Chapter 4

Simple models of Convection

In this chapter we discuss various simple models of convection. These include plume and thermal models as well as the vertical mixing model for shallow moist convection. Though three-dimensional numerical models of atmospheric convection have become quite common, the simple models provide a context for developing conceptual understanding which is difficult to obtain from the more complex models. They can also provide a "sanity check" on three-dimensional numerical models in simple cases. A good source for plume and thermal models is the paper by Morton, Taylor, and Turner (1956).

4.1 Plume or jet models

We begin with the Boussinesq mass continuity, vertical momentum, and buoyancy equations, written in time-independent form ($\partial(\ )/\partial t = 0$):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{4.1}$$

$$\frac{\partial v_x v_z}{\partial x} + \frac{\partial v_y v_z}{\partial y} + \frac{\partial v_z^2}{\partial z} + \frac{\partial \pi'}{\partial z} - b' = 0 \tag{4.2}$$

$$\frac{\partial v_x b}{\partial x} + \frac{\partial v_y b}{\partial y} + \frac{\partial v_z b}{\partial z} = 0. \tag{4.3}$$

We have split the kinematic pressure and the buoyancy into mean $(b_0(z), \pi_0(z))$ and perturbation (b', π') parts in the momentum equation, with the mean parts related hydrostatically.

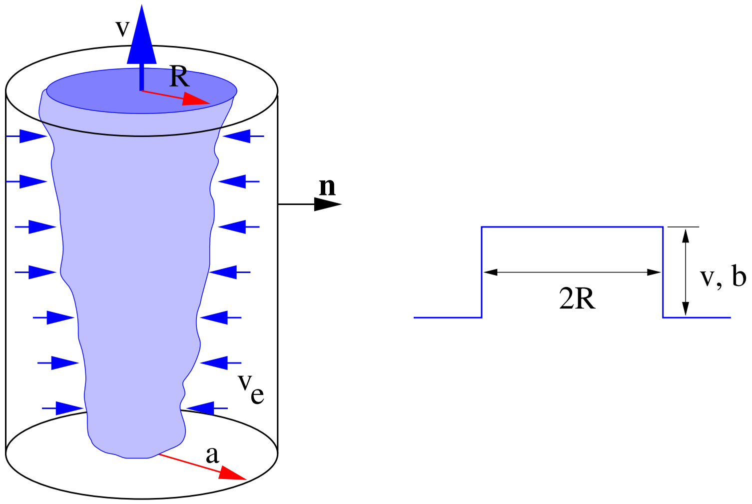

Figure 4.1: Sketch of a plume and the cylindrical control volume containing it. Shown at right is the assumed "top hat" profile of vertical velocity and buoyancy inside the plume.

We assume an ascending plume in the form of a column of radius R which varies with height z, as shown in figure 4.1. At each level the vertical velocity v and buoyancy b inside the plume take on constant values, as shown in the figure. Outside the plume we assume that $v = 0$ and $b = b_0(z)$.

We now integrate equations (4.1) - (4.3) over the horizontal area with constant radius a, as shown in figure 4.1. The mass continuity equation becomes

$$2\pi a v_r(a) + \frac{d\pi R^2 v}{dz} = 0, \tag{4.4}$$

where we have used Gauss's law to find that

$$\int \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dA = \oint (v_x, v_y) \cdot \hat{\mathbf{n}} dl = 2\pi a v_r(a), \tag{4.5}$$

where $\hat{\mathbf{n}}$ is the unit outward normal to the cylinder shown in figure 4.1 and $v_r(a)$ is the radial wind at radius a. Mass continuity outside of the plume requires that $2\pi a v_r(a) = 2\pi R v_r(R) \equiv -2\pi R v_e$, where the plume radius $R < a$, and where we have defined the *entrainment velocity* (positive inward) as $v_e = -v_r(R)$. The inward flow defined by this velocity is needed to account for the entrainment of environmental fluid by the turbulent flow in the plume.

Equation (4.4) thus becomes

$$\frac{dR^2 v}{dz} = 2Rv_e. \tag{4.6}$$

We now address the question of how to estimate the entrainment velocity. This is a difficult problem in general, but once the plume has evolved enough to forget its initial state, v_e can be expressed solely in terms of current conditions. In the simple case in which both the plume and the surrounding environment have zero buoyancy, the only variable or parameter with the units of velocity is the upward velocity of the plume itself, v. We thus set

$$v_e = (\alpha/2)v \tag{4.7}$$

where $\alpha/2$ is the constant of proportionality, with the factor of 1/2 included so as to cancel out the factor of 2 on the right side of equation (4.6). Our mass continuity equation thus becomes

$$\frac{dR^2 v}{dz} = \alpha R v. \tag{4.8}$$

A similar treatment of the vertical momentum equation (4.2) yields

$$\frac{d}{dz}[R^2(v^2 + \pi')] = R^2 b'. \tag{4.9}$$

The x and y derivative terms evaluate to zero because the vertical velocity is assumed to be zero outside of the plume:

$$\int \left(\frac{\partial v_x v_z}{\partial x} + \frac{\partial v_y v_z}{\partial y} \right) dA = \oint (v_x v_z, v_y v_z) \cdot \hat{\mathbf{n}} dl = 2\pi R v_r(a) v_z(a) = 0, \tag{4.10}$$

which implies that the plume entrains no momentum from the environment.

The pressure perturbation term π' is conventionally dropped from this equation, partly because a scale analysis is thought to show that this term is not important for a tall, skinny plume, but mainly because it is very difficult to evaluate. The scale analysis is probably incorrect, but we shall nevertheless drop the pressure term as well in order to follow historical precedent. In partial defense, the effect of the pressure term is to redistribute in space the effect of the buoyancy force, not to change its overall strength. Thus, we arrive at the simple equation for vertical momentum:

$$\frac{dR^2 v^2}{dz} = R^2 b'. \tag{4.11}$$

A similar treatment of the buoyancy equation produces

$$\frac{dR^2vb}{dz} = \alpha Rvb_0, \tag{4.12}$$

where the arguments made for the entrainment of mass are extended to the entrainment of environmental buoyancy $b_0(z)$. Equations (4.8), (4.11), and (4.12) constitute the fundamental governing equation for a steady, entraining plume.

4.1.1 Non-buoyant jet

A plume which is not buoyant with respect to its environment is generally called a jet. The simplest case we can consider is that of an initial jet of fluid which has the same buoyancy (or density) as its environment. In this case the buoyancy perturbation is zero and from the vertical momentum equation (4.11) we conclude that

$$R^2v^2 \equiv P = \text{constant}. \tag{4.13}$$

Under these conditions the mass flux equation (4.8) integrates to

$$R^2v = \alpha P^{1/2}z, \tag{4.14}$$

where we have adjusted the constant of integration so that zero radius occurs at $z = 0$. Thus the mass flux in the jet increases linearly with distance traveled. Equation (4.13) tells us that $v = P^{1/2}/R$, so that

$$R = \alpha z, \tag{4.15}$$

i. e., the radius of the jet also expands linearly with distance traveled and α is the tangent of the half-angle of expansion. We also note that the velocity decreases with distance according to

$$v = \frac{P^{1/2}}{\alpha z}, \tag{4.16}$$

which simply means that the jet slows down as mass with zero initial momentum is entrained.

4.1.2 Buoyant plume, neutral environment

In this case we assume that the plume starts out with non-zero buoyancy, but that the environment remains neutrally buoyant, with $b_0(z) = 0$, which implies that $b' = b$. The buoyancy equation (4.12) integrates trivially in this case to

$$R^2 vb \equiv B = \text{constant}. \tag{4.17}$$

Using this to eliminate the buoyancy from equation (4.11) results in

$$\frac{dR^2v^2}{dz} = \frac{B}{v}. \tag{4.18}$$

To make further progress, we assume that $R = Cz^\gamma$ and $v = Dz^\delta$ and substitute into equations (4.8) and (4.18). After some algebra, this results in $\gamma = 1$, $\delta = -1/3$, $C = 3\alpha/5$, and $D = [25B/(12\alpha^2)]^{1/3}$. Thus,

$$R = (3\alpha/5)z, \tag{4.19}$$

$$v = \left(\frac{25B}{12\alpha^2 z}\right)^{1/3}, \tag{4.20}$$

$$b = \frac{(25)^{2/3}(12)^{1/3}B^{2/3}}{9\alpha^{4/3}z^{5/3}}. \tag{4.21}$$

The plume radius expands linearly as in the case of the non-buoyant jet, but with a smaller angle of expansion. The velocity still decreases with distance, but at a lesser rate, due to the contribution of the buoyancy force to the momentum of the plume. In addition, the buoyancy decreases with distance as a result of the entrainment of zero-buoyancy fluid into the plume. There are other possible solutions to this problem, but this solution is the only one that exhibits simple power law behavior. Solutions of this type are called *similarity solutions*, since the solution at any value of $z > 0$ can be obtained from the solution at some standard level by simple rescaling with powers of z.

4.1.3 Plume in unstable environment

There are no similarity solutions for the case of a stable environment, since the plume terminates after a finite distance due to the development of negative buoyancy at some level. However, such solutions exist for the case of an unstable environment, i. e., in the case for which b_0 decreases with z. Let us assume that $b_0 = -\Gamma z$ where Γ is a positive constant. For similarity to hold, b' must also be proportional to z – otherwise $b = b_0 + b'$ would not take the form of a simple power law. We set $b' = \Gamma' z$ where $\Gamma' > 0$, since it is necessary to have positive buoyancy for similarity to hold. Thus, $b = b_0 + b' = -(\Gamma - \Gamma')z$. We assume as before that $R = Cz^\gamma$ and $v = Dz^\delta$.

Substituting these assumptions into equations (4.8), (4.11), and (4.12) results in $\gamma = 1$, $\delta = 1$, $C = \alpha/3$, $D^2 = \Gamma'/4$, and $\Gamma' = \Gamma/4$. Thus,

$$b = -(3\Gamma/4)z, \tag{4.22}$$

$$b' = (\Gamma/4)z, \tag{4.23}$$

$$R = (\alpha/3)z, \tag{4.24}$$

$$v = \Gamma^{1/2}z/4. \tag{4.25}$$

Unlike the previous cases, plumes in this environment start off with zero initial buoyancy perturbation, velocity, and radius at $z = 0$. Thus, they can be considered to form spontaneously from infinitesimal fluctuations in the environment. The opening angle of the plume is less than in either of the above cases, and the plume velocity increases with displacement rather than decreasing, as does the buoyancy perturbation b'.

The spontaneous generation of plumes in this case presents the following problem of interpretation; how can one determine the rate at which a particular environment produces such plumes? This question can only be answered if the effect of the plumes on reducing the instability of the environment is somehow included. If this is done, then one could imagine a balance between the creation of instability by some mechanism and its removal by the action of the resulting plumes. The number of plumes would then be just that required to counter the destabilization. The stabilizing effects of the plumes would come from descending motion surrounding the plume updraft. However, this descending motion is ignored as a part of the idealizations made in creating the plume model. Thus, with the current model we cannot answer the above question. We will address this issue later.

4.2 Thermal models

Sometimes convection does not occur continuously as in a steady plume, but transiently. In this case we can idealize the convective element as a homogeneous ascending parcel, which we call a *thermal*. If the thermal is also entraining environmental air, we can use the method of open systems to treat its evolution. In this method, any extensive quantity X possessed by the thermal is subject to an equation of the form

$$\frac{dX}{dt} = S_X + \left(\frac{dX}{dt}\right)_{in} - \left(\frac{dX}{dt}\right)_{out}, \tag{4.26}$$

where S_X is the source of the quantity X and the last two terms on the right side of the above equation represent the gain and loss of X as mass is transferred into or out of the thermal from the environment. As in the case of the plume, we assume that the thermal is turbulent and the environment is quiescent, so that the transfer of mass is only into the thermal from the environment.

For the case in which X is mass, we can write the mass conservation equation as

$$\frac{d}{dt}(4\pi R^3 \rho_R/3) = 4\pi R^2 \rho_R v_e, \tag{4.27}$$

where we idealize the thermal as being spherical of radius R and invoke the Boussinesq approximation in which the density is replaced by a constant reference density ρ_R except in the expression for buoyancy. The quantity v_e is the entrainment velocity, which is the velocity of environmental air adjacent to the thermal, relative to the motion of the surface of the thermal. We assume here for simplicity that v_e takes on a uniform, radially inward value, an assumption that we will later find to be rather poor.

For the case of momentum, we assume that the only source (which in this case is the external force) is buoyancy, ignoring pressure and drag forces to be consistent with plume theory:

$$\frac{d}{dt}(4\pi R^3 \rho_R v/3) = -(4\pi R^3/3)g[\rho - \rho_0(z)], \tag{4.28}$$

where v is the upward thermal velocity, g is the acceleration of gravity, ρ is the air density in the thermal, and $\rho_0(z)$ is the density of the surrounding environment.

We assume here an incompressible fluid, so that density is conserved by parcels. However, by employing the Boussinesq approximation, we implicitly extend the analysis to other situations in which the Boussinesq approximation is valid. The buoyancy equation can be written

$$\frac{d}{dt}[4\pi R^3(\rho - \rho_R)/3] = 4\pi R^2(\rho_0 - \rho_R)v_e. \tag{4.29}$$

An equation is needed to relate the vertical position of the thermal to the vertical velocity and the elapsed time:

$$\frac{dz}{dt} = v. \tag{4.30}$$

Finally, we assume a simple similarity relationship between the entrainment velocity and the vertical velocity:

$$v_e = \alpha v, \tag{4.31}$$

where α is a constant.

Equations (4.30) and (4.31) can be used to simplify the governing equations for a thermal to

$$\frac{dR^3}{dz} = 3\alpha R^2, \tag{4.32}$$

$$\frac{dR^3 v}{dz} = R^3 b'/v, \tag{4.33}$$

and

$$\frac{dR^3 b}{dz} = 3\alpha R^2 b_0, \tag{4.34}$$

where the buoyancy is defined $b = -g(\rho - \rho_R)/\rho_R$, the environmental buoyancy is $b_0 = -g(\rho_0 - \rho_R)/\rho_R$, and the buoyancy perturbation is $b' = b - b_0 = -g(\rho - \rho_0)/\rho_R$.

The mass equation can be immediately solved to yield

$$R = \alpha z, \tag{4.35}$$

which shows that the thermal expands linearly in radius with height under the similarity condition, just as does the plume. Furthermore, unlike the case of the plume, the opening angle is independent of any further information about the solution as long as similarity is maintained. Solutions to the two other equations depends on the assumptions made about the vertical structure of $b_0(z)$ and the starting values of b and v.

4.2.1 Non-buoyant parcel

As with the plume case, we begin with a non-buoyant "thermal" with an initial upward motion in a neutrally stable environment. In this case we

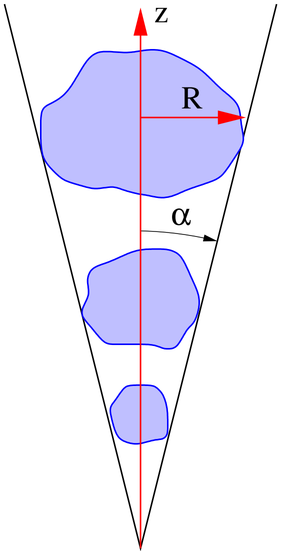

Figure 4.2: Expansion of a thermal under similarity conditions. We have assumed the small angle approximation $\alpha \approx \tan\alpha = \tan(R/z)$.

have $b_0 = b' = 0$, so that equation (4.34) is irrelevant and equation (4.33) integrates trivially to

$$R^3 v \equiv P = \text{constant}. \tag{4.36}$$

From equation (4.35) we conclude that

$$v = \frac{P}{\alpha^3 z^3}, \tag{4.37}$$

which means that the initial parcel velocity drops off very rapidly with distance.

4.2.2 Buoyant thermal, neutral environment

The second example is that of an initially buoyant parcel in a neutrally stable environment, i. e., $b_0 = 0$ and $b = b'$. Equation (4.34) integrates trivially in this case to

$$R^3 b \equiv B = \text{constant}, \tag{4.38}$$

which means that

$$b = b' = \frac{B}{\alpha^3 z^3}. \tag{4.39}$$

Assuming that $v = Dz^{\delta}$, where D and δ are constants, we find upon substitution of this and equations (4.35) and (4.39) into equation (4.33) that

$$v = \left(\frac{B}{2\alpha^3} \right)^{1/2} \left(\frac{1}{z} \right). \tag{4.40}$$

Thus, the velocity drops off less rapidly with height than in the non-buoyant case.

4.2.3 Thermal in unstable environment

For our final example we consider a thermal starting in an unstable environment with zero velocity and zero buoyancy perturbation. The environmental buoyancy takes the form $b_0 = -\Gamma z$ where Γ is a constant. As for the corresponding plume problem, we assume for the sake of similarity that $b' = \Gamma' z$, so that $b = b_0 + b' = -(\Gamma - \Gamma')z$. From these assumptions and equations (4.34) and (4.35) we conclude that $\Gamma' = \Gamma/4$, which means that

$$b' = \Gamma z/4. \tag{4.41}$$

We now assume that the vertical velocity takes the form $v = Dz^{\delta}$, where D and δ are constants. Substitution into equation (4.33) finally gives us $\delta = 1$ and $D = \Gamma^{1/2}/4$, so that

$$v = \Gamma^{1/2} z/4. \tag{4.42}$$

Thus, both the buoyancy perturbation and the velocity increase linearly with z, as does the thermal radius. These relations are very similar to those for a plume in an unstable environment.

4.3 Nonsimilar thermals

So far we have assumed that thermals and plumes develop in a self-similar fashion, which means that the tangent of the half-angle of expansion $\alpha = dR/dz$ is constant. However this does not allow us to study the initiation of thermals in the atmosphere. In order to make progress, we need to attack the problem by means other than simple theory.

Sánchez et al. (1989) reported a laboratory study and numerical simulations of a thermal starting from rest with an initial buoyancy in both neutral and stably stratified environments. The experiments were carried out in a water tank and the thermal consisted of a parcel of salty water mixed with food coloring. Stratification was produced by filling the tank with water of variable salt concentration.

Figure 4.3 shows the evolution of a typical thermal launched from rest into an unstratified environment. The behavior of the thermal is clearly not self-similar, at least until the last two pictures. The volumes of many thermals were measured from photographs as a function of distance from their starting point, assuming cylindrical symmetry. Converting these into equivalent radii for spheres of the same volume, the results of Sánchez et al. (1989) imply that the radius depends on distance traveled according to

$$R = R_0 \exp[\mu(z/R_0 - \zeta)], \tag{4.43}$$

where R_0 is the initial parcel radius starting from rest, $\mu = 0.095$, and $\zeta = 0.56$.

The opening angle of thermal expansion is clearly not constant in this case. We define an instantaneous opening angle (actually the tangent of the half-opening angle)

$$\alpha = \frac{dR}{dz} = \mu \exp[\mu(z/R_0 - \zeta)] = \mu R/R_0, \tag{4.44}$$

which shows that the expansion angle starts with a value near μ, and then increases as the thermal evolves. Similarity thermals typically have $\alpha \approx 0.25$, so the initial value of α is smaller than the similarity value by a factor of ≈ 2.5. The thermal must move more than 10 initial radii before the value of α approaches its similarity value. Meanwhile the thermal grows by more than a factor of 2.6 in radius, or 18 in volume.

Entrainment into thermals actually occurs in a conceptually different manner than is envisioned in the simple model described in the previous section. In that model, entrainment is basically a turbulent process which is assumed to occur locally over the surface of a roughly spherical turbulent blob. In laboratory experiments such as the one described here, entrainment is at least initially a laminar flow process in which environmental fluid is swept around the thermal as it rises through the environment, and is engulfed into the rear of the thermal, as shown in figure 4.4.

Sánchez et al. (1989) found that the growth of a thermal by entrainment obeyed equation (4.43) in a stably stratified as well as in a neutral environment. We can therefore compute the evolution of the buoyancy perturbation

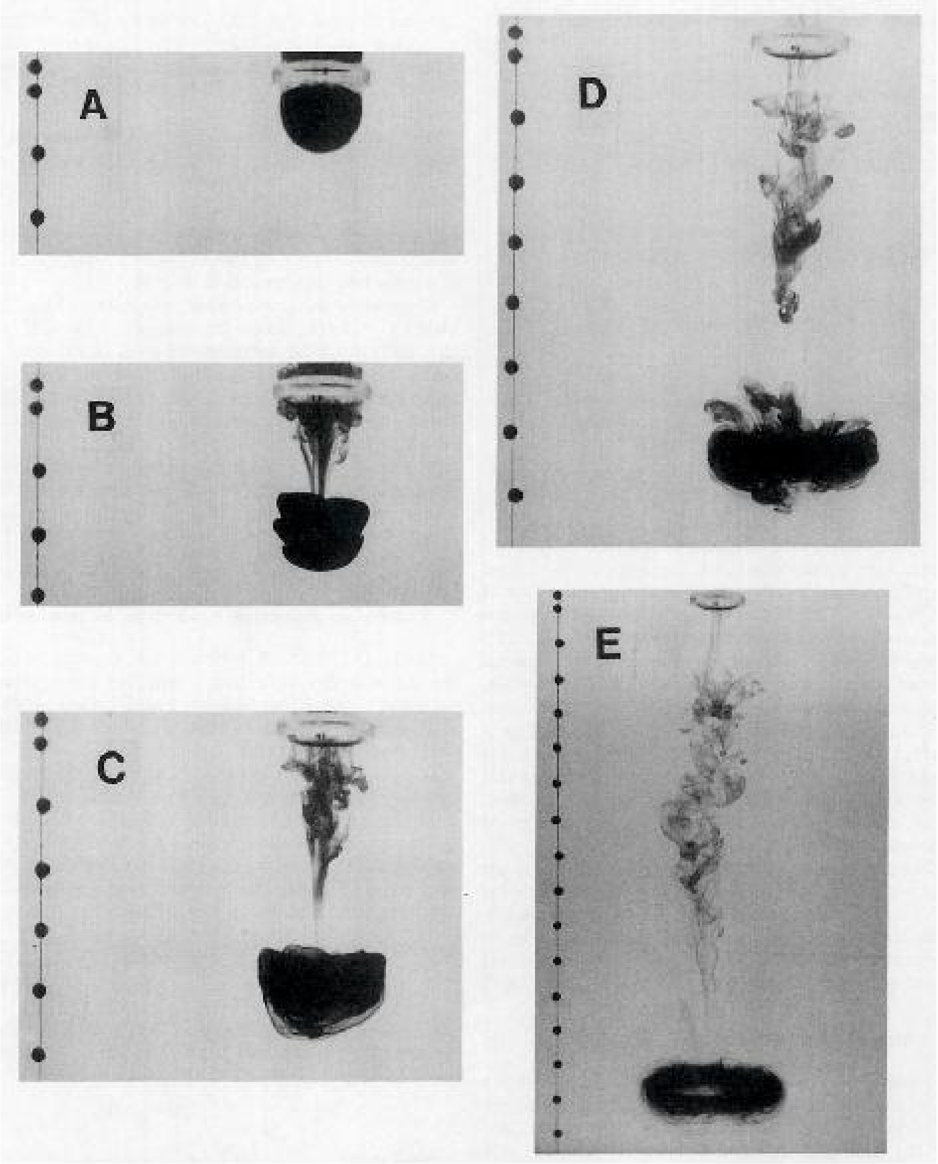

Figure 4.3: Laboratory thermal launched into an unstratified environment from rest (Sánchez et al., 1989). Notice that picture E is presented at a different scale than the others. The dot spacing on the left side of each picture is 1 cm.

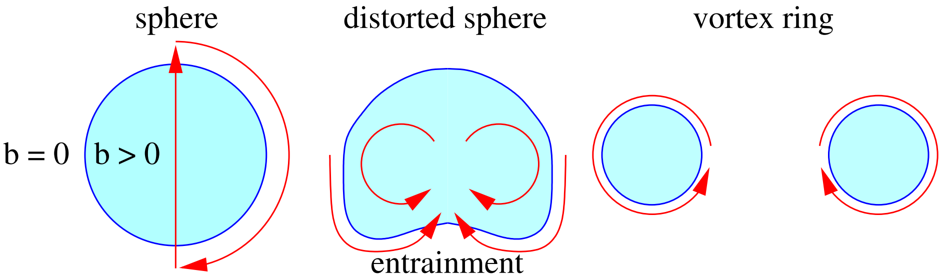

Figure 4.4: Initial evolution of a thermal. The buoyancy distribution causes the circulation around the loop shown in the left panel to increase, resulting in the distortion of the initial spherical blob of fluid as shown in the center panel. Entrainment occurs primarily via the bulk engulfment of air into the rear of the evolving thermal. Eventually the thermal takes the form of a vortex ring, as shown in the right panel.

of a thermal starting from rest in either type of environment using equations (4.32) and (4.34), which together result in a simple equation for the buoyancy of such a thermal,

$$\frac{db}{dz} = -\frac{3\alpha}{R}(b - b_0) = -\frac{3\mu}{R_0}(b - b_0), \tag{4.45}$$

where we have used equation (4.44) for the spreading angle. Letting $b_0 = \Gamma z$ to represent a stable environment and defining $b' = b - b_0 = b - \Gamma z$, we get an equation for b',

$$\frac{db'}{dz} + \frac{3\mu}{R_0} b' = -\Gamma, \tag{4.46}$$

which has the solution

$$b' = (\Gamma L_0 + \Gamma l)\exp(-z/l) - \Gamma l \tag{4.47}$$

where $l = R_0/(3\mu)$. The quantity ΓL_0 is the initial buoyancy perturbation of the parcel where L_0 is the distance the parcel would ascend to come into buoyancy equilibrium with its environment were there no entrainment. We call L_0 the *level of undilute neutral buoyancy*.

The actual level of neutral buoyancy L is the value of z for which $b' = 0$. This is found by solving equation (4.47) with b' set to zero:

$$L/l = \ln(1 + L_0/l). \tag{4.48}$$

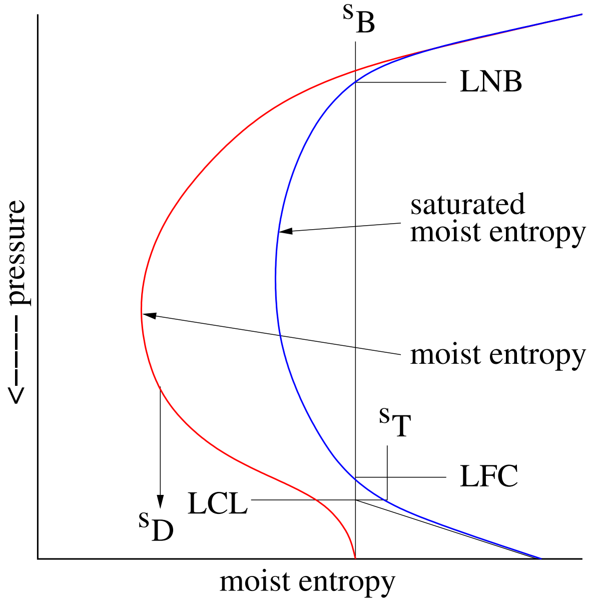

Figure 4.5: Schematic of a typical conditionally unstable atmospheric sounding. LCL is lifting condensation level, LFC is level of free convection, and LNB is level of neutral buoyancy. Plotted are the moist entropy and the saturated moist entropy as a function of pressure.

4.4 Conditional instability

Figure 4.5 shows a schematic of a conditionally unstable sounding expressed in terms of the moist entropy and the saturated moist entropy plotted as a function of height. *Conditional instability* means that finite energy must be expended to lift a parcel to the point where it is positively buoyant. The decrease in both saturated and unsaturated moist entropy up to some level, followed by an increase with height is universally found in conditionally unstable soundings.

The saturated moist entropy is the moist entropy with the vapor mixing ratio replaced by the saturated mixing ratio. The moist entropy is conserved in both moist and dry adiabatic processes, so the moist entropy of a surface parcel lifted through the troposphere follows the vertical trajectory shown in figure 4.5. The saturated moist entropy of the surface parcel initially follows a line of constant potential temperature, which low in the troposphere slants sharply to the left, as shown. However, when the saturated entropy line intersects the moist entropy line, the parcel becomes saturated. This level is called the *lifting condensation level* (LCL). Subsequently the saturated moist entropy equals the moist entropy, so both follow the vertical trajectory of the lifted surface parcel.

Neglecting virtual temperature effects, the difference between the saturated moist entropy of a parcel and its surrounding environment is a measure of the buoyancy of the parcel relative to the environment. Thus, when the saturated moist entropy of the parcel exceeds the saturated moist entropy of the environment, the buoyancy of the parcel, which typically is negative initially, becomes positive. This level is called the *level of free convection* (LFC). The parcel will eventually reach its *level of neutral buoyancy* (LNB), after which the buoyancy becomes negative.

The buoyancy of a lifted surface parcel is initially negative relative to the environment (with the exception of a possible shallow region of positive buoyancy over sun-heated land) in almost all cases. Thus, some energy source is required to lift the parcel initially to its level of free convection. The required energy per unit parcel mass is called the *convective inhibition* (CIN). We now derive an equation for CIN. Realizing that the net buoyancy force consists of the downward force of gravity on the parcel plus the upward pressure force which equals the weight of the displaced parcel of environmental air M_e, the work needed to lift the parcel from its initial level ($z = z_I$) to the level of free convection ($z = z_{LFC}$) is the CIN times the mass of the parcel M_p:

$$\mathrm{CIN} \times M_p = \int_{z_I}^{z_{LFC}} g(M_p - M_e)dz, \tag{4.49}$$

where g is the acceleration of gravity. Dividing by M_p results in

$$\mathrm{CIN} = \int_{z_I}^{z_{LFC}} g(1 - \rho_e/\rho_p)dz, \tag{4.50}$$

where ρ_p and ρ_e are the densities of the parcel and the environment.

It is more convenient to express equation (4.50) in the form of a pressure integral, converting from geometrical height to environmental pressure p using the hydrostatic equation $dp = -g\rho_e dz$:

$$\mathrm{CIN} = \int_{p_{LFC}}^{p_I} (1/\rho_e - 1/\rho_p)dp, \tag{4.51}$$

where p_I and p_{LFC} are the pressures at the initial level and the level of free convection.

A convective parcel may contain condensate. To the extent that the hydrometeors have reached terminal fall speed, the mass of this condensate must be included in the calculation of the density of the parcel. If the volume of the condensate in a parcel of volume V is V_c, then the volume of air equals $V_a = V - V_c$. The mass of condensate is $M_c = \rho_w V_c$ where ρ_w is the density of water or ice as appropriate, so the combined density of air and condensate in the parcel is

$$\rho_p = \frac{M_a}{V} + \frac{M_c}{V} = \frac{M_a}{V_a}\left(\frac{V - V_c}{V}\right) + \frac{M_c}{V} = \rho(1 - V_c/V) + \rho_c \tag{4.52}$$

where ρ is the density of the air and ρ_c is the mass of condensate per unit total volume. The quantity $V_c/V = \rho_c/\rho_w$. Since $\rho_w \approx 10^3$ kg m^{-3} and $\rho_c \approx 10^{-2}$ kg m^{-3} at most, $V_c/V \leq 10^{-5}$ and can be ignored, which means to an excellent approximation that

$$\rho_p = \rho + \rho_c. \tag{4.53}$$

We use the equation of state for air to compute the parcel density in terms of the pressure p, temperature T, water vapor mixing ratio r_V, and condensate density:

$$\rho_p = \frac{p}{R_D T(1 + 0.61 r_V)} + \rho_c \approx \frac{p}{R_D T(1 + 0.61 r_V - r_C)}. \tag{4.54}$$

The quantity $\rho_c R_D T/p \approx \rho_c/\rho \approx r_C$ is to good approximation the mixing ratio of condensate, i. e., the ratio of condensate density to dry air density. This series of approximations is valid as long as both r_V and r_C are very much less than unity.

The CIN thus becomes

$$\text{CIN} = R_D \int_{\ln p_{LFC}}^{\ln p_I} [T_e(1 + 0.61 r_{Ve}) - T_p(1 + 0.61 r_{Vp} - r_{Cp})] d \ln p, \tag{4.55}$$

where a subscripted e indicates environment, a subscripted p indicates parcel values, and where we have used $dp/p = d \ln p$. We have assumed that no condensate exists in the environment. An extended virtual temperature including the effects of condensate is defined

$$T_V = T(1 + 0.61 r_V - r_C), \tag{4.56}$$

so the integrand is simply the difference between the extended virtual temperatures of the environment and the parcel.

A similar integral yielding the energy released per unit mass in a parcel ascending from the level of free convection to the level of neutral buoyancy is called the *convective available potential energy* (CAPE):

$$\text{CAPE} = R_D \int_{\ln p_{LNB}}^{\ln p_{LFC}} [T_p(1 + 0.61 r_{Vp} - r_{Cp}) - T_e(1 + 0.61 r_{Ve})] d \ln p. \quad (4.57)$$

The order of environmental and parcel quantities is reversed in the integrand compared to the expression for CIN, since CAPE is energy released by the parcel ascent, whereas CIN is the external energy required to lift the parcel.

At a given pressure level the difference between the temperature of the parcel and the environment can be related to the difference between the saturated moist entropies of the parcel and the environment,

$$s_p^* - s_e^* = \frac{\partial s^*}{\partial T}(T_p - T_e), \quad (4.58)$$

where the partial derivative of the saturated moist entropy is taken at constant pressure. (Note that the saturation mixing ratio in the definition of saturated moist entropy is a function of temperature, the effect of which must be included in the partial derivative.) Thus, to the extent that the difference between real and virtual temperatures can be ignored, the CIN can be written

$$\text{CIN} \approx R_D \int_{\ln p_{LFC}}^{\ln p_I} \left(\frac{\partial s^*}{\partial T}\right)^{-1} \left[s_e^*(p) - s_p^*\right] d \ln p. \quad (4.59)$$

Similarly, the CAPE can be approximated

$$\text{CAPE} \approx R_D \int_{\ln p_{LNB}}^{\ln p_{LFC}} \left(\frac{\partial s^*}{\partial T}\right)^{-1} \left[s_p^* - s_e^*(p)\right] d \ln p. \quad (4.60)$$

Above the lifting condensation level the saturated entropy of the parcel may be replaced by the normal moist entropy in these two equations, since parcel is saturated there and the two quantities are identical.

4.5 Mixing moist parcels

For dry convection in the atmosphere, we have implicitly assumed that density mixes linearly, i.e., the density resulting from mixing two parcels, ρ_m, is intermediate between the densities of the original parcels, ρ_1 and ρ_2, according to the equation

$$\rho_m = \epsilon\rho_1 + (1-\epsilon)\rho_2, \tag{4.61}$$

where ϵ is the mass fraction of parcel 1 in the mixture. When one of the parcels contains condensed water, this isn't necessarily true, as we now demonstrate.

If a thermal contains condensed water, then mixing the thermal with an unsaturated environment will result in the evaporation of at least some of the condensed water. As a consequence of the evaporation, the mixed parcel may be cooler, and therefore less buoyant than either of the initial parcels. To understand this more quantitatively, we utilize the fact that the enthalpy is conserved in constant pressure processes in which no heat is added or removed.

For a mixture of dry air, water vapor, and liquid, the enthalpy per unit mass of dry air is

$$h = [C_{PD} + r_V C_{PV} + r_L C_L]T - \mu_{BL} r_L, \tag{4.62}$$

where r_L is the condensed water mixing ratio and r_V is the water vapor mixing ratio. For our purposes it is sufficient to approximate this by

$$h \approx C_{PD}T - \mu_{BL} r_L. \tag{4.63}$$

Let us now consider two parcels of equal temperature T, but the first with liquid water mixing ratio r_{L1}, and the other unsaturated with vapor mixing ratio r_{V2}. The vapor mixing ratio of the first parcel is equal to the saturation value,

$$r_{V1} = r_S(T,p) = \frac{\rho_S}{\rho_D} = \frac{m_V e_S(T)}{m_D p_D} \approx \frac{m_V e_S(T)}{m_D p}, \tag{4.64}$$

where in the last step we ignore the difference between the partial pressure of dry air and the total pressure. The enthalpy of the first parcel is

$$h_1 = C_{PD}T - \mu_{BL} r_{L1} \tag{4.65}$$

while the enthalpy of the second parcel is

$$h_2 = C_{PD}T. \tag{4.66}$$

The enthalpy of the mixture will be

$$h_m = C_{PD}T_m - \mu_{BL} r_{Lm} \tag{4.67}$$

where T_m is the temperature of the mixture and r_{Lm} is its liquid water mixing ratio. Since enthalpy is conserved in this mixing process and thus mixes linearly, we have

$$\epsilon h_1 + (1-\epsilon)h_2 = h_m \tag{4.68}$$

where ϵ is the fraction of the first parcel (the cloud fraction) included in the mixture. This becomes

$$h_m = C_{PD}T - \epsilon\mu_{BL}r_{L1} = C_{PD}T_m - \mu_{BL}r_{Lm} \tag{4.69}$$

upon substituting equations (4.65)-(4.67). Similarly, the total water mixing ratio in the mixture is given by

$$r_{Tm} = \epsilon r_{T1} + (1-\epsilon)r_{T2} = \epsilon[r_{L1} + r_S(T,p)] + (1-\epsilon)r_{V2} \tag{4.70}$$

since total water mixes linearly. The total cloud water mixing ratio in parcel two has been replaced by the vapor mixing ratio since this parcel is unsaturated by hypothesis.

If the fraction ϵ of parcel one is small enough, then its liquid component will evaporate completely upon mixing. In this case the right side of equation (4.69) is just $C_{PD}T_m$ and the temperature T_m of the mixture is

$$T_m = T - \epsilon\mu_{BL}r_{L1}/C_{PD} \quad \text{unsaturated.} \tag{4.71}$$

However, for some critical value of $\epsilon = \epsilon_C$, the liquid will all just barely evaporate, leaving a saturated mixture. At this point we have $r_S(T_m,p) = r_{Tm}$. For larger values of ϵ, some of the liquid will remain. In this case equation (4.69) yields

$$T_m = T + \mu_{BL}[\epsilon r_S(T,p) + (1-\epsilon)r_{V2} - r_S(T_m,p)]/C_{PD} \quad \text{saturated,} \tag{4.72}$$

where we have set $r_{Lm} = r_{Tm} - r_S(T_m,p)$ and used equation (4.70) for r_{Tm}. This must be solved iteratively for T_m, as no analytical solution exists. An effective procedure for obtaining T_m is to solve both equations (4.71) and (4.72) for candidate mixture temperatures. The larger of the two will be the actual T_m.

Figure 4.6 illustrates the variation in the temperature of the mixed parcel as a function of the mixing fraction ϵ. The mixture is coldest at the boundary between a saturated and unsaturated mixture. The density is related to the parcel temperature by the ideal gas law.

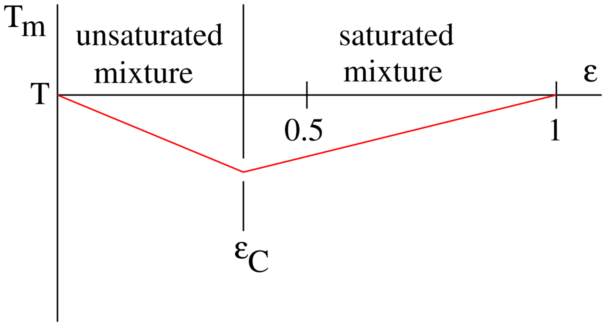

Figure 4.6: Temperature of mixture T_m as a function of the mixing fraction ϵ for the mixing of cloudy and clear parcels at the same initial temperature T. The value $\epsilon = \epsilon_C$ marks the boundary between an unsaturated and a saturated mixture.

4.6 Moist plumes and thermals

UNDER CONSTRUCTION.

4.7 Vertical mixing models

Entraining plume and thermal models of cumulus clouds were stimulated by the observation of Stommel (1947) that the liquid water content of clouds was typically much less than could be explained by the adiabatic lifting of a parcel of sub-cloud layer air from cloud base. This decrease from the adiabatic value was thought to be the result of entrainment of air from the surrounding environment, followed by its mixing with the cloudy air. As we saw in the above analysis, this results in a reduction in the condensate mixing ratio in the cloud.

A great deal of effort was put into matching the measured properties of clouds to laboratory models of plumes and thermals. A major problem with this effort is that laboratory experiments consisting typically of salt thermals in a freshwater tank could not emulate the thermodynamic effects of the evaporation of condensate.

As early as the late 1950s, Squires (1958) pointed out that cooling by condensate evaporation could lead to downdrafts originating at the top of small clouds, which penetrate downward through the body of the cloud. This implies a cloud structure which is much more complex than that representable

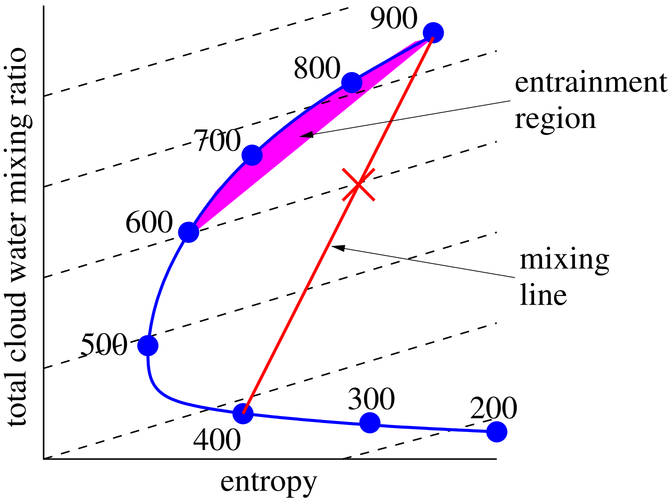

Figure 4.7: Atmospheric sounding represented by labeled curve, with labels being pressure in hPa. The solid straight line is a mixing line between 400 hPa and 900 hPa (see text) and the dashed lines are contours of constant potential temperature. The cross indicates the characteristics of the mixture which has potential temperature equal to the environmental potential temperature at 600 hPa. The shaded region indicates possible values of cloud water mixing ratio and entropy in the cloud at 600 hPa that could result from entrainment between 600 hPa and 900 hPa.

by a single thermal or plume.

An elegant pioneering paper by Fraser (1968) postulated that the net effect of a cloud is to mix air from the bottom and top of the cloud and eject it from the sides of the cloud. The argument was based on thermodynamic calculations of mixing and buoyancy presented in the previous section.

Paluch (1979) demonstrated, using conserved variables to trace the origin of cloud parcels, that the process envisioned by Fraser was essentially correct. The conserved variables used by Paluch were total cloud water (assuming that the cloud was not precipitating) and a form of equivalent potential temperature.

Paluch's technique is illustrated by figure 4.7. The total cloud water mixing ratio of the environmental sounding of the convection of interest is plotted versus the (moist) entropy of the environment. Mixing environmental parcels from different altitudes (really pressures) results in parcel characteristics which lie along a nearly straight line connecting the points on the sounding representing these altitudes. This is because the values of cloud water mixing ratio and entropy in the mixture are averages of the values in the original parcels, with weighting factors in proportion to the relative amounts of mass from each level in the mixture. Thus, if a cumulus cloud has its base at 900 hPa and its top at 400 hPa, all parcels resulting from mixing cloud top and cloud base air in varying proportions lie along the *mixing line* shown in figure 4.7.

In contrast, if all cloud parcels reaching 600 hPa come from this level or below, they would have entropy and total cloud water mixing ratios which put them in the shaded region of figure 4.7.

Paluch found that measurements in real clouds tended to place observed parcels along mixing lines rather than in the region representing mixtures of air from levels below the observation level. This is definitive evidence that vertical mixing does indeed occur, with mixtures descending considerable distances in clouds.

Raymond and Blyth (1986) developed a model of shallow, nonprecipitating convection which incorporates the ideas of Frasier (1968) and Paluch (1979). The predictions of this model were then compared against observations of detrainment in cumulus clouds. The predictions of the model were in reasonable agreement with the levels at which detrainment actually occurred in the observed clouds.

4.8 Downdrafts

UNDER CONSTRUCTION. (Show Byers-Braham model of thunderstorm cell.)

4.9 References

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4.10 Problems

1. Obtain a similarity solution to the plume problem for $b_0 = K_0/z$ where K_0 is a specified constant. Hint: To maintain similarity, assume that $b' = K'/z$ where K' is a constant to be determined.

2. Obtain a non-similarity solution, but using the similarity entrainment assumption, for the buoyancy perturbation in a thermal starting from rest at $z = 0$ with initial radius R_0 in a stable environment with $b_0(z) = \Gamma z$ with initial buoyancy anomaly ΓL_0. Recall that L_0 is the undilute level of neutral buoyancy. Compare your level of neutral buoyancy to that obtained using the Sánchez et al. treatment for $L_0 = 10000$ m, $\Gamma = 2 \times 10^{-4}$ s^{-2}, and $R_0 = 1000$ m.

3. Using the simplified moist entropy $s = C_{PD} \ln(T/T_F) - R_D \ln(p/p_R) + Lr_V/T_F$ (see the chapter on thermodynamics), compute the factor $\partial s^*/\partial T$ in equation (4.58). (We get s^* from s by replacing the vapor mixing ratio r_V with its saturation value r_S.) Since we are using the simplified form for s in which the latent heat L is constant, this constant value should be used in the differential form of the saturation vapor pressure equation $d \ln e_s/dT = L/(R_V T^2)$.

4. Write a computer program to solve for the temperature of a parcel obtained by mixing two parcels at initial temperature T and pressure p, the first with liquid water mixing ratio r_{L1} and the second with vapor mixing ratio r_{V2}. Assume that the parcels mix with fraction ϵ for the first parcel and $1-\epsilon$ for the second. As a check, find and plot the values of $T_m(\epsilon)$, $0 < \epsilon < 1$, for $T = 283$ K, $p = 700$ hPa, and with $r_{L1} = 0.005$ and $r_{V2} = 0.003$.