Chapter 3

Sound, Inertia-Gravity Waves, and Lamb Waves

We now examine fundamental modes of the geophysical fluid dynamics equations at small scales and amplitudes, which allows a linearized treatment. We first consider the full modes of an isothermal, non-rotating atmosphere at rest. This is followed by the modes of a rotating atmosphere at rest with constant Brunt-Väisälä frequency under the Boussinesq approximation.

3.1 Resting, non-rotating atmosphere

The full atmospheric governing equations with no rotation, heating, or friction are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{3.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + \frac{1}{\rho} \boldsymbol{\nabla} p + g\boldsymbol{k} = 0 \tag{3.2}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = 0. \tag{3.3}$$

We first define an isothermal base state in hydrostatic balance, with base state profiles indicated by a subscripted zero. From the ideal gas law, the base state density can be written

$$\rho_0 = \frac{p_0}{RT_0} \tag{3.4}$$

where T_0 is constant. After minor manipulation, the hydrostatic equation becomes

$$\frac{d \ln p_0}{dz} = -\frac{g}{RT_0} = -\frac{1}{z_S} \tag{3.5}$$

where $z_S = RT_0/g \approx 8$ km is the *scale height* of the atmosphere. This has the solution

$$p_0 = p_R \exp(-z/z_S) \tag{3.6}$$

where p_R is a reference pressure, typically 1000 hPa. From equation (3.4) we get the base state density profile

$$\rho_0 = \frac{p_0}{RT_0} = \frac{p_0}{gz_S} = \frac{p_R \exp(-z/z_S)}{gz_S}. \tag{3.7}$$

The base state potential temperature profile is

$$\theta_0 = T_0 \left(p_R/p_0\right)^\kappa = T_0 \exp(\kappa z/z_S). \tag{3.8}$$

We now linearize the governing equations, assuming that $p = p_0(z)+p'$, $\rho = \rho_0(z)+\rho'$, and $\theta = \theta_0(z) + \theta'$, where the primed quantities are assumed to be small enough that quadratic and higher terms in them can be ignored. Since we are linearizing about a state of rest, the velocity components $\boldsymbol{v} = (u, v, w)$ are also assumed to be small.

Two somewhat complex parts of the linearization are first discussed. The pressure gradient and gravity terms in the momentum equation become

$$\frac{1}{\rho_0 + \rho'}\boldsymbol{\nabla}(p_0 + p') + g\boldsymbol{k} \approx \frac{1}{\rho_0}\boldsymbol{\nabla} p_0 - \frac{\rho'}{\rho_0^2}\boldsymbol{\nabla} p_0 + \frac{1}{\rho_0}\boldsymbol{\nabla} p' + g\boldsymbol{k} = \frac{1}{\rho_0}\boldsymbol{\nabla} p' + \frac{g\rho'}{\rho_0}\boldsymbol{k} \tag{3.9}$$

where the hydrostatic equation has been invoked to eliminate $\boldsymbol{\nabla} p_0$. The potential temperature is rewritten in terms of the pressure and density and then linearized:

$$\theta = T\left(\frac{p_R}{p}\right)^\kappa = \frac{p_R^\kappa}{R}\frac{p^{1-\kappa}}{\rho} \approx \theta_0\left[1 + \frac{1}{\gamma}\left(\frac{p'}{p_0}\right) - \frac{\rho'}{\rho_0}\right] \tag{3.10}$$

implies that

$$\frac{\theta'}{\theta_0} = \frac{1}{\gamma}\left(\frac{p'}{p_0}\right) - \frac{\rho'}{\rho_0}, \tag{3.11}$$

where we recall that $1 - \kappa = 1/\gamma$.

We assume a plane wave form for all perturbation variables $\exp[i(kx - \omega t)]$ where k is the wave vector component in the x direction and ω is the angular frequency of the wave. No generality is lost by assuming wave motion in the x direction due to isotropy in the $x-y$ plane, and this assumption simplifies matters by rendering the y component of the momentum equation irrelevant. Simplifying the notation by defining $P = p'/p_0$ and $N = \rho'/\rho_0$, the linearized governing equations are

$$-i\omega N - \frac{w}{z_S} + iku + \frac{\partial w}{\partial z} = 0 \tag{3.12}$$

$$-i\omega u + ikgz_S P = 0 \tag{3.13}$$

$$-i\omega w + g z_S \frac{\partial P}{\partial z} + g(N - P) = 0 \tag{3.14}$$

$$-i\omega \left[P/\gamma - N\right] + \frac{\kappa w}{z_S} = 0 \tag{3.15}$$

where θ' has been eliminated between equations (3.3) and (3.11).

Eliminating u and N using equations (3.13) and (3.15) results in a system of two, first-order differential equations for w and P:

$$\left(\frac{\partial}{\partial z} - \frac{1}{\gamma z_S}\right) w + i\omega \left(\frac{k^2 g z_S}{\omega^2} - \frac{1}{\gamma}\right) P = 0 \tag{3.16}$$

$$\left(\frac{\partial}{\partial z} - \frac{\kappa}{z_S}\right) P - \frac{i\omega}{g z_S} \left(1 - \frac{\kappa g}{\omega^2 z_S}\right) w = 0. \tag{3.17}$$

By further elimination we arrive at a single second-order differential equation for P:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{z_S}\frac{\partial}{\partial z} + \left(\frac{\kappa g}{z_S \omega^2} - 1\right) k^2 + \frac{\omega^2}{\gamma g z_S}\right] P = 0. \tag{3.18}$$

The second order structure of this equation suggests oscillatory behavior in the vertical with some modification due to the first derivative in z. No obvious energy sources or dissipation exist in this system, so we expect the frequency ω to be real. We try a solution of the form $P \propto \exp[(im + \mu)z]$, with real m and μ, whereupon the first two terms become

$$\left[(im + \mu)^2 - \frac{im + \mu}{z_S}\right] P. \tag{3.19}$$

Demanding that the imaginary part of the coefficient of P be zero implies that $\mu = 1/(2z_S)$, which means that the amplitude of the pressure perturbation and other perturbation quantities increase exponentially with height. The physical reason for this is that the ambient density of air decreases with height, and for a wave to carry the same amount of energy as it moves upward, the amplitude has to increase.

Substituting $P \propto \exp\left[imz + z/(2z_S)\right]$ into equation (3.18) and rearranging results in a quadratic equation for ω^2

$$\omega^4 - \gamma g z_S l^2 \omega^2 + \gamma \kappa g^2 k^2 = 0 \tag{3.20}$$

where $l^2 = k^2 + m^2 + 1/(4z_S^2)$. This equation has the solution

$$\omega^2 = \frac{\gamma g z_S l^2}{2} \left[1 \pm \left(1 - \frac{4\kappa k^2}{\gamma z_S^2 l^4}\right)^{1/2}\right]. \tag{3.21}$$

The branch with the positive root represents sound waves, or at low frequencies, infrasound. The negative root represents gravity waves, a type of wave which we will study extensively.

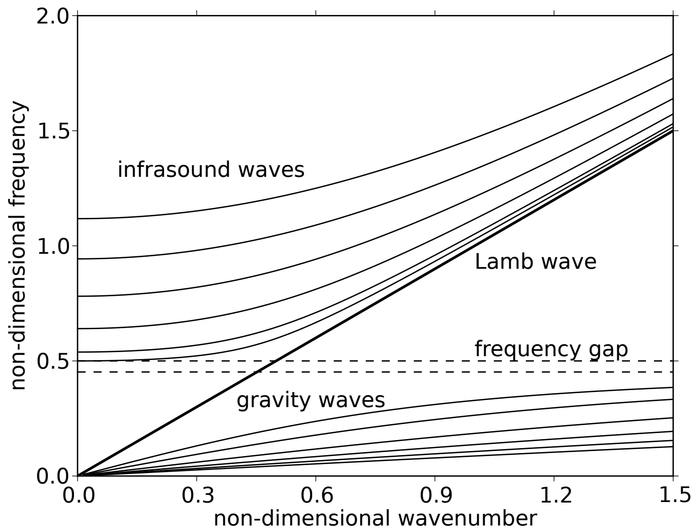

Figure 3.1: Non-dimensionalized dispersion relations for sound, gravity, and Lamb waves in an isothermal, resting atmosphere. The non-dimensional horizontal wavenumber is $z_S k$, the non-dimensional frequency is $z_S\omega/c$, and the non-dimensional vertical wavenumber is $z_S m$. Curves for gravity modes with vertical wavenumber values of 0, 1, 2, 3, 4, and 5 are shown while curves for infrasound modes with values 0, 0.2, 0.4, 0.6, 0.8, and 1 are plotted. In both cases curves with $m = 0$ are closest to the frequency gap.

3.1.1 Approximations

This solution exhibits interesting limits:

1. If gravity is zero, then $z_S \to \infty$. However, $gz_S = RT_0$, and we find two solutions,

$$\omega^2 \to \gamma RT_0(k^2 + m^2),\ 0. \tag{3.22}$$

   The first solution corresponds to pure sound waves with phase speed squared of $c^2 = \gamma RT_0$. The second is the degenerate case for a gravity wave with no gravity.

2. For vertical wavelengths short compared to the scale height, $z_S^4 l^4 \gg z_S^2 k^2$, and the second term inside the square root is much smaller than unity, so we can perform a binomial expansion on the square root, resulting in

$$\omega^2 \to c^2(k^2 + m^2),\ \frac{N^2 k^2}{(k^2 + m^2)}. \tag{3.23}$$

   In this case the first solution represents sound as before, while the second is the dispersion relation for small-scale gravity waves in the special case of an isothermal environment. We have written g/z_S in terms of the ambient potential temperature profile using equation (3.8) $\kappa g/z_S = gd(\ln\theta_0)/dz \equiv N^2$. The quantity N is called the Brunt-Väisälä frequency. Notice that the dispersion relation for small-scale sound waves in an isothermal atmosphere is isotropic in the $x-z$ plane even in the presence of gravity, whereas the dispersion relation for gravity waves is highly anisotropic. This has great significance for atmospheric dynamics.

3.1.2 Lamb wave

In eliminating w to obtain equation (3.18), we have inadvertently eliminated a solution to equations (3.16) and (3.17) which the vertical velocity is identically zero. Setting $w = 0$ in equation (3.16) yields the dispersion relation for these modes, which are called Lamb waves:

$$\omega^2 = \gamma g z_S k^2 = \gamma R T_0 k^2 = c^2 k^2, \tag{3.24}$$

which is just the dispersion relation for horizontally propagating sound waves. Setting $w = 0$ in equation (3.17) yields an equation for the vertical structure of Lamb waves:

$$\left(\frac{\partial}{\partial z} - \frac{\kappa}{z_S}\right) P = 0, \tag{3.25}$$

which has the solution $P \propto \exp(\kappa z/z_S)$. These waves have purely horizontal motion, and are thus purely longitudinal, and they extend through the full depth of the atmosphere. They travel at the speed of ordinary sound.

Figure 3.1 summarizes what we have learned, showing plots of non-dimensional frequency versus wavelength for various values of the non-dimensional vertical wavenumber, $z_S m$. The maximum non-dimensional frequency for gravity waves is $(\kappa/\gamma)^{1/2}$ while the minimum frequency for infrasound waves is 0.5. Only the Lamb mode has frequencies in the gap between these two values. Interestingly, the horizontal phase speed of infrasound ω/k exceeds that of sound, going to infinity as the horizontal wavenumber goes to zero.

3.2 Boussinesq inertia-gravity waves

We now extend our analysis of gravity waves to the case with rotation, but subject to the Boussinesq approximation in order to make the equations tractable. These are called inertia-gravity waves. (It is not necessary to discuss infrasound waves with rotation because the minimum infrasound frequency is so much greater than the Coriolis parameter.) The Boussinesq equations linearized about a state of rest are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.26}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \pi'}{\partial x} - fv = 0 \tag{3.27}$$

$$\frac{\partial v}{\partial t} + \frac{\partial \pi'}{\partial y} + fu = 0 \tag{3.28}$$

$$\frac{\partial w}{\partial t} + \frac{\partial \pi'}{\partial z} - b' = 0 \tag{3.29}$$

$$\frac{\partial b'}{\partial t} + N^2 w = 0 \tag{3.30}$$

where we have ignored molecular transfer and the buoyancy source term. The buoyancy is split into a mean part which increases linearly with height and a perturbation part $b = N^2 z + b'$, where N is the Brunt-Väisälä frequency introduced in section 3.1. The kinematic pressure is correspondingly split $\pi = \pi_0(z) + \pi'$, with π_0 in hydrostatic balance with $b_0 = N^2 z$.

We assume a plane wave moving in the $x - z$ plane, with all variables proportional to $\exp[i(kx + mz - \omega t)]$. However, unlike the rotating case, we cannot neglect the y component of the momentum equation. This assumption reduces the partial differential governing equations (3.26)-(3.30) to algebraic equations

$$iku + imw = 0 \tag{3.31}$$

$$-i\omega u + ik\pi' - fv = 0 \tag{3.32}$$

$$-i\omega v + fu = 0 \tag{3.33}$$

$$-i\omega w + im\pi' - b' = 0 \tag{3.34}$$

$$-i\omega b' + N^2 w = 0. \tag{3.35}$$

This is a set of five linear, homogeneous equations in five dependent variables, and a consistent solution only exists if the determinant of the coefficients of the dependent variables equals zero. The resulting secular equation is cubic in ω

$$\left[\omega^2(k^2 + m^2) - (N^2 k^2 + f^2 m^2)\right]\omega = 0, \tag{3.36}$$

yielding the dispersion relation for inertia-gravity waves

$$\omega = \pm\left(\frac{N^2 k^2 + f^2 m^2}{k^2 + m^2}\right)^{1/2} \tag{3.37}$$

and the (not so) trivial dispersion relation $\omega = 0$.

3.2.1 "Trivial" mode

For the $\omega = 0$ case, the relations between the dependent variables, or *polarization* relations, are $u = w = 0$ as well as hydrostatic balance

$$b' = im\pi' \tag{3.38}$$

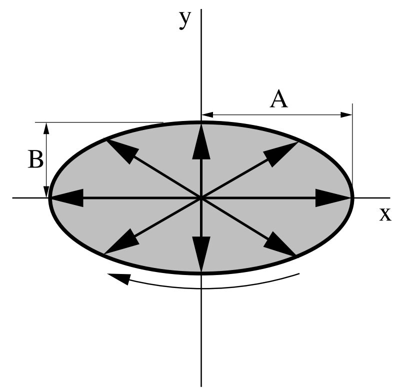

Figure 3.2: Anti-cyclonic (clockwise looking down in the northern hemisphere) rotation of the horizontal component of the wind perturbation in inertia-gravity waves. The ratio of $A/B = \omega/f$.

and geostrophic balance of the y component of the flow with the pressure perturbation

$$v = \frac{ik}{f}\pi'. \tag{3.39}$$

This case corresponds to steady, geostrophically balanced jets in the y direction. Though somewhat trivial in the present case, this mode becomes interesting when base states with wind shear or the curvature of the earth's surface are added.

3.2.2 Inertia-gravity mode

The inertia-gravity wave polarization relations for $\omega^2 > 0$ are

$$u = -\frac{m}{k}w \tag{3.40}$$

$$v = -\frac{if}{\omega}u = \frac{ifm}{\omega k}w \tag{3.41}$$

$$b' = -\frac{iN^2}{\omega}w \tag{3.42}$$

$$\pi' = -\frac{m\left(\omega^2 - f^2\right)}{k^2\omega}w. \tag{3.43}$$

From equation (3.40) we see that $ku + mw = 0$, which means that the perturbation velocity in the $x - z$ plane (u, w) is normal to the wave vector (k, m). Gravity waves, at least in the Boussinesq approximation, are thus pure transverse waves. In the horizontal plane, the perturbation wind vector rotates anti-cyclonically around an elliptical trajectory in one wave period, as illustrated in figure 3.2.

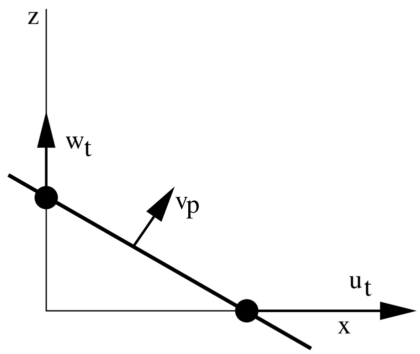

Figure 3.3: Wave front in the $x - z$ plane with phase speed v_p and the x and z trace speeds u_t and w_t illustrated.

The dispersion relation (3.37) shows that as long as k and m are real, the wave frequency is bounded by $f^2 \leq \omega^2 \leq N^2$. This dispersion relation agrees with the approximate dispersion relation (3.23) derived from the exact analysis of section 3.1 in the limit of $f = 0$. It also shows that inertia-gravity waves are highly anisotropic in the vertical.

3.2.3 Speeds and velocities

The *phase speed* of a wave in three-dimensional space is the wave frequency divided by the total wavenumber. In the case of Boussinesq inertia-gravity waves, it is

$$v_p = \frac{\omega}{(k^2 + m^2)^{1/2}} = \frac{(k^2 N^2 + m^2 f^2)^{1/2}}{k^2 + m^2}. \tag{3.44}$$

The phase speed is not the magnitude of a vector, and in particular it is not the square root of the sum of the squares of the speeds obtained by dividing the frequency by the components of the wave vector, k and m. These quantities

$$u_t = \frac{\omega}{k} \qquad w_t = \frac{\omega}{m} \tag{3.45}$$

are speeds with which the intersection of wave fronts with the respective Cartesian axes move. (See figure 3.3.) There is a relationship between the phase speed and the trace speeds:

$$\frac{1}{v_p^2} = \frac{1}{u_t^2} + \frac{1}{w_t^2}. \tag{3.46}$$

It is easy to show that the phase speed is less than that of any of the trace speeds.

The *group velocity* is a true velocity with Cartesian components

$$\boldsymbol{v}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) \tag{3.47}$$

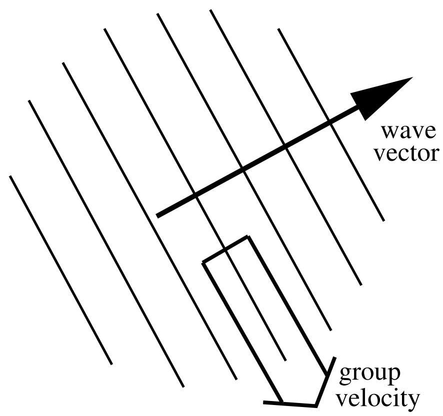

Figure 3.4: Schematic illustration of an inertia-gravity wave wave packet showing that the group velocity is oriented (in the vertical plane) at a right angle to the wave vector.

where $\boldsymbol{k} = (k, l, m)$ is the wave vector. In the case discussed here we have set $l = 0$. For Boussinesq inertia-gravity waves, the components of the group velocity are easily computed:

$$u_g = \frac{m^2 (N^2 - f^2)}{u_t(k^2 + m^2)^2} \qquad w_g = -\frac{k^2 (N^2 - f^2)}{w_t(k^2 + m^2)^2}. \tag{3.48}$$

Note that the x component of the group velocity has the same sign as the x trace speed, whereas the z component has the opposite sign from the z trace speed. This is a peculiarity of inertia-gravity waves which led to an early mis-interpretation of the vertical propagation of these waves in the ionosphere. Radar observations showed downward-moving wave fronts (vertical trace speed), which led to questions about the possible origin of these waves high in the atmosphere. In reality, the wave energy, as represented by the group velocity, was moving upward, with origins in various types of tropospheric disturbances.

Curiously, the group velocity of the Boussinesq inertia-gravity wave is normal to the wave vector, with the vertical components of the two being of opposite sign. This is illustrated in figure 3.4.

3.3 References

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics.* Cambridge University Press, 745 pp. The current material is discussed in chapter 2 of Vallis. Note that figure 2.9 in Vallis is incorrect (in the first edition).

3.4 Questions and problems

1. We noted that combining equations (3.16) and (3.17) by eliminating w results in the omission of a solution with $w \equiv 0$. By combining these equations while eliminating P, is a solution with $P \equiv 0$ omitted? Explain.

2. Make a qualitative sketch of the horizontal trace speed and the horizontal component of the group velocity of a typical infrasound wave. (No equations, please!)

3. Show that $\boldsymbol{v}_g \cdot \boldsymbol{k} = 0$ for Boussinesq gravity waves.

4. Standing gravity waves over sinusoidal terrain: Find the steady, non-rotating, flow over terrain with terrain elevation of the form $h = h_0 \sin(kx)$. Do this by carrying out the following steps:

   (a) Linearize the inviscid, adiabatic, non-rotating ($f = 0$) Boussinesq equations about a state of constant, uniform flow U in the x direction and uniform Brunt-Väisälä frequency, so that $b_0(z) = N^2 z$. The base state kinematic pressure $\pi_0(z)$ should be in hydrostatic equilibrium with the base state buoyancy b_0. Assume steady flow ($\partial/\partial t = 0$) and slab symmetry ($\partial/\partial y = 0$).

   (b) Assume that all dependent variables are proportional to $\exp\left[i\left(kx + mz\right)\right]$ and substitute this into the linearized equations, resulting in a set of linear, homogeneous equations in the dependent variables u', w, π', and b'. Find the relationship between m^2 and k^2.

   (c) Choose the sign of m which makes wave packets move upward. This is called the upward radiation boundary condition. Hint: View the motion of wave packets with various orientations of the wave fronts in the reference frame of the moving flow.

   (d) Find u', π', and b' in terms of $w(x, z)$. The real parts of these variables are the physically interesting parts, so compute these.

   (e) Air adjacent to the surface moves up and down as it flows over the undulating terrain. This vertical velocity equals $w_s(x) = (U + u')(dh/dx)$. The computed vertical velocity pattern must match this at the surface $z = h(x)$, i.e., $w(x, h(x)) = w_s(x)$. In order to solve this problem, we must also assume that the variations in terrain height are small, so that we can approximate $w(x, h) \approx w(x, 0)$. We also must assume that $|u'| \ll U$, so that the actual boundary condition applied is $w(x, 0) = U(dh/dx)$. Apply this boundary condition and make plots of u', w, π', and b' as a function of x and z. You may wish to use the computer graphics tool of your choice to make contour plots showing these variables.