Chapter 9

Barotropic Instability

The Rossby wave is the building block of low Rossby number geophysical fluid dynamics. In this chapter we learn how Rossby waves can interact with each other to produce a dynamical instability. These instabilities are the origin of the day-to-day variability in the flow that we call weather. Two types of instability are generally distinguished, barotropic and baroclinic instability. For barotropic instability the component Rossby waves are displaced horizontally, whereas they are displaced vertically for baroclinic instability. However, as we shall show, these are just two special cases of low Rossby number instability.

We investigate low Rossby number instabilities using quasi-geostrophic theory. A common feature of these instabilities is that they require either vertical or meridional (north-south) shear of the zonal (east-west) wind. In this chapter we confine our attention to barotropic instability in shallow water flow. Baroclinic instability doesn't occur in this environment as there is no vertical dimension.

9.1 Linearized governing equations

In the shallow water case only meridional shear exists, and we divide the "starred" fractional thickness perturbation, zonal wind, and potential vorticity into zonally symmetric and disturbance components, $\eta^* = \eta_Z(y) + \eta'$, $u_g = U + u_g'$, and $q^* = q_Z(y) + q'$. The zonal wind U is in geostrophic balance with the zonal fractional thickness perturbation η_Z

$$U(y) = -f_0 L_R^2 \frac{d\eta_Z}{dy} \tag{9.1}$$

and geostrophic balance for the primed quantities is

$$u_g' = -f_0 L_R^2 \frac{\partial \eta'}{\partial y} \qquad v_g = f_0 L_R^2 \frac{\partial \eta'}{\partial x}. \tag{9.2}$$

The zonal flow part of the potential vorticity is

$$q_Z = q_0 \left(L_R^2 \frac{d^2\eta_V}{dy^2} - \eta_V \right) \tag{9.3}$$

and the total geostrophic potential vorticity is $q_g = q_A + q_Z + q'$ where

$$q_A = q_0(1 + \beta y / f_0) \tag{9.4}$$

is the previously defined geographic potential vorticity with terrain ignored.

The disturbance fractional thickness perturbation is now diagnosed using

$$L_R^2 \nabla^2 \eta' - \eta' = q'/q_0 \tag{9.5}$$

and the potential vorticity evolution equation becomes

$$\frac{\partial q'}{\partial t} + \left(U - f_0 L_R^2 \frac{\partial \eta'}{\partial y} \right) \frac{\partial q'}{\partial x} + f_0 L_R^2 \frac{\partial \eta'}{\partial x} \left(\frac{\partial (q_A + q_Z + q')}{\partial y} \right) = 0. \tag{9.6}$$

Linearization in terms of primed quantities yields

$$\frac{\partial q'}{\partial t} + U \frac{\partial q'}{\partial x} + f_0 L_R^2 \left(\frac{d(q_A + q_Z)}{dy} \right) \frac{\partial \eta'}{\partial x} = 0 \tag{9.7}$$

and substitution of equations (9.3)-(9.5) results in

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left(L_R^2 \nabla^2 \eta' - \eta' \right) + f_0 L_R^2 \frac{dQ}{dy} \frac{\partial \eta'}{\partial x} = 0 \tag{9.8}$$

where $q_0 Q$ is the geographic plus zonal wind potential vorticity:

$$Q(y) = (q_A + q_Z)/q_0 = 1 + \frac{\beta y}{f_0} + L_R^2 \frac{d^2 \eta_Z}{dy^2} - \eta_Z \tag{9.9}$$

9.2 Non-dimensionalization

We now write the governing equations in non-dimensional form, scaling all lengths by the Rossby radius L_R and all times by the inverse of the base Coriolis parameter f_0^{-1}. This results in a substantial reduction in clutter:

$$U(y) = -\frac{d\eta_Z}{dy} \qquad u_g' = -\frac{\partial \eta'}{\partial y} \qquad v_g = \frac{\partial \eta'}{\partial x} \tag{9.10}$$

$$\nabla^2 \eta' - \eta' = q'/q_0 \tag{9.11}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\nabla^2\eta'-\eta'\right)+\frac{dQ}{dy}\frac{\partial\eta'}{\partial x}=0 \tag{9.12}$$

$$Q(y)=(q_A+q_Z)/q_0=1+\beta y+\frac{d^2\eta_Z}{dy^2}-\eta_Z. \tag{9.13}$$

The dimensional forms of the variables can always be recovered by multiplying by $L_R^m f_0^{-n}$ where m and n are selected to give the variable the correct dimensionality, e.g., $m=1$ and $n=-1$ for velocity.

9.3 Interfacial Rossby waves

All Rossby waves propagate on a gradient in potential vorticity. We have seen cases in which this gradient is caused by the beta effect and a uniform meridional thickness gradient resulting from either a gradient in terrain height or a uniform zonal flow. Here we consider as an idealization cases in which the gradient is very large, but confined to a tiny interval in the meridional direction y. Aside from a constant multiplier, $Q(y)$ is the undisturbed meridional distribution of potential vorticity. Arranging our coordinate system so that the gradient occurs at $y=0$, we let

$$\frac{dQ}{dy}=\Delta Q\delta(y) \tag{9.14}$$

where $\Delta Q=Q_+-Q_-$ is the discontinuity in Q across the interface and $\delta(y)$ is the Dirac delta function. We further assume that all dependent variables are proportional to the plane wave form $\exp\left[i(kx-\omega t)\right]$. In this idealization $dQ/dy=0$ for $y\neq 0$. The advantage is that the governing equation (9.12) takes the simple form

$$-(c-U)\left(\frac{\partial^2}{\partial y^2}-k^2-1\right)\eta'+\Delta Q\delta(y)\eta'=0 \tag{9.15}$$

where $c=\omega/k$ is the phase speed of the wave.

Except possibly at critical levels where $c-U=0$, equation (9.15) tells us for $y\neq 0$ that

$$\frac{\partial^2\eta'}{\partial y^2}-l^2\eta'=0 \tag{9.16}$$

where

$$l=(k^2+1)^{1/2} \tag{9.17}$$

which has solutions

$$\eta'\propto\exp\left(\pm ly\right). \tag{9.18}$$

Treating the regions of positive and negative y separately and requiring that $\eta' \to 0$ as $|y| \to \infty$ yields

$$\eta' = \begin{cases} A_+ \exp(-ly), & y > 0 \\ A_- \exp(ly), & y < 0 \end{cases} \tag{9.19}$$

where A_+ and A_- are constants.

In order to avoid infinite u_g at $y = 0$, we must have $A_+ = A_- \equiv A$. Integrating equation (9.15) in y over the interval $-\epsilon \le y \le \epsilon$ where ϵ is very small, results in

$$-(c - U_0)\Delta(\partial\eta'/\partial y) + \Delta Q A = 0 \tag{9.20}$$

where $U_0 = U(0)$ and where $\Delta(\partial\eta'/\partial y)$ is the jump in $\partial\eta'/\partial y$ across $y = 0$:

$$\Delta\left(\frac{\partial\eta'}{\partial y}\right) = \left(\frac{\partial\eta'}{\partial y}\right)_{\epsilon} - \left(\frac{\partial\eta'}{\partial y}\right)_{-\epsilon} = -2lA. \tag{9.21}$$

Substituting this into equation (9.20) results in the dispersion relation for the wave

$$c = \frac{\omega}{k} = U_0 - \frac{\Delta Q}{2(k^2 + 1)^{1/2}}. \tag{9.22}$$

As with other Rossby waves for which the potential vorticity increases to the north (positive y), the wave propagates to the west (negative x) relative to the ambient wind. The dynamics of an interfacial wave is governed by processes at the interface, which means that the wind at the interface U_0 plays this role. Taking the real parts of the complex solutions yields the physical solutions

$$\begin{aligned} \eta^* =& \eta_Z(y) + A\cos(kx - \omega t)\exp(-l|y|) \\ u_g =& U(y) + Al\mathrm{sgn}(y)\cos(kx - \omega t)\exp(-l|y|) \\ v_g =& - Ak\sin(kx - \omega t)\exp(-l|y|) \end{aligned} \tag{9.23}$$

where $\mathrm{sgn}(y) = y/|y|$.

The final element of the solution is to determine forms of $Q(y)$ and $U(y)$ consistent with $dQ/dy = 0$ and the value of ΔQ. The first condition in conjunction with equation (9.13) leads to the differential equation for U

$$\frac{d^2U}{dy^2} - U - \beta = 0, \tag{9.24}$$

which has the solution

$$U = (U_0 + \beta)\exp(-|y|) - \beta \tag{9.25}$$

where we have arranged for $U(0) = U_0$ and have demanded that U not blow up as $y \to \pm\infty$. From equation (9.10) we infer that

$$\eta_Z(y) = (U_0 + \beta)\,\mathrm{sgn}(y)\,[\exp(-|y|) - 1] + \beta y \tag{9.26}$$

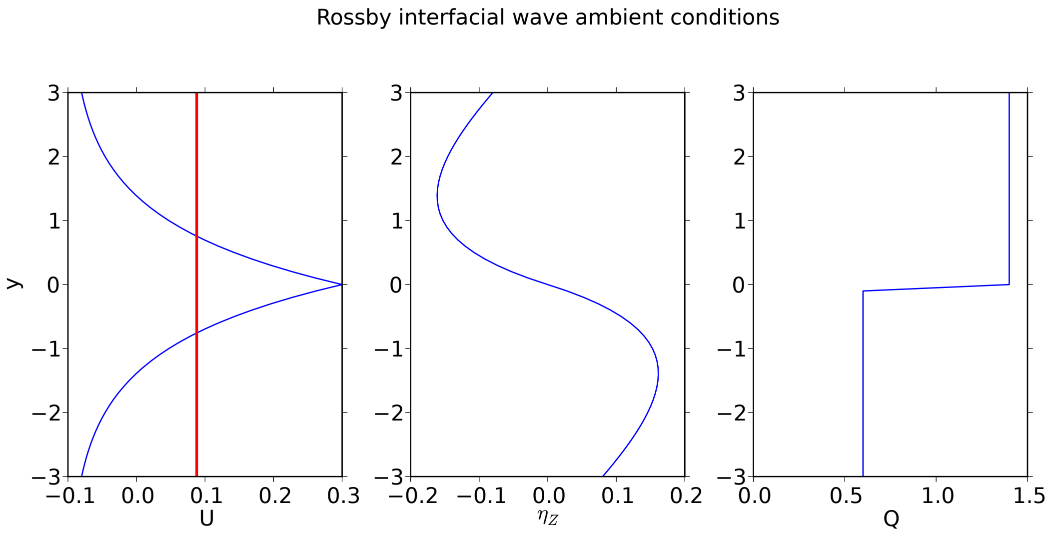

Figure 9.1: Plot of the ambient meridional profiles of wind (left panel), fractional thickness perturbation (middle panel), and the ambient potential vorticity (right panel) for the interfacial Rossby wave. All are presented in non-dimensionalized form with dimensionless $U_0 = 0.3$ and $\beta = 0.1$. The vertical line in the left panel represents the propagation velocity of the Rossby wave.

where we have neglected an overall additive constant of integration and have further insisted that η_Z be continuous at $y = 0$. From equation (9.13) we find that

$$Q = 1 + (U_0 + \beta)\,\mathrm{sgn}(y) \tag{9.27}$$

satisfies all of our conditions. Thus, $\Delta Q = 2\,(U_0 + \beta)$.

Figure 9.1 shows ambient meridional profiles of wind, fractional thickness perturbation, and potential vorticity in non-dimensionalized form. The discontinuity in potential vorticity is associated with a zonal wind jet centered on the discontinuity. Negative thickness anomalies and positive relative vorticity exist north of the discontinuity while the reverse occurs to the south.

Figure 9.2 shows the perturbation potential vorticity and wind associated with the wave. The potential vorticity perturbation comes from the meridional displacement of the interface, with positive potential vorticity anomalies occurring where the interface is displaced to the south and negative anomalies for northward displacement.

The interfacial Rossby wave propagates to the west relative to the zonal wind U_0 at $y = 0$, as with other Rossby waves in which the ambient potential vorticity increases to the north. The propagation mechanism described in the previous chapter is evident in figure 9.2. Southward flow to the west of positive potential vorticity anomalies and northward flow to

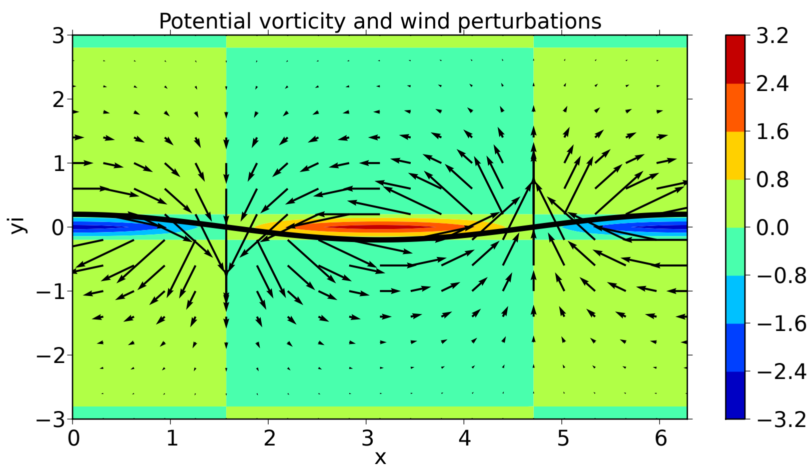

Figure 9.2: Perturbation wind and potential vorticity fields associated with the Rossby wave with dimensionless wave amplitude $A = 0.2$ and wavenumber $k = 1$. The thick, curved line represents the displacement of the interface due to the wave.

the east tend respectively to enhance the anomaly to the west and destroy it to the east, thus shifting the whole pattern to the west.

9.4 Example of barotropic instability

We now consider the interaction between two interfacial Rossby waves separated meridionally. The waves occur on potential vorticity discontinuities of opposite sign. Since the beta effect plays no essential role in this interaction, we set $\beta = 0$.

Ambient meridional profiles of U, η_Z, and Q are shown in figure 9.3. The meridional wind structure consists of oppositely directed zonal jets separated by a non-dimensional distance 2ξ with velocity maxima of $\pm U_0$. A negative thickness anomaly exists between the jets. The patterns of $U(y)$ and $\eta_Z(y)$ are obtained from $Q(y)$ by methods used in the last section and take the form

$$U(y) = \begin{cases} -U_0 \text{sgn}(y) \exp[-(|y| - \xi)], & |y| > \xi \\ -U_0 \sinh(y)/\sinh(\xi), & |y| < \xi \end{cases} \tag{9.28}$$

and

$$\eta_Z(y) = \begin{cases} -U_0 \exp[-(|y| - \xi)], & |y| > \xi \\ U_0 \left[\{\cosh(y) - \cosh(\xi)\}/\sinh(\xi) - 1\right], & |y| < \xi \end{cases} . \tag{9.29}$$

The potential vorticity anomaly which generates this pattern is in the form of a “top-hat”

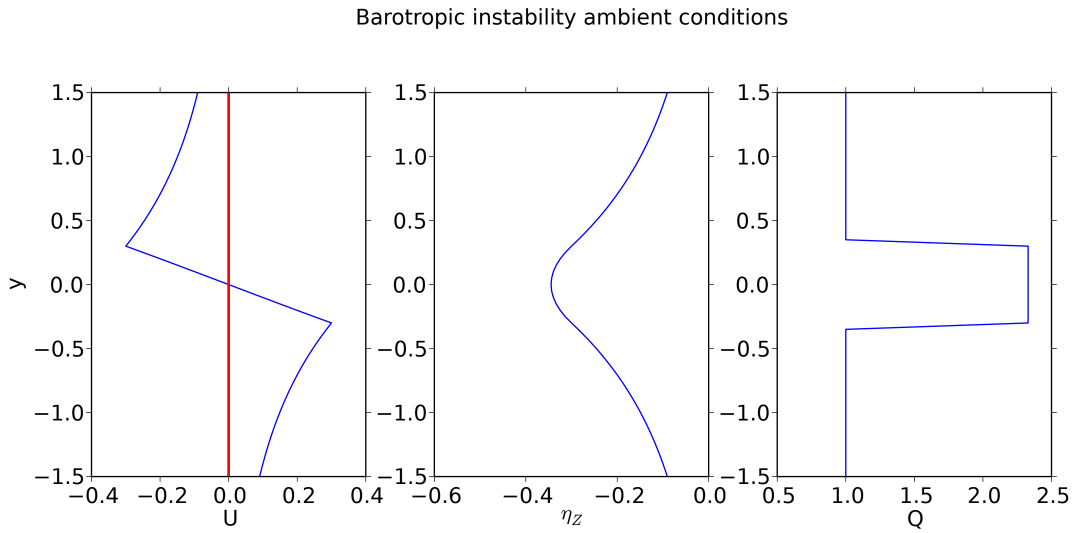

Figure 9.3: Plot of the ambient meridional profiles of wind (left panel), fractional thickness perturbation (middle panel), and the ambient potential vorticity (right panel) for coupled interfacial waves. All are presented in non-dimensionalized form with dimensionless $U_0 = 0.3$ and $\xi = 0.3$. The vertical line in the left panel represents the propagation velocity of the resulting instability.

profile of width 2ξ and magnitude

$$\Delta Q = U_0[1 + \cosh(\xi)/\sinh(\xi)] = 2U_0/[1 - \exp(-2\xi)]. \tag{9.30}$$

The non-dimensionalized governing equation for a wave disturbance is obtained from equation (9.12) upon the substitution of the usual plane wave form $\exp[i(kx - \omega t)]$:

$$-(c - U)\left(\frac{\partial^2}{\partial y^2} - k^2 - 1\right)\eta' + \Delta Q\left[\delta(y + \xi) - \delta(y - \xi)\right]\eta' = 0. \tag{9.31}$$

For $y^2 \neq \xi^2$, the solutions are superpositions of $\exp(\pm ly)$ where for convenience we define

$$l = (k^2 + 1)^{1/2}. \tag{9.32}$$

In particular we assume that

$$\eta' = \left\{ \begin{array}{cc} A\exp[l(\xi + y)], & y < -\xi \\ C\exp(ly) + D\exp(-ly), & -\xi < y < \xi \\ B\exp[l(\xi - y)], & y > \xi \end{array} \right. , \tag{9.33}$$

where A, B, C, and D are constants, and where solutions that blow up at infinity for $y^2 > \xi^2$ have been omitted.

Two conditions on η' apply at each of the interfaces. First, η' must be continuous at the interfaces in order to avoid infinite geostrophic zonal winds there. This leads to two conditions

$$A = C\exp(-l\xi) + D\exp(l\xi) \qquad B = C\exp(l\xi) + D\exp(-l\xi). \tag{9.34}$$

second, integrations over small intervals around each of the interfaces, as was done in the previous section, results in

$$\nu_- A - C\exp(-l\xi) + D\exp(l\xi) = 0 \tag{9.35}$$

and

$$-\nu_+ B - C\exp(l\xi) + D\exp(-l\xi) = 0 \tag{9.36}$$

where

$$\nu_+ = 1 - \frac{\Delta Q}{l(U_0 + c)} \qquad \nu_- = 1 - \frac{\Delta Q}{l(U_0 - c)}. \tag{9.37}$$

Direct substitution of the equations in (9.34) into equations (9.35) and (9.36) results in two coupled, linear, homogeneous equations for C and D, leading to the matrix equation

$$\begin{pmatrix} (\nu_- - 1)\exp(-l\xi) & (\nu_- + 1)\exp(l\xi) \\ (\nu_+ + 1)\exp(l\xi) & (\nu_+ - 1)\exp(-l\xi) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = 0. \tag{9.38}$$

A non-trivial solution only exists when the determinant of the coefficients in the above equation equals zero, which leads to the dispersion relation for the disturbance:

$$c^2 = U_0^2 \left[\left(\frac{\Delta Q}{2lU_0} - 1 \right)^2 - \left(\frac{\Delta Q}{2lU_0} \right)^2 \exp\left(-4l\xi\right) \right]. \tag{9.39}$$

By inspection, we see that $c^2 < 0$ is guaranteed if

$$l = l_C = \Delta Q/(2U_0) = [1 - \exp(-2\xi)]^{-1}, \tag{9.40}$$

since for this choice of l the first term in the square brackets in equation (9.39) vanishes, leaving only the negative-definite second term. Thus, $c = \pm i c_I = \pm U_0 \exp(-2l_C\xi)$ and an instability exists with a phase speed of zero. The growth rate for $l = l_C$ is the imaginary part of the frequency

$$\omega_I = k_C c_I = k_C U_0 \exp(-2l_C\xi) \tag{9.41}$$

where where we have chosen the plus sign for c_I and $k_C = (l_C^2 - 1)^{1/2}$ is the value of k corresponding to $l = l_C$. This is close to, but not exactly equal to the maximum growth rate.

Computation of C and D is somewhat involved, though a few tricks help out in this regard. First of all, the first and second rows of equation (9.38)imply that

$$\frac{C}{D} = -\frac{\nu_- + 1}{\nu_- - 1}\exp(2l\xi) \qquad \frac{D}{C} = -\frac{\nu_+ + 1}{\nu_+ - 1}\exp(2l\xi). \tag{9.42}$$

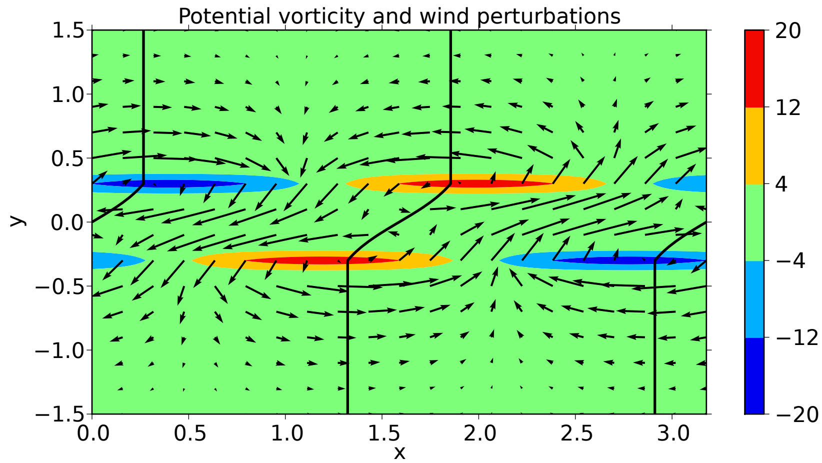

Figure 9.4: Perturbation wind and potential vorticity fields associated with the barotropic instability at the dimensionless wavenumber $k = k_C = 1.98$ with dimensionless wave amplitude $E = 0.2$. The thick, curved lines represent the contours of zero meridional wind v_g.

Note that when c is imaginary (the case of interest to us), $\nu_- = \nu_+^*$ where the asterisk indicates complex conjugate. Thus, $C/D = D^*/C^*$, from which we conclude that $CC^* = DD$; in other words, C and D are complex, but with the same absolute value. Since we are free to set the overall phase of A, B, C, and D, let us require that $D = C^*$. If $E = |C| = |D|$, then we can define the phase angle ϕ such that

$$C = E\exp(-i\phi) \qquad D = E\exp(i\phi). \tag{9.43}$$

From equations (9.37) and (9.42) we can therefore infer

$$\exp(2i\phi) = \frac{D}{C} = -\frac{\nu_+ + 1}{\nu_+ - 1}\exp(2l\xi) = \frac{2l(U_0 + ic_I) - \Delta Q}{\Delta Q}\exp(2l\epsilon), \tag{9.44}$$

whence

$$\phi = \frac{1}{2}\arctan\left(\frac{2lc_I}{2lU_0 - \Delta Q}\right). \tag{9.45}$$

When $l = l_C$, the denominator inside the inverse tangent function is zero, which means that $\phi = \phi_C = \pi/2$.

The full spatial and temporal dependence of η', u_g', and v_g' are easily calculated given the above results. From equation (9.34) we find that

$$A = E\left[\exp(-i\phi - l\xi) + \exp(i\phi + l\xi)\right] \tag{9.46}$$

and

$$B = E\left[\exp(-i\phi + l\xi) + \exp(i\phi - l\xi)\right]. \tag{9.47}$$

Defining two y profiles

$$F_+(y) = \exp\left(-l|y-\xi| + l\xi\right) \qquad F_-(y) = \exp\left(-l|y+\xi| + l\xi\right), \tag{9.48}$$

after some algebra we find that

$$\begin{aligned} \eta' =& E\left[F_+\cos(kx-\phi) + F_-\cos(kx+\phi)\right]\exp(kc_I t) \\ u_g' =& -E\left[\frac{dF_+}{dy}\cos(kx-\phi) + \frac{dF_-}{dy}\cos(kx+\phi)\right]\exp(kc_I t) \\ v_g =& -kE\left[F_+\sin(kx-\phi) + F_-\sin(kx+\phi)\right]\exp(kc_I t). \end{aligned} \tag{9.49}$$

Figure 9.4 shows the potential vorticity perturbations and the geostrophic wind associated with the baroclinic instability analyzed here. The disturbance consists of two interfacial waves located at $y = \pm\xi = \pm 0.3$. The upper wave propagates to the east relative to the ambient wind at the upper discontinuity, $-U_0$, whereas the lower wave propagates to the west relative to the ambient wind $+U_0$ at the lower discontinuity. (See figure 9.3.) Each wave propagates under the influence of the potential vorticity advection produced by its own induced winds. However, the phase lag between the two waves results in an interaction between the two waves which not only causes them to become phase-locked, but also to increase each other's amplitude. Thus, the meridional wind from the lower wave acts to increase the amplitude of the potential vorticity anomalies for the upper wave and vice versa. This interactive growth mechanism is a common theme in low Rossby number flow in the atmosphere.

9.5 References

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 745 pp. Barotropic instability is covered in chapter 6.

9.6 Questions and problems

1. Charney-Stern theorem for shallow water equations:

   (a) Starting with equation (9.12), assume $x-t$ dependence of the form $\exp\left[ik(x-ct)\right]$, divide by $c-U$, and then multiply the equation by η'^*, the complex conjugate of η', and integrate in y from $-\infty$ to ∞.

   (b) Do an integration by parts on the first term and assume that $\eta' \to 0$ as $y \to \pm\infty$.

   (c) Take the imaginary part of the equation and determine a necessary condition for the imaginary part of c to be non-zero.

(d) Determine whether the modes discussed in sections 9.3 and 9.4 satisfy this condition.

2. Compute the flux of x momentum in the y direction averaged over a wavelength in the x direction
$$\overline{u_g v_g} = \frac{1}{\lambda}\int_0^\lambda \mathrm{Re}(u_g)\mathrm{Re}(v_g)dx$$
for the barotropic instability mode presented in section 9.4. You may start with equation (9.49). Separate calculations need to be done in each of the three regions $y < -\xi$, $-\xi < y < \xi$, and $y > \xi$. Hint: Since $\overline{v_g} = 0$, $\overline{U v_g} = 0$.

3. Do the same calculation for the y flux of fractional thickness perturbation
$$\overline{\eta v_g} = \frac{1}{\lambda}\int_0^\lambda \mathrm{Re}(\eta)\mathrm{Re}(v_g)dx.$$

4. Show that the potential vorticity flux in the y direction can be written in terms of the fluxes of momentum and fractional thickness perturbation:
$$\overline{q v_g} = -q_0\left[\frac{1}{f_0}\frac{\partial \overline{u_g v_g}}{\partial y} + \overline{\eta v_g}\right].$$
From the results of the last two problems, compute the flux of potential vorticity in the y direction in the case of the barotropic instability mode.