Chapter 1

Governing Equations of GFD

The fluid dynamical governing equations consist of an equation for mass continuity, one for the momentum budget, and one or more additional equations to account for the characteristics of the fluid medium. These so-called constituent equations can be simple or elaborate depending on the degree of detail considered. Unlike in ordinary classical mechanics, we tend to address what happens in a stationary control volume, with the fluid flowing through this volume. Fluid dynamics thus becomes an open systems problem, with fluid elements flowing in and out of the system.

1.1 Mass continuity

Let us first consider the flow of mass in and out of a control volume, as illustrated in figure 1.1. The mass flowing out of the area element dA per unit time is $\rho\boldsymbol{v}\cdot\boldsymbol{n}dA$ where ρ is the fluid mass density and $\boldsymbol{v}$ is the fluid velocity, so the time rate of change of mass in the control volume is

$$\frac{d}{dt}\int \rho dV = \int \frac{\partial\rho}{\partial t}dV = -\oint \rho\boldsymbol{v}\cdot\boldsymbol{n}dA, \tag{1.1}$$

since mass is conserved. Applying the divergence theorem to the right side of equation (1.1)

$$\oint \rho\boldsymbol{v}\cdot\boldsymbol{n}dA = \int \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})dV \tag{1.2}$$

and putting equations (1.1)-(1.2) together, we find that

$$\int\left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\right)dV = 0. \tag{1.3}$$

Since this result is true for all control volumes, the integrand of equation (1.3) is zero, resulting in the *flux form* of the mass continuity equation:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0. \tag{1.4}$$

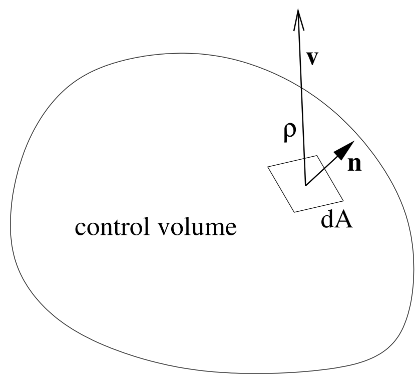

Figure 1.1: Control volume with surface area element dA. The quantity $\boldsymbol{n}$ is an outward-pointing unit vector normal to the area element, $\boldsymbol{v}$ is the velocity of the fluid flowing through the area element, and ρ is its mass density.

An alternate form of this equation, called the *advective form*, is obtained by expanding the second term using the product rule $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v}$ and combining the first term of this expansion with the density time derivative,

$$\frac{1}{\rho} \frac{d\rho}{dt} + \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \tag{1.5}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{d\boldsymbol{x}}{dt} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \tag{1.6}$$

is called the *material* or *parcel* derivative. This comes from the identification of $d\boldsymbol{x}/dt$ with $\boldsymbol{v}$ where $\boldsymbol{x}$ is the position of a parcel of fluid. The material derivative is thus the time derivative of some property (e.g., density) associated with a parcel. It is used broadly in fluid dynamics.

1.2 Momentum equation

The momentum per unit mass in a fluid is just the velocity $\boldsymbol{v}$ and the momentum flux due to the bulk motion of the fluid is the tensor $\rho \boldsymbol{v}\boldsymbol{v}$. The momentum per unit time flowing through the area element dA in figure 1.1 is $\rho \boldsymbol{v}\boldsymbol{v} \cdot \boldsymbol{n} dA$, so the bulk flow of momentum out of the control volume is the closed area integral of this over the surface of the control volume

$$\oint \rho \boldsymbol{v}\boldsymbol{v} \cdot \boldsymbol{n} dA = \int \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v}) dV \tag{1.7}$$

in analogy with equation (1.2). The time rate of change of momentum in the control volume is

$$\frac{d}{dt} \int \rho \boldsymbol{v} dV = \int \frac{\partial \rho \boldsymbol{v}}{\partial t} dV. \tag{1.8}$$

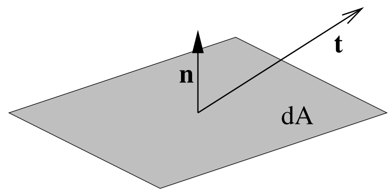

Figure 1.2: Illustration of traction vector $\boldsymbol{t}$ relative to a surface element dA with unit normal $\boldsymbol{n}$. Note that the direction of $\boldsymbol{t}$ is unrelated to the direction of $\boldsymbol{n}$.

Momentum is not conserved in a non-isolated system; the source of momentum is simply the force. We divide the total force applied to a fluid into two parts, body forces such as gravity, and short-range, inter-molecular forces. The total body force on the control volume is

$$\boldsymbol{F}_B = \int \rho \boldsymbol{B} dV \tag{1.9}$$

where $\boldsymbol{B}$ is the body force per unit mass, e.g., $-g\boldsymbol{k}$ for gravity, where g is the gravitational acceleration and $\boldsymbol{k}$ is the upward-pointing unit vector.

The short-range forces are represented in terms of a *stress tensor* $\mathbf{T}$. The physical meaning of the stress tensor is derived from the *traction vector* $\boldsymbol{t}$, which is defined relative to a surface element dA. If $\boldsymbol{n}$ is the unit normal to the surface element, the traction is

$$\boldsymbol{t} = \mathbf{T} \cdot \boldsymbol{n}. \tag{1.10}$$

The traction is the force per unit area exerted by short-range forces acting across the surface. In particular, it is the force per unit area exerted *by* molecules on the side of the surface penetrated by the unit vector $\boldsymbol{n}$ *on* molecules on the other side of the surface. (This convention is common in the continuum mechanics community but is opposite that used for the stress tensor in general relativity.) Thus, the force on the control volume due to short-range forces acting across the surface of the volume is represented by the surface integral

$$\boldsymbol{F}_T = \oint \boldsymbol{t} dA = \oint \mathbf{T} \cdot \boldsymbol{n} dA = \int \boldsymbol{\nabla} \cdot \mathbf{T} dV \tag{1.11}$$

where $\boldsymbol{n}$ is the outward unit normal on the surface of the control volume.

Both air and water are Newtonian fluids, which means that the stress tensor has parts related to pressure p, fluid strain rate $\mathbf{D}$, and the trace of the strain rate $\mathrm{Tr}(\mathbf{D})$

$$\mathbf{T} = -[p - (\eta - 2\mu/3)\mathrm{Tr}(\mathbf{D})]\mathbf{I} + 2\mu \mathbf{D} \tag{1.12}$$

where $\mathbf{I}$ is the unit tensor and μ and η are positive parameters denoting the first and second coefficients of viscosity. In component notation the strain rate is

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{1.13}$$

and the trace of $\mathbf{D}$ is

$$\mathrm{Tr}(\mathbf{D}) = D_{ii} = \frac{\partial v_i}{\partial x_i} = \boldsymbol{\nabla} \cdot \boldsymbol{v}. \tag{1.14}$$

(The Einstein convention in which repeated indices indicate summation is used.) Thus, $\eta - 2\mu/3$ is related to the resistance of the fluid to expansion and contraction. This can be understood if $p - (\eta - 2\mu/3)\mathrm{Tr}(\mathbf{D})$ is considered to be an "effective pressure". The effective pressure is less than the actual pressure when the fluid is expanding ($\mathrm{Tr}(\mathbf{D}) > 0$) and greater when the fluid is compressing ($\mathrm{Tr}(\mathbf{D}) < 0$). Thus, hysteresis occurs in expansion-compression cycles, which dissipates energy. The effects of the $\mathrm{Tr}(\mathbf{D})\mathbf{I}$ term are significant only when rapid expansion and contraction occur on very small scales, such as in shock waves. They are of little interest in geophysical fluid dynamics, and for this reason this term generally ignored. On the other hand, the $\mathbf{D}$ term is important on small scales in geophysical flows. This term is most important in non-divergent shear and strain flows.

Putting all of this together and applying the same arguments as in the discussion of the mass continuity equation results in the flux form of the *Navier-Stokes* equation:

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = \rho \boldsymbol{B} - \boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{v}. \tag{1.15}$$

Doing a product rule expansion of the terms on the left side of this equation and invoking the mass continuity equation results in the advective form of the Navier-Stokes equation:

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{B} - \frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{v}. \tag{1.16}$$

The quantity $\nu = \mu/\rho$ is called the *kinematic viscosity.* Both μ and ν are functions of temperature and pressure. These quantities vary only slightly over spatial scales for which the viscosity term is important, they are generally treated as constants.

The factors which distinguish geophysical fluid dynamics from ordinary fluid dynamics are the body forces involved. These are gravity and the inertial forces which arise from working in the non-inertial reference frame of the rotating earth (see any advanced mechanics text):

$$\boldsymbol{B} = \boldsymbol{g}^* - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) - 2\boldsymbol{\Omega} \times \boldsymbol{v} \tag{1.17}$$

where $\boldsymbol{g}^*$ is the true gravitational force per unit mass, $\boldsymbol{\Omega}$ is the rotation vector of the earth, $\boldsymbol{r}$ is the position of the test point relative to the center of the earth, and $\boldsymbol{v}$ is the fluid velocity in the earth's rotating frame. The second term on the right side of equation (1.17) is the centrifugal force due to the earth's rotation. It has magnitude $\Omega^2 r \cos\theta$, where θ is the latitude of the test point, and it points away from the rotation axis of the earth. The combination of the first two terms in equation (1.17), $\boldsymbol{g} = \boldsymbol{g}^* - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$, is commonly called gravity, even though it includes the centrifugal force. The earth is slightly ellipsoidal due to its rotation, but to an excellent approximation we can treat it as being spherical with $\boldsymbol{g}$ pointing toward the earth's center. Furthermore, the vertical extent of the atmosphere and

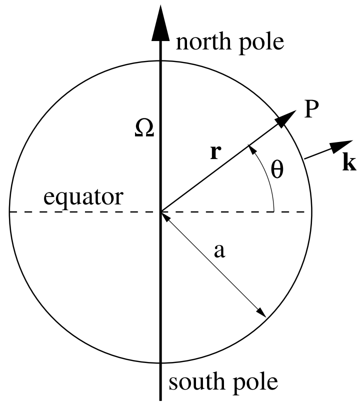

Figure 1.3: Schematic for determining the body forces acting in the frame of the rotating earth at test point P. The quantity a is the radius of the earth and $\boldsymbol{\Omega}$ is the rotation vector of the earth.

oceans is much smaller than the radius a of the earth, and we can approximate its magnitude as being constant. The the third term on the right side of equation (1.17) is the *Coriolis force*. Though weak, this force plays a central role in geophysical fluid dynamics. Combining all the approximations, we have

$$\boldsymbol{B} = -g\boldsymbol{k} - 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\boldsymbol{\nabla}\Phi - 2\boldsymbol{\Omega} \times \boldsymbol{v} \tag{1.18}$$

where $\boldsymbol{k}$ is a unit vector pointing away from the center of the earth and g is taken to be constant. The quantity $\Phi = gz$ is called the *geopotential*, where $z = r - a$ is the height above sea level.

An additional approximation is often made with the Coriolis force, by dividing the rotation vector into locally horizontal and vertical parts: $\boldsymbol{\Omega} = \boldsymbol{\Omega}_h + \Omega \sin\theta \boldsymbol{k}$. The vertical component of the Coriolis force $2\boldsymbol{\Omega}_h \times \boldsymbol{v}$ competes with much stronger vertical forces. Therefore it is often neglected, so that $\boldsymbol{B}$ can be further approximated as

$$\boldsymbol{B} = -g\boldsymbol{k} - f\boldsymbol{k} \times \boldsymbol{v} \tag{1.19}$$

where $f = 2\Omega \sin\theta$ is called the *Coriolis parameter*. We may or may not make this approximation, as appropriate.

1.3 The ocean

Determining the density of ocean water is rather complex, involving not only temperature and pressure, but salinity as well. However, we adopt a very simple approximation, namely that the ocean is an incompressible fluid of variable density. More specifically, the density of parcels of sea water is assumed not to change with time except possibly as the result of diffusion. This condition is expressed compactly in terms of the material derivative of density,

$$\frac{d\rho}{dt} = \sigma \nabla^2 \rho \tag{1.20}$$

where we have defined a diffusivity σ. Combining this with equation (1.5) shows that mass continuity reduces to

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{1.21}$$

where the effect of diffusivity on this equation is generally considered to be negligible, and is therefore omitted. The density of sea water near the surface ranges roughly from 1022 kg m^{-3} to 1028 kg m^{-3}.

The advective form of the momentum equation becomes

$$\frac{d\boldsymbol{v}}{dt} + \frac{1}{\rho}\boldsymbol{\nabla} p + g\boldsymbol{k} + f\boldsymbol{k} \times \boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} = 0 \tag{1.22}$$

where we have ignored the vertical component of the Coriolis force.

We now explore two balance conditions. The first is hydrostatic balance. If the ocean is at rest so that $\boldsymbol{v} = 0$, then the momentum equation reduces to

$$\frac{1}{\rho}\boldsymbol{\nabla} p = -g\boldsymbol{k}. \tag{1.23}$$

Considering first the horizontal components of this vector equation shows us that

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \tag{1.24}$$

which implies that $p = p(z)$. The vertical component can therefore be written

$$\frac{dp}{dz} = -g\rho, \tag{1.25}$$

which further implies that $\rho = \rho(z)$ is a function only of z as well. Thus, in static equilibrium, the ocean is horizontally homogeneous. Equation (1.25) is called the *hydrostatic equation*.

Let us now assume that the pressure and density can be split into base parts in hydrostatic balance plus small perturbations, $p = p_0(z) + p'$ and $\rho = \rho_R + \rho'$. The base pressure p_0 is a function of z, but we take the base density ρ_R to be a constant reference value close to the mean density of the fluid. Substituting in the momentum equation and dropping terms quadratic in primed quantities results in

$$\frac{d\boldsymbol{v}}{dt} + \boldsymbol{\nabla}\pi - b\boldsymbol{k} - f\boldsymbol{k} \times \boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} = 0 \tag{1.26}$$

where $\pi = p'/\rho_R$ is the *kinematic pressure* and $b = -g\rho'/\rho_R$ is the *buoyancy*.

Expanding the incompressibility condition (1.20) and eliminating ρ' in favor of the buoyancy results in

$$\frac{\partial b}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} b = S_b \tag{1.27}$$

where the right side has been generalized into an arbitrary source of buoyancy. The set of approximations giving rise to equations (1.21), (1.26), and (1.27) collectively constitute the *Boussinesq* approximation. This is a very good approximation in the ocean as long as compressibility effects which increase the density in the deep ocean are not important in the problem under consideration.

If the vertical parcel acceleration is small, then the *hydrostatic* approximation can be made. This comes from ignoring the vertical acceleration and viscosity terms in the vertical component of the momentum equation, resulting in

$$\frac{\partial \pi}{\partial z} - b = 0. \tag{1.28}$$

This approximation is valid when the horizontal scale of the flow pattern under consideration is much greater than the vertical scale. This condition is frequently satisfied for oceanic and atmospheric motions of large horizontal scale.

The second balance condition occurs when the flow is horizontal and does not depend on x, y, or t. Defining the horizontal velocity components as (u, v), equation (1.26) reduces to

$$\boldsymbol{\nabla} \pi - f \boldsymbol{k} \times \boldsymbol{v} = 0. \tag{1.29}$$

The horizontal velocity which satisfies this condition exactly regardless of the structure of the actual flow is called the *geostrophic* velocity, and is given by

$$(u_G, v_G) = \left(-\frac{1}{f}\frac{\partial \pi}{\partial y}, \frac{1}{f}\frac{\partial \pi}{\partial x} \right) \tag{1.30}$$

in the context of the Boussinesq approximation. Significant insight into large-scale flows arises from considering flow patterns which differ only slightly from geostrophic balance.

1.4 The atmosphere

The entropy of dry air is a key variable in atmospheric fluid dynamics. The specific entropy s can be expressed in terms of the temperature T and pressure p:

$$s = C_p \ln\left(\frac{T}{T_R}\right) - R \ln\left(\frac{p}{p_R}\right). \tag{1.31}$$

The quantity $C_p = 1005$ J K^{-1} kg^{-1} is the specific heat of air (per unit mass) at constant pressure, $R = 287$ J K^{-1} kg^{-1} is the universal gas constant divided by the molecular weight of dry air. Recall that $C_p = C_v + R$ where $C_v = 718$ J K^{-1} kg^{-1} is the specific heat of air at constant volume. We define the ratios $\gamma = C_p/C_v$ and $\kappa = R/C_p$ and note for later use that $1 - \kappa = 1/\gamma$. For air, $\gamma \approx 1.40$ and $\kappa \approx 0.286$. The quantities $T_R = 300$ K and $p_R = 1000$ hPa are constant reference values for temperature and pressure. The entropy is conserved by air parcels in slow, adiabatic processes. "Slow" basically means flows in the absence of shock waves and "adiabatic" means with no heating. Since shock waves are not a part of geophysical fluid dynamics, we can write the governing equation for specific dry entropy as

$$\frac{ds}{dt} = \frac{Q}{T} \tag{1.32}$$

where Q is the heating rate per unit mass. This expresses the thermodynamic relation that the change in entropy due to heating is the amount of heat added divided by the temperature.

The heating can be divided into two parts, an external contribution Q_X, due, perhaps, to the absorption of radiation or to latent heat release associated with condensation, and an internal part due to the conduction of heat:

$$Q = Q_X + k\nabla^2 T. \tag{1.33}$$

The second term on the right comes from the realization that the conductive flux of heat is $-K\boldsymbol{\nabla}T$ where K is the heat conductivity of air, and that the convergence of heat conduction per unit mass of air is $-[\boldsymbol{\nabla}\cdot(-K\boldsymbol{\nabla}T)]/\rho$. We have defined $k = K/\rho$ and have assumed that k varies slowly enough with environmental conditions that it can be drawn out of the divergence operator.

Though processes involving condensation and freezing of atmospheric water substance and the formation and fallout of precipitation are important atmospheric processes, we will mostly ignore them here, or present them in highly simplified contexts.

An alternate thermodynamic variable equivalent to the specific dry entropy is used extensively in meteorology, namely, the potential temperature θ. The potential temperature of a parcel is the temperature it would attain if it were compressed or expanded isentropically to a standard reference temperature:

$$\theta = T(p_R/p)^{\kappa}. \tag{1.34}$$

It is easy to show that

$$s = C_p \ln\left(\frac{\theta}{T_R}\right). \tag{1.35}$$

It is also straightforward to show, using equations (1.31) and (1.35), that

$$\frac{d\theta}{dt} = \frac{\theta Q}{C_p T} = \frac{Q}{\Pi} \tag{1.36}$$

where $\Pi = C_p T/\theta$ is the *Exner function.* From equation (1.34) it is clear that we can also write the Exner function in terms of pressure: $\Pi = C_p(p/p_R)^{\kappa}$.

We further note that

$$\theta d\Pi = \theta R \left(\frac{p}{p_R}\right)^{\kappa-1} \frac{dp}{p_R} = \frac{RT}{p} dp = \frac{dp}{\rho} \tag{1.37}$$

where the definition of potential temperature and the ideal gas law $p/\rho = RT$ are used. This relationship facilitates the elimination of density and pressure in favor of potential temperature and Exner function in the momentum equation:

$$\frac{d\boldsymbol{v}}{dt} + \theta \boldsymbol{\nabla}\Pi + g\boldsymbol{k} + f\boldsymbol{k} \times \boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} = 0. \tag{1.38}$$

Unfortunately, the density is not so easy to eliminate from the mass continuity equation (1.4). However, the *anelastic* approximation replaces the density by a fixed, ambient density profile $\rho_0(z)$, and the continuity equation simplifies to

$$\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v}) = 0. \tag{1.39}$$

The anelastic approximation eliminates sound waves while retaining (with some distortion) the most interesting meteorological modes. Using the ideal gas law and the definition of potential temperature, the density may be written in terms of the pressure and potential temperature:

$$\rho = \frac{p}{R\theta} \left(\frac{p_R}{p}\right)^{\kappa}. \tag{1.40}$$

The Boussinesq approximation may be used for the atmosphere, though with considerably less justification than for the ocean. To do so we assume that $\theta = \theta_R + \theta'$, where θ_R is a constant reference potential temperature and set $\Pi = \Pi_0(z) + \Pi'$. We then define buoyancy as $b = g\theta'/\theta_R$ and kinematic pressure as $\pi = \theta_R \Pi'$. Making similar approximations to the ocean case, the momentum equation takes a form identical to (1.26). Ignoring heating term, equation (1.36) can be transformed in the buoyancy equation (1.27). Finally, the least justifiable approximation we need to make is to assume that ρ_0 is constant in the mass continuity equation (1.39), leading us to the simplified mass continuity equation (1.21). The Boussinesq approximation is hard to justify quantitatively for atmospheric problems. However, it is useful for obtaining a qualitative understanding of many meteorologically significant disturbances.

1.5 References

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 745 pp. The material here is mostly covered in chapter 1 of Vallis.

1.6 Questions and problems

1. Explain why the mass continuity equation has no diffusion term.

2. Image a container with salt water in the lower half and fresh water in the upper half. As this container sits on the shelf over the years, salt gradually diffuses upward until ultimately the salinity is uniform through the container. In this process is there fluid motion? Explain.

3. In hydrostatic balance with the Boussinesq approximation, show that the horizontal gradient of buoyancy is proportional to the vertical derivative of the geostrophic wind. This is called the *thermal wind* equation. Investigate how this plays out in the exact atmospheric equations.

4. Using the hydrostatic equation, compute the pressure as a function of depth for a constant-density ocean and as a function of height for an isothermal and an isentropic atmosphere.