Physics 322: Common EM integrals
Sept. 2009

1.

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}| + C = \sinh^{-1}\left(\frac{x}{a})\right) + \ln a + C \quad (1)$$

The trick to this one is the inverse trig substitution $x = a\tan\theta$ followed by the "there is no way you would guess this" substitution $u = \tan\theta + \sec\theta$ (my 1st year calc textbook says that this should be committed to memory and I have it underlined). You then need trig functions to work out the final indefinite integral.

Here is what it looks like

$$\text{let } x = a\tan\theta = a\frac{\sin\theta}{\cos\theta} \quad (2)$$

$$dx = ad\theta\frac{(\cos\theta)(\cos\theta)-(\sin\theta)(-\sin\theta)}{\cos^2\theta} = \frac{a}{\cos^2\theta}d\theta = a\sec^2\theta\, d\theta \quad (3)$$

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{d\theta\, a\sec^2\theta}{\sqrt{a^2\tan^2\theta+a^2}} = \int\frac{d\theta\sec^2\theta}{\sec\theta} = \int d\theta\sec\theta \quad (4)$$

Prior to the 2nd substitution multiply by $(\sec\theta+\tan\theta)/(\sec\theta+\tan\theta)$.

$$\text{let } u = \tan\theta+\sec\theta,\ du = (\sec^2\theta+\sec\theta\tan\theta) \quad (5)$$

$$\int d\theta\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta} = \int\frac{du}{u} = \ln|u|+C \quad (6)$$

Often the absolute value signs aren't included. To a certain extent they must be if you are ever evaluating the integral for $u<0$. If you are in the $u<0$ region the absolute value signs imply an extra negative sign.

$$\frac{d}{du}\ln|u| = \frac{d}{du}\ln(-u) = \frac{1}{-u}(-1) = \frac{1}{u}\ (u<0). \quad (7)$$

So we see that $\ln|u|$ is an antiderivative of $1/u$ in the $u<0$ region.

Now $u = \sec(\arctan(x/a)) + \tan(\arctan(x/a))$. I find the easiest way to evaluate these combinations of forward and inverse trig functions is to draw a triangle. So in the first term if the tangent is x divided by a draw a right-angled triangle with the "opposite" side equal to x and the "adjacent" side equal to a. This means the hypotenuse must be $\sqrt{x^2+a^2}$ and the secant (hypotenuse over adjacent) is $\sqrt{x^2+a^2}/a$. The other one is easy. So

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right| + C = \ln\left|x+\sqrt{x^2+a^2}\right| - \ln a + C' \quad (8)$$

The inverse hyperbolic sine form was pointed out to me by T. Huard who found it through Maple. I admit I had no idea what an inverse hyperbolic sine was. Here is the definition from my 1st year calculus book

$$\sinh^{-1}x = ln(x+\sqrt{x^2+1}) \quad (9)$$

which you would obtain from solving

$$\sinh y = \frac{e^y-e^{-y}}{2} = x \quad (10)$$

And guess what... we use hyperbolic sines and cosines as separable solutions of Laplace's equation in Cartesian coordinates!

2.

$$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C \tag{11}$$

An inverse trig substitution with $x = a\tan\theta$. Then the integral is just $\frac{1}{a^2}\sin\theta$.

3.

$$\int \frac{dx\, x}{(x^2+a^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2+a^2}} + C \tag{12}$$

The substitution $u = x^2$ reduces this to "elementary" form. You have taken this derivative when you calculate the gradient of $1/r$ in Cartesian coordinates.

4.

$$\int \frac{dx\, x^2}{(x^2+a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2+a^2}} + \ln\left|x + \sqrt{x^2+a^2}\right| \tag{13}$$

Can get it by parts using a couple of the previous results.

5.

$$\int \frac{dx\, x}{(a^2-bx)^{3/2}} = \frac{2x}{b\sqrt{a^2-bx}} + \frac{4\sqrt{a^2-bu}}{b^2} \tag{14}$$

This comes up in Newton's integral. Again integrate by parts to eliminate $U = x$ in the numerator.