative for a concave lens. This matrix and those previously derived are summarized for quick reference in Table 1.

TABLE 1 SUMMARY OF SOME SIMPLE RAY-TRANSFER MATRICES

Translation matrix:	$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \mathfrak{T}$	
Refraction matrix, spherical interface:	$M = \begin{bmatrix} 1 & L \\ \dfrac{n-n'}{Rn'} & \dfrac{n}{n'} \end{bmatrix} = \mathfrak{R}$	 $(+R)$: convex $(-R)$: concave
Refraction matrix, plane interface:	$M = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n}{n'} \end{bmatrix}$	
Thin-lens matrix:	$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$ $\dfrac{1}{f} = \dfrac{n'-n}{n}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$	 $(+f)$: convex $(-f)$: concave
Spherical mirror matrix:	$M = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$	 $(+R)$: convex $(-R)$: concave

7 SYSTEM RAY-TRANSFER MATRIX

By combining appropriate individual matrices in the proper order, according to Eq. (14), it is possible to express any optical system by a single 2×2 matrix, which we call the *system matrix*.

Example 2

Find the system matrix for the thick lens of Figure 8, whose matrix before multiplication is expressed by Eq. (15), and specify the thick lens exactly by choosing $R_1 = 45$ cm, $R_2 = 30$ cm, $t = 5$ cm, $n_L = 1.60$, and $n = n' = 1$.

Solution

$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{50} & 1.6 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{120} & \frac{1}{1.6} \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} \frac{23}{24} & \frac{25}{8} \\ -\frac{7}{1200} & \frac{17}{16} \end{bmatrix}$$

The elements of this composite ray-transfer matrix, usually referred to in the symbolic form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

describe the relevant properties of the optical system, as we shall see. Be aware that the particular values of the matrix elements of a system depend on the location of the ray at input and output. In the case of the thick lens just calculated, the *input plane* was chosen at the left surface of the lens, and the *output plane* was chosen at its right surface. If each of these planes is moved some distance from the lens, the system matrix will also include an initial and a final translation matrix incorporating these distances. The matrix elements change and the system matrix now represents this enlarged "system." In any case, the determinant of the system matrix has a very useful property:

$$\text{Det } M = AD - BC = \frac{n_0}{n_f} \tag{19}$$

where n_0 and n_f are the refractive indices of the initial and final media of the optical system. The proof of this assertion follows upon noticing first that the determinant of all the individual ray-transfer matrices in Table 1 have values of either n/n' or unity and then making use of the theorem[2] that the determinant of a product of matrices is equal to the product of the determinants. Symbolically, if $M = M_1 M_2 M_3 \cdots M_N$, then

$$\text{Det}(M) = (\text{Det } M_1)(\text{Det } M_2)(\text{Det } M_3)\cdots(\text{Det } M_N) \tag{20}$$

In forming this product, using determinants of ray-transfer matrices, all intermediate refractive indices cancel, and we are left with the ratio n_0/n_f, as stated in Eq. (19). Most often, as in the case of the thick-lens example, n_0 and n_f both refer to air, and Det (M) is unity. The condition expressed by Eq. (19) is useful in checking the correctness of the calculations that produce a system matrix.

[2]The theorem can easily be verified for the product of two matrices and generalized by induction to the product of any number of matrices. Formal proofs can be found in any standard textbook on matrices and determinants, for example, E. T. Browne, *Introduction to the Theory of Determinants and Matrices* (Chapel Hill: University of North Carolina, 1958).

8 SIGNIFICANCE OF SYSTEM MATRIX ELEMENTS

We examine now the implications that follow when each of the matrix elements in turn is zero. In symbolic form, we have, from Eq. (13),

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \tag{21}$$

which is equivalent to the algebraic relations

$$\begin{aligned} y_f &= Ay_0 + B\alpha_0 \\ \alpha_f &= Cy_0 + D\alpha_0 \end{aligned} \tag{22}$$

1. $D = 0$. In this case, $\alpha_f = Cy_0$, independent of α_0. Since y_0 is fixed, this means that all rays leaving a point in the input plane will have the same angle α_f at the output plane, independent of their angles at input. As shown in Figure 9a, the input plane thus coincides with the first focal plane of the optical system.
2. $A = 0$. This case is much like the previous one. Here $y_f = B\alpha_0$ implies that y_f is independent of y_0, so that all rays departing the input plane at the same angle, regardless of altitude, arrive at the same altitude y_f at the output plane. As shown in Figure 9b, the output plane thus functions as the second focal plane.
3. $B = 0$. Then $y_f = Ay_0$, independent of α_0. Thus, all rays from a point at height y_0 in the input plane arrive at the same point of height y_f in the output plane. The points are then related as object and image points, as shown in Figure 9c, and the input and output planes correspond to

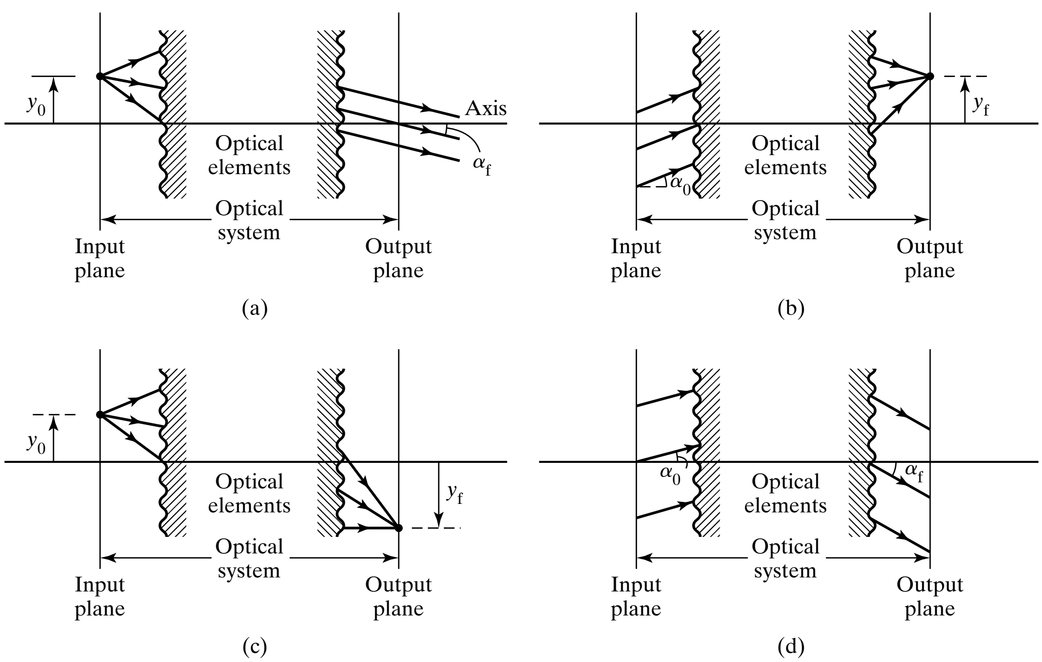


Figure 9 Diagrams illustrating the significance of the vanishing of specific system matrix elements. (a) When $D = 0$, the input plane corresponds to the first focal plane of the optical system. (b) When $A = 0$, the output plane corresponds to the second focal plane of the optical system. (c) When $B = 0$, the output plane is the image plane conjugate to the input plane and A is the linear magnification. (d) When $C = 0$, a parallel bundle of rays at the input plane is parallel at the output plane and D is the angular magnification.

conjugate planes for the optical system. Furthermore, since $A = y_f/y_0$, the matrix element A represents the linear magnification.

4. $C = 0$. Now $\alpha_f = D\alpha_0$, independent of y_0. This case is analogous to case 3, with directions replacing ray heights. Input rays, all of one direction, now produce parallel output rays in some other direction. Moreover, $D = \alpha_f/\alpha_0$ is the angular magnification. A system for which $C = 0$ is sometimes called a "telescopic system," because a telescope admits parallel rays into its objective and outputs parallel rays for viewing from its eyepiece.

Example 3

We illustrate case 3 in this example. We place a small object on axis at a distance of 16 cm from the left end of a long, plastic rod with a polished spherical end of radius 4 cm, as indicated in Figure 10. The refractive index of the plastic is 1.50 and the object is in air. Let the unknown image be formed at the output reference plane, a distance x from the spherical cap. We wish to determine the image distance x and the lateral magnification m. The system matrix connecting the object and image planes consists of the product of three matrices, corresponding to (1) a translation $\mathfrak{T}_1$ in air from object to the rod, (2) a refraction $\mathfrak{R}$ at the spherical surface, and (3) a translation $\mathfrak{T}_2$ in plastic to the image.

Solution

Remembering to take the matrices in "reverse" order and working in cm, we have

$$M = \mathfrak{T}_2\mathfrak{R}\mathfrak{T}_1 = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1 - 1.50}{4(1.50)} & \dfrac{1}{1.50} \end{bmatrix} \begin{bmatrix} 1 & 16 \\ 0 & 1 \end{bmatrix}$$

or

$$M = \begin{bmatrix} 1 - \dfrac{x}{12} & 16 - \dfrac{2x}{3} \\ -\dfrac{1}{12} & -\dfrac{2}{3} \end{bmatrix}$$

with the unknown quantity x incorporated in the matrix elements. According to this discussion, when $B = 0$, the output plane is the image plane, so that the image distance x is determined by setting

$$16 - \frac{2x}{3} = 0 \quad \text{or} \quad x = 24 \text{ cm}$$

Further, the linear magnification m is then given by the value of element A:

$$m = A = 1 - \frac{x}{12} = -1$$

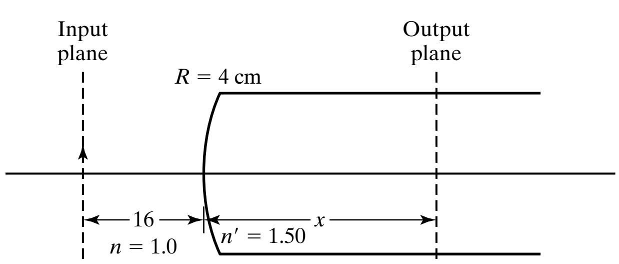


Figure 10 Schematic defining an example for ray-transfer matrix methods.

We conclude that the image occurs 24 cm inside the rod, is inverted, and has the same lateral size as the object. This illustrates how the system matrix can be used to find image locations and sizes, although this may usually be done more quickly by using the Gaussian image formulas derived earlier.

9 LOCATION OF CARDINAL POINTS FOR AN OPTICAL SYSTEM

Since the properties of an optical system can be deduced from the elements of the system ray-transfer matrix, it follows that relationships must exist between the matrix elements, A, B, C, and D and the cardinal points of the system. In Figure 11, we generalize Figure 3 by defining distances locating the six cardinal points relative to the input and output planes that define the limits of an optical system. The focal points F_1 and F_2 are located at distances f_1 and f_2 from the principal points H_1 and H_2 and at distances p and q from the reference input and output planes, respectively. Further, measured from the input and output planes, the distances r and s locate the principal points, and the distances v and w locate the nodal points. Distances measured to the right of their reference planes are considered positive and to the left, negative. The principal points and nodal points often occur outside the optical system, that is, outside the region defined by the input and output planes.

We now derive the relationships between the distances defined in Figure 11 and the system matrix elements. Consider Figure 12a, which highlights distances p, r, and f_1 as they are determined by the positions of the first focal point and the first principal plane. Input coordinates of the given ray are (y_0, α_0) and output coordinates are $(y_f, 0)$. Thus, the ray equations, Eq. (22), become for this ray

$$y_f = Ay_0 + B\alpha_0$$

and

$$0 = Cy_0 + D\alpha_0 \quad \text{or} \quad y_0 = -\left(\frac{D}{C}\right)\alpha_0 \tag{23}$$

For small angles, Figure 12a shows that

$$\alpha_0 = \frac{y_0}{-p}$$

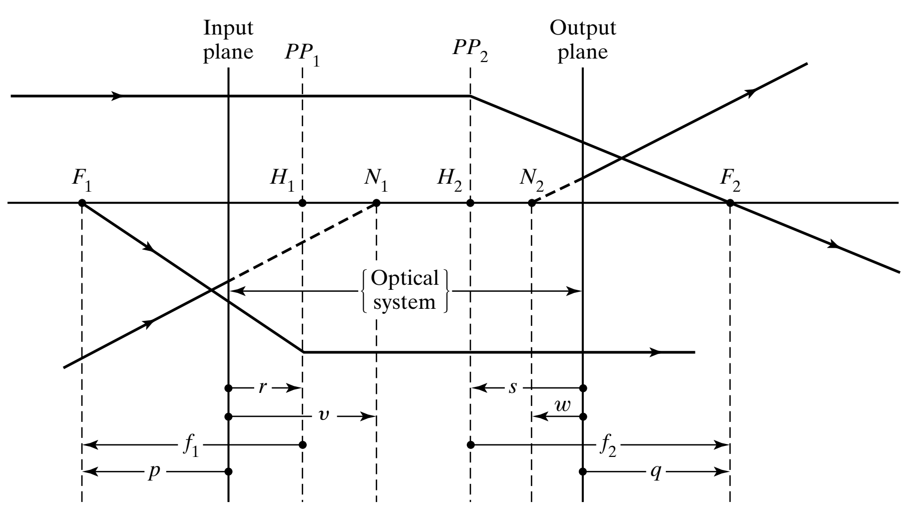


Figure 11 Location designations for the six cardinal points of an optical system. Rays associated with the nodal points and principal planes are also shown.

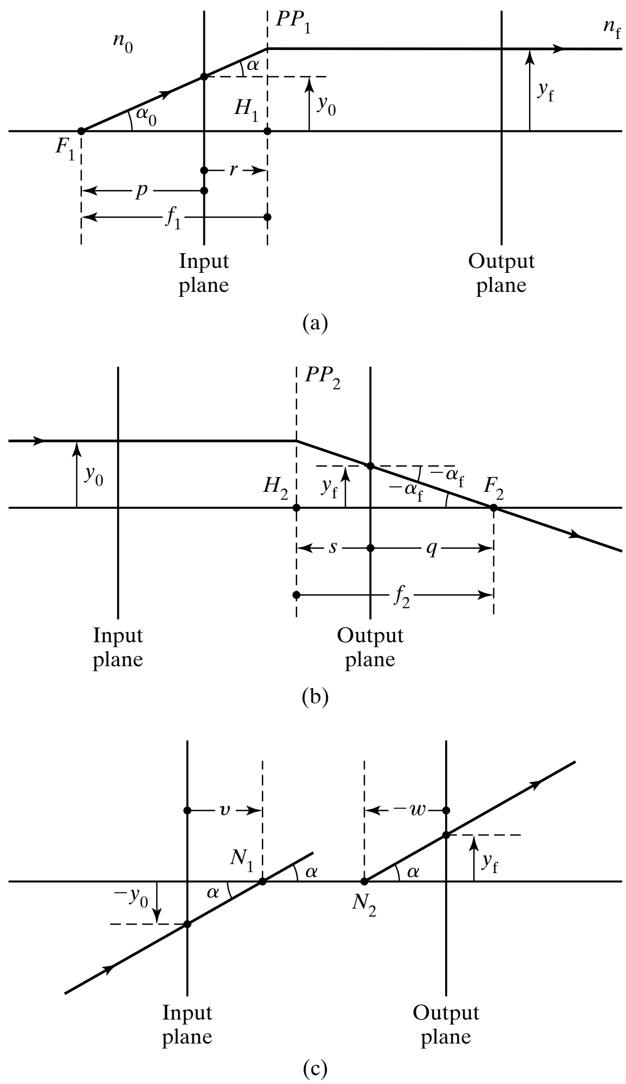


Figure 12 (a) Construction used to relate distances p, r, and f_1 to matrix elements. (b) Construction used to relate distances q, s, and f_2 to matrix elements. (c) Construction used to relate distances v and w to matrix elements.

where the negative sign indicates that F_1 is located a distance p to the left of the input plane. Incorporating Eq. (23),

$$p = \frac{-y_0}{\alpha_0} = \frac{D}{C} \tag{24}$$

Similarly, $\alpha_0 = y_f/(-f_1)$, and thus

$$f_1 = \frac{-y_f}{\alpha_0} = \frac{-(Ay_0 + B\alpha_0)}{\alpha_0} = \frac{AD}{C} - B$$

$$f_1 = \frac{AD - BC}{C} = \frac{\text{Det}(M)}{C} = \left(\frac{n_0}{n_f}\right)\frac{1}{C} \tag{25}$$

Finally, using Eqs. (24) and (25), the positive distance r can be expressed in terms of p and f_1:

$$r = p - f_1 = \frac{D}{C} - \frac{n_0}{n_f}\frac{1}{C} = \frac{1}{C}\left(D - \frac{n_0}{n_f}\right) \tag{26}$$

Using Figure 12b, one can similarly discover relations for the output distances q, f_2, and s. The results, together with those just derived for p, f_1,

TABLE 2 CARDINAL POINT LOCATIONS IN TERMS OF SYSTEM MATRIX ELEMENTS

$p = \frac{D}{C}$	F_1	Located relative to input (1) and output (2) reference planes
$q = -\frac{A}{C}$	F_2	
$r = \frac{D - n_0/n_f}{C}$	H_1	
$s = \frac{1 - A}{C}$	H_2	
$v = \frac{D - 1}{C}$	N_1	
$w = \frac{n_0/n_f - A}{C}$	N_2	
$f_1 = p - r = \frac{n_o/n_f}{C}$	F_1	Located relative to principal planes
$f_s = q - s = -\frac{1}{C}$	F_2	

and r, are listed in Table 2. With the help of Figure 12c, the nodal plane distances v and w may also be determined. For example, for small angle α,

$$\alpha = -\frac{y_0}{v} \tag{27}$$

where the negative sign indicates that the ray intersects the input plane below the axis. Input and output rays make the same angle relative to the axis. From Eq. (22), with $\alpha_0 = \alpha_f = \alpha$,

$$\alpha = Cy_0 + D\alpha \quad \text{or} \quad \frac{y_0}{\alpha} = \frac{1 - D}{C} \tag{28}$$

Combining Eqs. (27) and (28),

$$v = \frac{D - 1}{C} \tag{29}$$

Similarly, one can show that

$$w = \frac{(n_0/n_f) - A}{C} \tag{30}$$

again using the fact that Det $(M) = AD - BC = n_0/n_f$. These results are also included in Table 2. The relationships listed there can be used to establish the following useful generalizations:

1. Principal points and nodal points coincide, that is, $r = v$ and $s = w$, when the initial and final media have the same refractive indices.
2. First and second focal lengths of an optical system are equal in magnitude when initial and final media have the same refractive indices.
3. The separation of the principal points is the same as the separation of nodal points, that is, $r - s = v - w$.

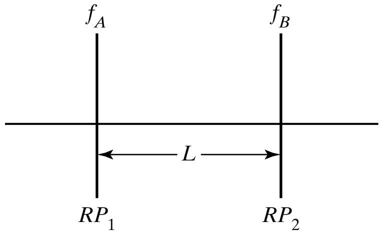


Figure 13 Optical system consisting of two thin lenses in air, separated by a distance L.

10 EXAMPLES USING THE SYSTEM MATRIX AND CARDINAL POINTS

As an example, consider an optical system that consists of two thin lenses in air, separated by a distance L, as shown in Figure 13. The lenses have focal lengths of f_A and f_B, which may be either positive or negative.

If input and output reference planes are located at the lenses, the system matrix includes two thin-lens matrices, $\mathfrak{L}_A$ and $\mathfrak{L}_B$, and a translation matrix $\mathfrak{T}$ for the distance L between them. The system matrix is $M = \mathfrak{L}_B\mathfrak{T}\mathfrak{L}_A$, or

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_B} & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_A} & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 - \dfrac{L}{f_A} & L \\ \dfrac{1}{f_B}\left(\dfrac{L}{f_A} - 1\right) - \dfrac{1}{f_A} & 1 - \dfrac{L}{f_B} \end{bmatrix} \tag{31}$$

Reference to Table 2 shows that the first and second focal lengths of this system are $f_1 = 1/C$ and $f_2 = -1/C$. We shall take the *equivalent focal length* of the two-lens system to be $f_{eq} = f_2 = -1/C$. So,

$$\frac{1}{f_{eq}} = \frac{1}{f_A} + \frac{1}{f_B} - \frac{L}{f_A f_B} \tag{32}$$

Furthermore, the first principal points and nodal points coincide at a distance given by $r = v = (D - 1)/C$ from the first lens, and the second principal points and nodal points coincide at a distance given by $s = w = (1 - A)/C$ from the second lens. Thus

$$r = v = \left(\frac{f_{eq}}{f_B}\right)L \quad \text{and} \quad s = w = -\left(\frac{f_{eq}}{f_A}\right)L \tag{33}$$

Example 4

Let us apply these results to the case of a Huygens eyepiece, which consists of two positive, thin lenses separated by a distance L equal to the average of their focal lengths. Suppose $f_A = 3.125$ cm and $f_B = 2.083$ cm, giving $L = 2.604$ cm and $f_{eq} = 2.5$ cm, by Eq. (32). Incidentally, the magnifying power of this eyepiece, given by 25/*f*, is therefore 10×. From Eq. (33), we conclude that $r = +3.125$ cm and $s = -2.083$ cm. The optical system, together with its cardinal points and sample rays, is shown roughly to scale in Figure 14. The converging incident rays 1, 2 and 3 determine an image location between the lenses, which acts as a virtual object *VO* for the optical system. An enlarged, virtual image (not shown) is formed by the diverging rays leaving the system, as seen by an eye looking into the eyepiece.

Solution

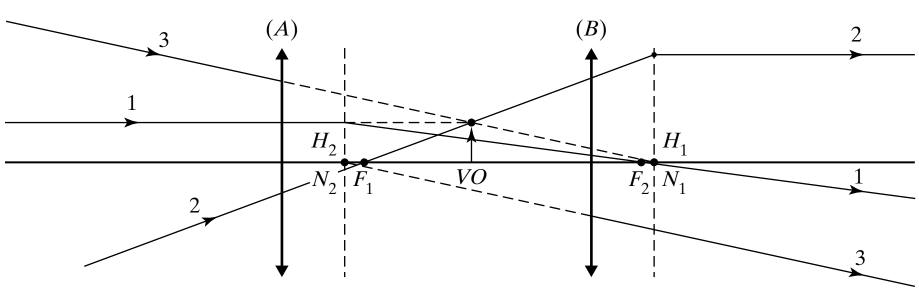


Figure 14 Ray construction for a Huygens eyepiece, using cardinal points.

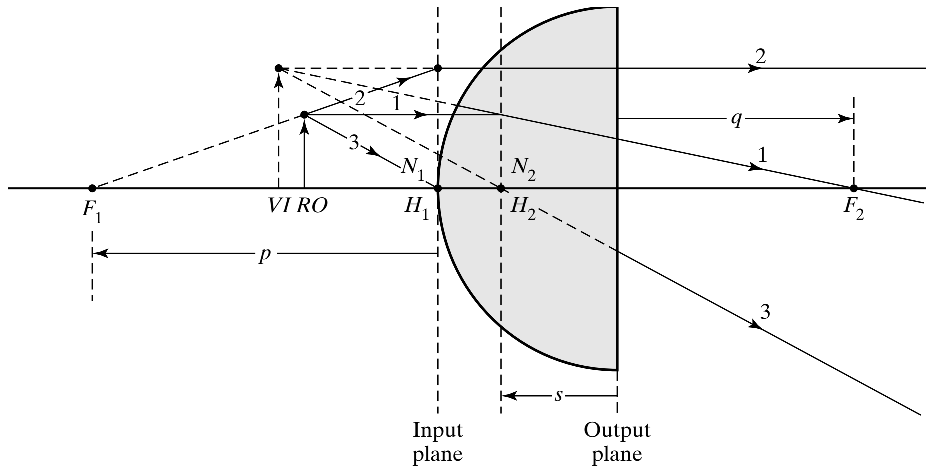


Figure 15 Ray construction for a hemispherical lens, using cardinal points.

Example 5

As a final calculation, let us find the cardinal points and sketch a ray diagram for the hemispherical glass lens shown in Figure 15. The radii of curvature are $R_1 = 3$ cm and $R_2 \to \infty$, and the lens in air has a refractive index of 1.50.

Solution

The system matrix, for input and output reference planes at the two surfaces of the lens, is, then,

$$M = \mathfrak{R}_2 \mathfrak{T} \mathfrak{R}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{-0.5}{1.5(3)} & \dfrac{1}{1.5} \end{bmatrix}$$

or

$$M = \begin{bmatrix} \dfrac{2}{3} & 2 \\ -\dfrac{1}{6} & 1 \end{bmatrix}, \quad \text{with } \mathrm{Det}(M) = 1$$

The relations in Table 2 then give values of $p = -6$ cm, $q = 4$ cm, $r = 0$, $s = -2$ cm, $f_1 = -6$ cm, and $f_2 = 6$ cm. Principal and nodal points coincide. The cardinal points are located, approximately to scale, in Figure 15. Ray diagrams using the principal planes and nodal points are constructed for an arbitrary real object. In this case the emerging rays determine a virtual image VI near the object RO erect and slightly magnified.

11 RAY TRACING

The assumption of paraxial rays greatly simplifies the description of the progress of rays of light through an optical system, because trigonometric terms do not appear in the equations. For many purposes, this treatment is sufficient. In practice, rays of light contributing to an image in an optical system are, in fact, usually rays in the near neighborhood of the optical axis. If the quality of the image is to be improved, however, ways must be found to reduce the ever-present aberrations that arise from the presence of rays deviating, more or less, from this ideal assumption. To determine the actual path of individual rays of light through an optical system, each ray must be *traced*, independently, using only the laws of reflection and refraction together with geometry. This technique is called *ray tracing* because it was formerly done by hand, graphically, with ruler and compass, in a step-by-step process through an accurate sketch of the optical system. Today, with the help of computers, the necessary calculations yielding the progressive changes in a ray's altitude and angle is done more easily and quickly. Graphic techniques are used to actually draw the optical system and to trace the ray's progress through the optical system on the monitor.[3]

Ray-tracing procedures, such as the one to be described here, are often limited to *meridional rays*, that is, rays that pass through the optical axis of the system. Since the law of refraction requires that refracted rays remain in the plane of incidence, a meridional ray remains within the same meridional plane throughout its trajectory. Thus the treatment in terms of meridional rays is a two-dimensional treatment,[4] greatly simplifying the geometrical relationships required. Rays contributing to the image that do not pass through the optical axis are called *skew rays* and require three-dimensional geometry in their calculations. The added complexity does not pose a problem for the computer, once the ray-tracing program is written. Analysis of various aberrations, such as spherical aberration, astigmatism, and coma, require knowledge of the progress of selected nonparaxial rays and skew rays. The design of a complex lens system, such as a photographic lens with four or five elements, is a combination of science and skill. By alternating ray tracing with small changes in the positions, focal lengths, and curvatures of the surfaces involved and in refractive indices of the elements, the design of the lens system is gradually optimized.

For our present purposes, it will be sufficient to show how the appropriate equations for meridional ray tracing can be developed and how they can be repeated in stepwise fashion to follow a ray through any number of spherical refracting surfaces that constitute an optical system. The technique is well adapted to iterative loops handled by computer programs.

Figure 16 shows a single, representative step in the ray-tracing analysis. By incorporating a sign convention, the equations developed from this diagram can be made to apply to any ray and to any spherical refracting surface. The ray selected originates at (or passes through) point A, making an angle α with the optical axis. The ray passes through the optical axis at O and then

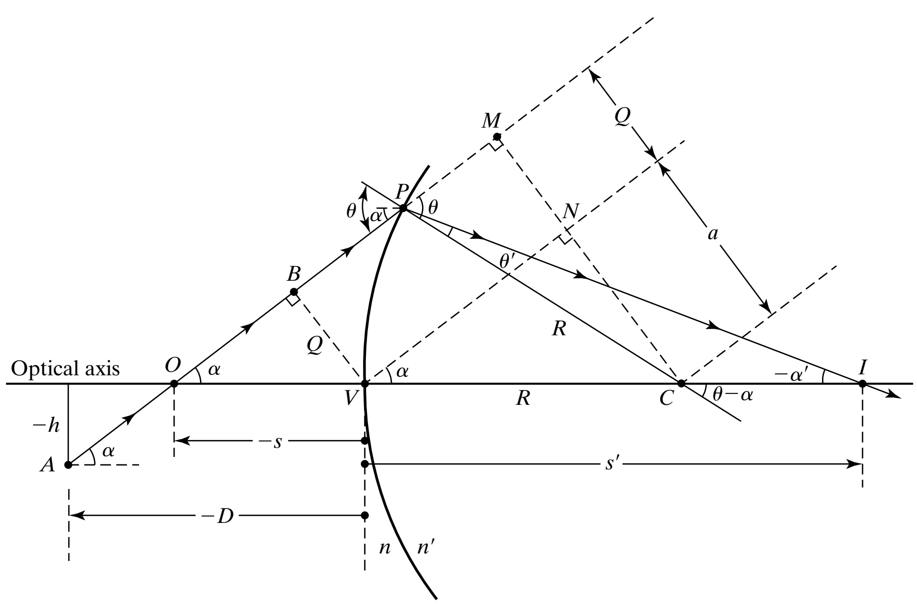


Figure 16 Single refraction at a spherical surface. The figure defines the symbols and shows the geometrical relationships that lead to ray-tracing equations for a meridional ray.

[3]A search of the Internet will reveal the existence of several free, high-quality, ray-tracing programs.

[4]The two dimensions are those of the page on which we have been drawing our ray diagrams. Without emphasizing this, we have been using meridional rays in all our diagrams.

intersects the refracting surface at P, where it is refracted into a medium of index n', cutting the axis again at I. The angles of incidence and refraction, θ and θ', are related by Snell's law. Points O and I are conjugate points with distances s and s' from the surface vertex at V. The radius R of the surface is also shown, passing through the center of curvature at C. Other points and lines are added to help in developing the necessary geometrical relationships.

The sign convention is the same as that used previously in this chapter. Distances to the left of the vertex V are negative, and to the right, positive. If we use light rays progressing from left to right, their angles have the same sign as their slopes. Distances measured above the axis are positive and below, negative. An important quantity in the calculations, also subject to this sign convention, is the parameter Q, the perpendicular distance VB from the vertex to the ray, as shown.

The input parameters for the ray are its elevation h, angle α, and distance D. Figure 16 shows that the "object distance," s, is related to D by

$$s = D - \frac{h}{\tan \alpha} \tag{34}$$

Also, in ΔOBV:

$$\sin \alpha = \frac{Q}{-s} \tag{35}$$

In ΔPMC:

$$\sin \theta = \frac{a + Q}{R}$$

In ΔVNC:

$$\sin \alpha = \frac{a}{R}$$

Eliminating the length a from the last two equations, we get

$$\sin \theta = \frac{Q}{R} + \sin \alpha \tag{36}$$

Snell's law at P:

$$n \sin \theta = n' \sin \theta' \tag{37}$$

In ΔCPI:

$$\theta - \alpha = \theta' - \alpha' \tag{38}$$

The Q parameter for the refracted ray is shown in Figure 17a as Q'. Analogous to the relations just found, we see that in ΔCMV:

$$\sin(-\alpha') = \frac{a'}{R}$$

in ΔPLC:

$$\sin \theta' = \frac{Q' - a'}{R}$$

As before, when a' is eliminated, there results

$$Q' = R(\sin \theta' - \sin \alpha') \tag{39}$$

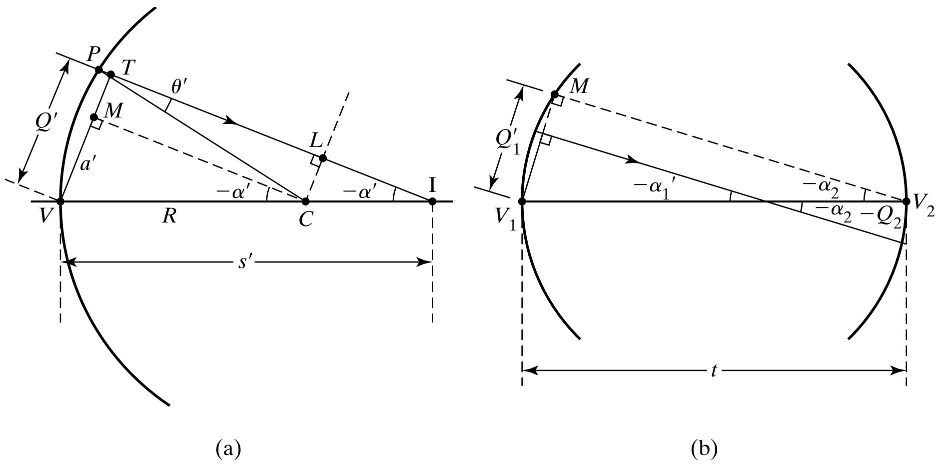


Figure 17 (a) Geometrical relationship of refracted-ray parameters with the distance Q'. (b) Geometrical relationships illustrating the transfer between Q and α after one refraction and before the next.

In ΔITV:

$$\sin(-\alpha') = \frac{Q'}{s'} \quad \text{or} \quad s' = \frac{-Q'}{\sin\alpha'} \tag{40}$$

The relevant equations describing the first refraction are included in Table 3 under the first column for the general case.

The calculations lead to new values of α, Q, and s (now primed), which prepare for the next refraction in the sequence. The geometrical *transfer* to the next surface, at distance t from the first, is shown in Figure 17b, where, in ΔV_2MV_1,

$$\sin(-\alpha_2) = \frac{V_1M}{t} = \frac{Q_1' - Q_2}{t}$$

TABLE 3 MERIDIONAL RAY-TRACING EQUATIONS (INPUT: $n, n', R, \boldsymbol{\alpha}, h, D$)

General case	Ray parallel to axis: $\alpha = 0$	Plane surface: $R \Rightarrow \infty$
$s = D - \dfrac{h}{\tan\alpha}$	—	$s = D - \dfrac{h}{\tan\alpha}$
$Q = -s\sin\alpha$	$Q = h$	$Q = -s\sin\alpha$
$\theta = \sin^{-1}\left(\dfrac{Q}{R} + \sin\alpha\right)$	$\theta = \sin^{-1}\left(\dfrac{Q}{R} + \sin\alpha\right)$	—
$\theta' = \sin^{-1}\left(\dfrac{n\sin\theta}{n'}\right)$	$\theta' = \sin^{-1}\left(\dfrac{n\sin\theta}{n'}\right)$	—
$\alpha' = \theta' - \theta + \alpha$	$\alpha' = \theta' - \theta + \alpha$	$\alpha' = \sin^{-1}\dfrac{n}{n'\sin\alpha}$
$Q' = R(\sin\theta' - \sin\alpha')$	$Q' = R(\sin\theta' - \sin\alpha')$	$Q' = Q\dfrac{\cos\alpha'}{\cos\alpha}$
$s' = \dfrac{-Q'}{\sin\alpha'}$	$s' = \dfrac{-Q'}{\sin\alpha'}$	$s' = \dfrac{-Q'}{\sin\alpha'}$
Transfer: Input: t		
$Q = Q' + t\sin\alpha'$		
$\alpha = \alpha'$		
$n = n'$		
Input: new n', R		
Return: to calculate θ		

or

$$Q_2 = Q_1' + t \sin \alpha_2 \tag{41}$$

Table 3 also shows how the equations must be modified for two special cases: (1) when the incident ray is parallel to the axis and (2) when the surface is plane, with an infinite radius of curvature.

Example 6

Do a ray trace for two rays through a *Rapid landscape* photographic lens of three elements. The parallel rays enter the lens from a distant object at altitudes of 1 and 5 mm above the optical axis. The lens specifications (all dimensions in mm) are as follows:

$$\begin{array}{lll} R_1 = -120.8 & & \\ R_2 = -34.6 & t_1 = 6 & n_1 = 1.521 \\ R_3 = -96.2 & t_2 = 2 & n_2 = 1.581 \\ R_4 = -51.2 & t_3 = 3 & n_3 = 1.514 \end{array}$$

Solution

Since the rays are parallel to the axis, the second column of Table 3 is used to calculate the progress of the ray. These can be tabulated as follows:

Input	Results: ray at $h = 1$	Results: ray at $h = 5$
First surface:		
$n = 1, n' = 1.521$	$Q = 1$	$Q = 5$
$\alpha = 0$	$\alpha' = 0.1625°$	$\alpha' = 0.8128°$
$h = 1$ or 5	$s' = -352.66$	$s' = -352.53$
$R = -120.8$	$Q' = 1.0000$	$Q' = 5.0010$
Second surface:		
$t = 6$	$Q = 1.0170$	$Q = 5.0861$
$n = 1.581$	$\alpha' = 0.2202°$	$\alpha' = 1.1041°$
$R = -34.6$	$s' = -264.59$	$s' = -264.03$
	$Q' = 1.0170$	$Q' = 5.0876$
Third surface:		
$t = 2$	$Q = 1.0247$	$Q = 5.1261$
$n = 1.514$	$\alpha' = 0.2030°$	$\alpha' = 1.0178°$
$R = -96.2$	$s' = -289.26$	$s' = -288.58$
	$Q' = 1.0247$	$Q' = 5.1260$
Final surface:		
$t = 3$	$Q = 1.0353$	$Q = 5.1793$
$n = 1$	$\alpha' = -0.2883°$	$\alpha' = -1.4520°$
$R = -51.2$	$s' = 205.72$	$s' = 203.91$
	$Q' = 1.0353$	$Q' = 5.1672$

Thus the two rays intersect the optical axis at 205.72 and 203.91 mm beyond the final surface, missing a common focus by 1.8 mm.

PROBLEMS

1 A biconvex lens of 5 cm thickness and index 1.60 has surfaces of radius 40 cm. If this lens is used for objects in water, with air on its opposite side, determine its effective focal length and sketch its focal and principal points.

2 A double concave lens of glass with $n = 1.53$ has surfaces of 5 D (diopters) and 8 D, respectively. The lens is used in air and has an axial thickness of 3 cm.

a. Determine the position of its focal and principal planes.

b. Also find the position of the image, relative to the lens center, corresponding to an object at 30 cm in front of the first lens vertex.

c. Calculate the paraxial image distance assuming the thin-lens approximation. What is the percent error involved?

3 A biconcave lens has radii of curvature of 20 cm and 10 cm. Its refractive index is 1.50 and its central thickness is 5 cm. Describe the image of a 1-in.-tall object, situated 8 cm from the first vertex.

4 An equiconvex lens having spherical surfaces of radius 10 cm, a central thickness of 2 cm, and a refractive index of 1.61 is situated between air and water ($n = 1.33$). An object 5 cm high is placed 60 cm in front of the lens surface. Find the cardinal points for the lens and the position and size of the image formed.

5 A hollow glass sphere of radius 10 cm is filled with water. Refraction due to the thin glass walls is negligible for paraxial rays.

- **a.** Determine its cardinal points and make a sketch to scale.
- **b.** Calculate the position and magnification of a small object 20 cm from the sphere.
- **c.** Verify your analytical results by drawing appropriate rays on your sketch.

6 Light rays enter the plane surface of a glass hemisphere of radius 5 cm and refractive index 1.5.

- **a.** Using the system matrix representing the hemisphere, determine the exit elevation and angle of a ray that enters parallel to the optical axis and at an elevation of 1 cm.
- **b.** Enlarge the system to a distance x beyond the hemisphere and find the new system matrix as a function of x.
- **c.** Using the new system matrix, determine where the ray described above crosses the optical axis.

7 Using Figure 12b and c, verify the expressions given in Table 2 for the distances q, f_2, s, and w.

8 A lens has the following specifications:
$R_1 = 1.5 \text{ cm} = R_2$, $d(\text{thickness}) = 2.0$ cm,
$n_1 = 1.00, n_2 = 1.60, n_3 = 1.30$.
Find the principal points using the matrix method. Include a sketch, roughly to scale, and do a ray diagram for a finite object of your choice.

9 A positive thin lens of focal length 10 cm is separated by 5 cm from a thin negative lens of focal length -10 cm. Find the equivalent focal length of the combination and the position of the foci and principal planes using the matrix approach. Show them in a sketch of the optical system, roughly to scale, and use them to find the image of an arbitrary object placed in front of the system.

10 A glass lens 3 cm thick along the axis has one convex face of radius 5 cm and the other, also convex, of radius 2 cm. The former face is on the left in contact with air and the other in contact with a liquid of index 1.4. The refractive index of the glass is 1.50. Find the positions of the foci, principal planes, and focal lengths of the system. Use the matrix approach.

11
- **a.** Find the matrix for the simple "system" of a thin lens of focal length 10 cm, with input plane at 30 cm in front of the lens and output plane at 15 cm beyond the lens.
- **b.** Show that the matrix elements predict the locations of the six cardinal points as they would be expected for a thin lens.
- **c.** Why is $B = 0$ in this case? What is the special meaning of A in this case?

12 A gypsy's crystal ball has a refractive index of 1.50 and a diameter of 8 in.

- **a.** By the matrix approach, determine the location of its principal points.
- **b.** Where will sunlight be focused by the crystal ball?

13 A thick lens presents two concave surfaces, each of radius 5 cm, to incident light. The lens is 1 cm thick and has a refractive index of 1.50. Find (a) the system matrix for the lens when used in air and (b) its cardinal points. Do a ray diagram for some object.

14 An achromatic doublet consists of a crown glass positive lens of index 1.52 and of thickness 1 cm, cemented to a flint glass negative lens of index 1.62 and of thickness 0.5 cm. All surfaces have a radius of curvature of magnitude 20 cm. If the doublet is to be used in air, determine (a) the system matrix elements for input and output planes adjacent to the lens surfaces; (b) the cardinal points; (c) the focal length of the combination, using the lensmaker's equation and the equivalent focal length of two lenses in contact. Compare this calculation of f, which assumes thin lenses, with the previous value.

15 Enlarge the optical system of Figure 15 to include an object space to the left and an image space to the right of the lens. Let the new input plane be located at distance s in object space and the new output plane at distance s' in image space.

- **a.** Recalculate the system matrix for the enlarged system.
- **b.** Examine element B to determine the general relationship between object and image distances for the lens. Also determine the general relationship for the lateral magnification.
- **c.** From the results of (b), calculate the image distance and lateral magnification for an object 20 cm to the left of the lens.
- **d.** What information can you find for the system by setting matrix elements A and D equal to zero? (See Figure 9.)

16 Find the system matrix for a Cooke triplet camera lens. Light entering from the left encounters six spherical surfaces whose radii of curvature are, in turn, r_1 to r_6. The thickness of the three lenses are, in turn, t_1 to t_3, and the refractive indices are n_1 to n_3. The first and second air separations between lens surfaces are d_1 and d_2. Sketch the lens system with its cardinal points. How far behind the last surface must the film plane occur to focus paraxial rays?

Data:	$r_1 = 19.4$ mm	$t_1 = 4.29$ mm	$n_1 = 1.6110$
	$r_2 = -128.3$ mm	$t_2 = 0.93$ mm	$n_2 = 1.5744$
	$r_3 = -57.8$ mm	$t_3 = 3.03$ mm	$n_3 = 1.6110$
	$r_4 = 18.9$ mm		
	$r_5 = 311.3$ mm	$d_1 = 1.63$ mm	
	$r_6 = -66.4$ mm	$d_2 = 12.90$ mm	

17 Process the product of matrices for a thick lens, as in Eq. (15), without assuming the special conditions, $n = n'$ and $t = 0$. Thus find the general matrix elements A, B, C, and D for a thick lens.

18 Using the cardinal point locations (Table 2) in terms of the matrix elements for a general thick lens (problem 17), verify that f_1 and f_2 are given by Eqs. (1) and (2).

19 Using the cardinal point locations (Table 2) in terms of the matrix elements for a general thick lens (problem 17), verify that the distances r, s, v, and w are given by Eqs. (3) and (4).

20 Write a computer program that incorporates Eqs. (34) to (41) for ray tracing through an arbitrary number of refracting, spherical surfaces. The program should allow for the special cases of rays from far-distant objects and for plane surfaces of refraction.

21 Trace two rays through the hemispherical lens of Figure 15. The rays originate from the same object point, 2 cm above the optical axis and an axial distance of 10 cm from the first surface. One ray is parallel to the axis and the other makes an angle of −20° with the axis.

22 Trace a ray originating 7 mm below the optical axis and 100 mm distant from a doublet. The ray makes an angle of +5° relative to the horizontal. The doublet is an equiconvex lens of radius 50 mm, index 1.50, and central thickness 20 mm, followed by a matched meniscus lens of radii −50 mm and −87 mm, index 1.8, and central thickness 5 mm. Determine the final values of s, α, and Q.

23 Trace two rays, both from far-distant objects, through a *Protor photographic lens*, one at altitude of 1 mm and the other at 5 mm. Determine where and at what angle the rays cross the optical axis. The specifications of this four-element lens, including an intermediate air space of 3 mm, are as follows, with distances in mm:

$R_1 = 17.5$	$t_1 = 2.9$	$n_1 = 1.6489$
$R_2 = 5.8$	$t_2 = 1.3$	$n_2 = 1.6031$
$R_3 = 18.6$	$t_3 = 3.0$	$n_3 = 1$
$R_4 = -12.8$	$t_4 = 1.1$	$n_4 = 1.5154$
$R_5 = 18.6$	$t_5 = 1.8$	$n_5 = 1.6112$
$R_6 = -14.3$		