Chapter 1

Fluid equations

1.1 Stress

Two kinds of forces act on molecules in a fluid, short-range intermolecular forces, and long-range forces such as gravity. The two types of forces require very different treatment in the fluid dynamical governing equations.

Let us consider the intermolecular forces acting across the faces of a tetrahedron of fluid, as shown in figure 1.1. Think of the tetrahedron as a corner cut off of a cube. We assume that this chunk of fluid is embedded in a larger fluid body. The force per unit area on the $y-z$ face of the tetrahedron is $\mathbf{t}_x$, on the $x-z$ face is $\mathbf{t}_y$, and on the $x-y$ face is $\mathbf{t}_z$. On the oblique face it is simply $\mathbf{t}$. The corresponding areas of each face are A_x, A_y, A_z, and A. Assuming that the tetrahedron is small, so that the force per unit area doesn't vary much over each face, Newton's second law in the static case with no other forces acting tells us that

$$A_x\mathbf{t}_x + A_y\mathbf{t}_y + A_z\mathbf{t}_z + A\mathbf{t} = 0. \tag{1.1}$$

Noting that A_x is just the projection of A on the $y-z$ plane, we find that $A_x/A = \hat{\mathbf{x}} \cdot \hat{\mathbf{n}}$, where $\hat{x}$ is the unit vector in the x direction and $\hat{\mathbf{n}}$ is the outward normal unit vector to the oblique surface. Similar relations hold for the other surfaces, resulting in

$$\mathbf{t} = -(\mathbf{t}_x\hat{\mathbf{x}} + \mathbf{t}_y\hat{\mathbf{y}} + \mathbf{t}_z\hat{\mathbf{z}}) \cdot \hat{\mathbf{n}} \equiv \mathbf{T} \cdot \hat{\mathbf{n}}. \tag{1.2}$$

The quantity $\mathbf{T}$ is called the *stress tensor*.

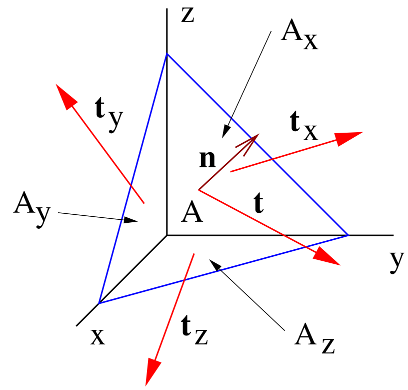

Figure 1.1: Molecular forces on a tetrahedron of fluid.

Note a convention that we have implicitly established: the sense of the force per unit area $\mathbf{t}$ across the oblique surface is that the fluid penetrated by the unit normal $\hat{\mathbf{n}}$ acts on the fluid on the other side of the surface. This force per unit area is called the *traction* across the surface. Note further that $\mathbf{t}_x$ indicates the traction across the $y-z$ plane, not the x component of a traction. Representation of components requires a second subscript: T_{yx} is the y component of the traction $\mathbf{t}_x$ across the $y-z$ face of the tetrahedron, whereas T_{xy} is the x component of the traction $\mathbf{t}_y$ across the $x-z$ face. Thus, the first subscript represents the vector component of the force and the second subscript indicates the face on which the force is acting.

We now demonstrate that the stress tensor is symmetric. In matrix form the stress tensor is

$$\mathbf{T} = -\begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix}, \tag{1.3}$$

given the above convention on indices, so that $T_{ij} = -t_{ij}$. Imagine a cube of fluid small enough that the stress tensor does not vary significantly over the cube. The shear or face-parallel components of the tractions on the $x-z$ and $y-z$ faces are shown in figure 1.2, the normal components being omitted for clarity. Recall that we assume a static situation with the tractions being the only forces applied to the cube. Under these conditions the torque on the cube must be zero. However, computing the torque about the center of the cube quickly leads us to the conclusion that $T_{xy} = T_{yx}$, etc. (The normal components of the traction do not contribute to the torque.)

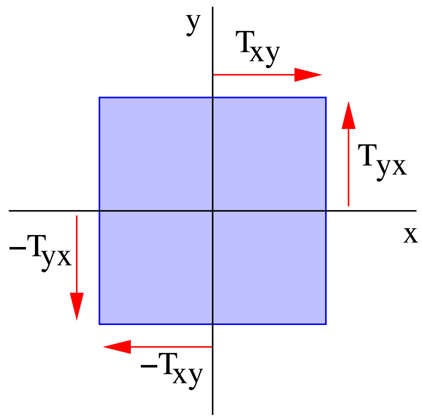

Figure 1.2: Shear tractions acting on a cube.

We now relax the assumptions of static equilibrium and no other forces. In this case the mass of the tetrahedron in figure 1.1 times the acceleration equals the total traction force plus the long range or body force on the tetrahedron. If L is a typical dimension of the tetrahedron, then the mass scales as L^3. The body force should also scale as L^3, since it should have a magnitude proportional to the total number of molecules in the tetrahedron. However, the traction forces scale as L^2, since they are proportional to the area of the tetrahedron faces. Therefore the ratio of the traction forces to the mass times the acceleration or to the body force scales as L^{-1}. If we now take the limit of very small L, it is clear that the traction forces dominate over the other two terms. Thus, for an infinitesimal tetrahedron, the analysis for static conditions with no body forces carries over to the more general case.

1.2 Newtonian fluids

In fluid dynamics we generally take what is called the *Eulerian* point of view, in which we consider our dependent variables such as fluid velocity and density to be a function of position rather than being identified with the individual parcels making up the fluid. Thus, the velocity at some point refers at different times to different parcels of fluids. The alternative point of view, in which variables are referenced to fluid parcels, is called the *Lagrangian* point of view. It has occasional uses in fluid dynamics, but is much less commonly seen than the Eulerian representation, which we use here exclusively.

We now investigate the ways in which the fluid velocity $\mathbf{v}(\mathbf{x})$ can vary in the vicinity of some test point $\mathbf{x}_0$. Representing vectors in component notation, we make a Taylor series expansion in v_i about the point x_{0i}:

$$v_i(x_j) = v_i(x_{0j}) + \frac{\partial v_i}{\partial x_j}\delta x_j, \tag{1.4}$$

where $\delta x_j = x_j - x_{0j}$ and where we use the Einstein convention that repeated indices are summed.

The quantity $\partial v_i/\partial x_j$ is called the *deformation rate tensor.* It can be split into symmetric and antisymmetric parts

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \equiv D_{ij} + R_{ij}, \tag{1.5}$$

where D_{ij} is called the *strain rate tensor* and R_{ij} is the *rotation tensor.*

To understand the origin of the latter name, suppose that the fluid is locally in rigid body rotation about the point $\mathbf{x}_0$. In this case the fluid velocity field will locally take the form $\mathbf{v} = \mathbf{v}_0 - \delta\mathbf{x} \times \Omega$, where Ω is the angular rotation rate vector. The cross product can be written in component form as $\epsilon_{ijk}\delta x_j \Omega_k$ where $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$, and ϵ_{ijk} equals zero if any two indices are equal. The deformation rate tensor thus becomes

$$\frac{\partial v_i}{\partial x_j} = R_{ij} = -\epsilon_{ijk}\Omega_k, \tag{1.6}$$

which is antisymmetric. The strain rate tensor is therefore zero in this case, and the deformation rate tensor equals the rotation tensor. Furthermore, Ω is called the *dual vector* of the rotation tensor.

In the general case, the rotation tensor represents the rotational part of the local deformation while the strain rate tensor represents that part of the deformation which changes the shape of the local fluid element.

A Newtonian fluid is one in which the components of the stress tensor are related linearly to the components of the strain rate tensor. This relationship cannot be arbitrary, but must be expressible in terms of a tensor expression, which makes the relationship independent of coordinate system, and thus physically viable. This condition puts strong limits on the possibilities, and the most general linear relationship is

$$T_{ij} = A\delta_{ij} + BD_{kk}\delta_{ij} + CD_{ij}, \tag{1.7}$$

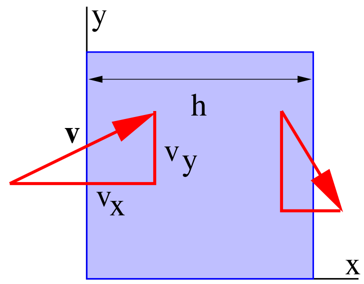

Figure 1.3: Side view of a stationary 3-D cube of side h with fluid flowing through it.

where A, B, and C are scalars which may be functions of fluid properties.

In the static case, $\mathbf{D} = 0$ and only the first term on the right side of equation (1.7) is important. A traction $\mathbf{t} = \mathbf{T} \cdot \hat{\mathbf{n}} = A\hat{\mathbf{n}}$ occurs in this case. Experimentally, an inward force equal to the pressure force is generally needed to keep a fluid parcel in place in this situation. Since $\mathbf{n}$ points outward, that means that A must be negative. We therefore write $A = -p$, where p is the pressure, i. e., the force per unit area needed to confine the fluid parcel.

By convention (the reasons for which we will explore later), we write $B = \eta - 2\mu/3$ and $C = 2\mu$, where μ is the *coefficient of viscosity* and η is the *second coefficient of viscosity*. Thus, we rewrite equation (1.7) as

$$T_{ij} = -p\delta_{ij} + (\eta - 2\mu/3)\delta_{ij}D_{kk} + 2\mu D_{ij}. \tag{1.8}$$

1.3 Mass continuity

Referring to figure 1.3, the mass of fluid per unit time flowing into the cube from the left side is $\rho(x)v_x(x)h^2$, whereas the mass flowing out of the right side is $\rho(x+h)v_x(x+h)h^2$, where ρ is the mass density of the fluid and v_x is the x component of the fluid velocity. Similar expressions can be developed for flow of mass in and out of the other faces of the cube.

The net flow of mass per unit time out of the box from the left and right faces is

$$\left(\frac{\rho(x+h)v_x(x+h) - \rho(x)v_x(x)}{h}\right) h^3 \approx \frac{\partial \rho v_x}{\partial x} h^3.$$

Since the cube is fixed in volume and location, the time rate of change of mass in the box is $(\partial\rho/\partial t)h^3$. Equating this to minus the rate at which mass is flowing out of the cube through all six faces results in

$$\frac{\partial\rho}{\partial t}h^3 \approx -\frac{\partial\rho v_x}{\partial x}h^3 - \frac{\partial\rho v_y}{\partial y}h^3 - \frac{\partial\rho v_z}{\partial z}h^3.$$

The approximation becomes exact when h is allowed to become very small, resulting in the *mass continuity equation* when h^3 is canceled:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{1.9}$$

An alternate form of this equation may be obtained by noting that $\nabla\cdot(\rho\mathbf{v}) = \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v}$:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0, \tag{1.10}$$

where

$$\frac{d\rho}{dt} \equiv \frac{\partial\rho}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla\rho = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho \tag{1.11}$$

is the time derivative of the density following a parcel in its flow. This is sometimes called the *parcel derivative* or the *material derivative*. It can also be applied to quantities other than the density.

1.4 Momentum

The corresponding equation for momentum is more complex in two respects: first, momentum is a vector, so the flux of momentum per unit area per unit time $\rho\mathbf{v}\mathbf{v}$ is a tensor; second, forces as well as mass transport can change the amount of momentum in the cube. The momentum balance in the cube is therefore governed by

$$\left(\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v})\right)h^3 = \text{force on cube}. \tag{1.12}$$

The stress force on the left and right x faces of the cube is

$$[\mathbf{T}(x)\cdot(-\hat{\mathbf{x}}) + \mathbf{T}(x+h)\cdot(\hat{\mathbf{x}})]h^2 \approx \frac{\partial\mathbf{T}}{\partial x}\cdot\hat{\mathbf{x}}h^3, \tag{1.13}$$

with similar expressions for the y and z faces. The total force is therefore $\nabla\cdot\mathbf{T}h^3$. The earth exerts a gravitational force on the mass in the cube.

In addition, the earth is rotating, resulting in a centrifugal force and a Coriolis force in the rotating reference frame of the earth. We combine gravity and centrifugal force per unit mass into an effective gravity $\mathbf{g} = \mathbf{g}' - \Omega \times (\Omega \times \mathbf{r})$, where $\mathbf{g}'$ is the actual gravitational field and Ω is the rotation vector for the earth, i. e., it has magnitude equal to 2π divided by the earth's rotation period and direction parallel to the rotation axis of the earth. The Coriolis force per unit mass is given by $-2\Omega \times \mathbf{v}$.

Note that the effective gravity has the potential energy per unit mass (called the *geopotential*) equal to $\phi = \phi' - \Omega^2 r_n^2/2$ where ϕ' is the gravitational potential energy per unit mass and r_n is the component of $\mathbf{r}$ normal to the rotation vector Ω.

Combining all of these effects and using the stress for a Newtonian fluid given in equation (1.8) results in the following *momentum equation*:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} - \mathbf{T}) = \rho \mathbf{g} - 2\rho\Omega \times \mathbf{v}. \tag{1.14}$$

To a very good approximation in atmospheric convection, η and μ can be considered constants. Furthermore, since atmospheric flow is highly subsonic, it is nearly incompressible on small spatial scales over which viscous effects are important, which means that terms involving both the divergence of the velocity and viscous constants can be ignored. Under these conditions the equations simplify to

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p - \rho \mathbf{g} + 2\rho\Omega \times \mathbf{v} - \mu \nabla^2 \mathbf{v} = 0. \tag{1.15}$$

An alternate form of this equation may be obtained by noting that

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \mathbf{v}. \tag{1.16}$$

The second term on the right side is zero by virtue of the mass continuity equation (1.9), which means that equation (1.15) can be rewritten

$$\frac{d\mathbf{v}}{dt} + \frac{1}{\rho}\nabla p - \mathbf{g} + 2\Omega \times \mathbf{v} - \nu \nabla^2 \mathbf{v} = 0, \tag{1.17}$$

where $\nu \equiv \mu/\rho$ is the *kinematic viscosity*.

1.5 Incompressible fluid

We now consider an incompressible fluid, which nevertheless may vary in density with position. An example of such a fluid is salty water with the value of the salinity varying from parcel to parcel. Ignoring diffusion for now, the density obeys the equation

$$\frac{d\rho}{dt} = 0, \tag{1.18}$$

which simply indicates that the density of a parcel doesn't change. Combining this with the mass continuity equation (1.10) yields the condition

$$\nabla \cdot \mathbf{v} = 0. \tag{1.19}$$

We now have all the ingredients needed to compute the evolution of an incompressible fluid. However, in many cases the density variations in a fluid are fractionally small, which allows certain simplifying approximations to be made. Let us assume that $\rho = \rho_0 + \rho'$, where ρ_0 is a constant reference density and ρ' is small in magnitude compared to ρ_0. The incompressibility equation therefore becomes

$$\frac{d\rho'}{dt} = 0. \tag{1.20}$$

The application of this approximation to the momentum equation is slightly more complex. In an equilibrium situation with a fluid at rest, equation (1.17) reduces to hydrostatic balance:

$$\frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = 0. \tag{1.21}$$

The horizontal component of this equation tells us that the horizontal gradient of pressure is zero, i. e., pressure is a function only of z. The vertical component obeys

$$\frac{1}{\rho}\frac{dp}{dz} + g = 0, \tag{1.22}$$

where the conversion to a total z derivative reflects the lack of dependence of p on x and y.

We now obtain the pressure distribution $p_0(z)$ hydrostatically consistent with our constant reference density ρ_0:

$$p_0(z) = g\rho_0(z_0 - z) \tag{1.23}$$

where z_0 is the elevation at which $p_0 = 0$. Assuming that $p = p_0(z) + p'$, where the magnitude of p' is taken to be much less than p_0, then the pressure gradient and gravity terms can be approximated by

$$\frac{1}{\rho_0 + \rho'}\nabla(p_0 + p') - \mathbf{g} \approx \frac{1}{\rho_0}\nabla p_0 - \mathbf{g} + \frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0^2}\nabla p_0 = \nabla\pi - b\hat{\mathbf{z}}, \quad (1.24)$$

where we have dropped nonlinear terms in primed quantities, and where the *kinematic pressure* is defined $\pi \equiv p'/\rho_0$ and where the *buoyancy* is given by $b = -g\rho'/\rho_0$. In addition we have used equation (1.21) to convert ∇p_0 to $-\rho_0 g\hat{\mathbf{z}}$. The momentum equation thus becomes

$$\frac{d\mathbf{v}}{dt} + \nabla\pi - b\hat{\mathbf{z}} + 2\Omega \times \mathbf{v} = 0 \quad (1.25)$$

where we have ignored viscosity as well as diffusion. Equation (1.20) can be rewritten in terms of the buoyancy:

$$\frac{db}{dt} = 0. \quad (1.26)$$

The approximations used in equations (1.19), (1.25), and (1.26) are jointly called the *Boussinesq approximation.*

1.6 Ideal gas

For the subsonic, non-turbulent flow of an ideal gas with no radiative processes or latent heat release, the specific entropy s written in terms of temperature T and pressure p

$$s = C_p \ln(T/T_0) - (R/m)\ln(p/p_0) \quad (1.27)$$

is conserved to a high degree of accuracy. The constants C_p and R are the (mass) specific heat of the gas at constant pressure and the universal gas constant, m is the molecular weight of the gas, and T_0 and p_0 are constant reference values of temperature and pressure. Again ignoring diffusive processes, this implies

$$\frac{ds}{dt} = 0. \quad (1.28)$$

A related variable is the *potential temperature*

$$\theta \equiv T_0 \exp(s/C_p) = T(p_0/p)^{\kappa}, \quad (1.29)$$

where $\kappa = R/(mC_p)$. The potential temperature is also conserved by parcels under these conditions,

$$\frac{d\theta}{dt} = 0, \tag{1.30}$$

and it can be thought of as a surrogate for the entropy. By using the ideal gas law $p/\rho = RT/m$, we can rewrite the pressure term in the momentum equation:

$$\frac{1}{\rho}\nabla p = \frac{RT}{mp}\nabla p = \frac{R\theta}{mp}(p/p_0)^\kappa \nabla p = \theta C_p \nabla (p/p_0)^\kappa \equiv \theta \nabla \Pi. \tag{1.31}$$

The quantity $\Pi = C_p(p/p_0)^\kappa$ is called the *Exner function*.

The Boussinesq approximation can be made in the case of an ideal gas. Density variations in the earth's atmosphere are related primarily to the hydrostatic variation in pressure with height and with sound waves. Subsonic flows and waves with vertical scales small compared to the vertical scale of hydrostatic pressure changes still exhibit density variations, but these variations are small. Thus, we can ignore $d\rho/dt$ compared to the individual terms of $\rho \nabla \cdot \mathbf{v}$, allowing us to use equation (1.19).

For the momentum equation we assume that $\theta = \theta_0 + \theta'$, where θ_0 is a constant reference potential temperature, and $\Pi = \Pi_0 + \Pi'$, where $\Pi_0(z)$ is hydrostatically related to θ_0, i. e.,

$$\theta_0 \nabla \Pi_0 + g\hat{\mathbf{z}} = 0. \tag{1.32}$$

Following the same procedure used for the incompressible fluid case, we come up with the following momentum equation:

$$\frac{d\mathbf{v}}{dt} + \nabla(\theta_0 \Pi') - \frac{g\theta'}{\theta_0} + 2\Omega \times \mathbf{v} = 0. \tag{1.33}$$

If we define $\pi = \theta_0 \Pi'$ and $b = g\theta'/\theta_0$, then equation (1.33) takes precisely the same form as equation (1.25). Starting from equation (1.30), it is easy to show that the buoyancy as defined here also satisfies equation (1.26). Thus, under the conditions employed in this section, an ideal gas obeys the same set of equations as an incompressible fluid under the Boussinesq approximation. This demonstrates that the ocean and the atmosphere exhibit essentially similar behavior in many circumstances.

A somewhat more accurate approximation for the atmosphere takes into account the great variations in density with height while ignoring deviations from this density profile everywhere except in the buoyancy term. In particular, we assume that $\rho = \rho_0(\theta_0, \Pi_0) + \rho'$ where θ_0 as well as Π_0 is now a function of z, and write the mass continuity equation

$$\nabla \cdot (\rho_0 \mathbf{v}) = 0. \tag{1.34}$$

This is called the *anelastic approximation.* In this approximation the θ_0 cannot be drawn inside the gradient operator in equation (1.33).

1.7 Reference

Kundu, P. K., 1990: *Fluid Mechanics.* Academic Press, New York. This is a good overall reference on fluid dynamics, and it contains a discussion of the Boussinesq approximation.

Landau, L. D., and E. M. Lifshitz, 1959: *Fluid Mechanics.* Pergamon Press, Oxford. The "gold standard" in fluid dynamics.

1.8 Problems

1. Show by a scale analysis that the argument for the symmetry of the stress tensor carries over to the non-static case with body forces when the cube in figure 1.2 is of infinitesimal size.

2. Given the stress tensor and the pressure, find the rate of strain tensor.

3. Given a fluid velocity field $\mathbf{v} = Sy\hat{\mathbf{x}}$ where S is a constant:

   (a) Compute the strain rate and rotation rate tensors.

   (b) Find the rotation vector associated with the rotation rate tensor.

   (c) Find the principal axes and eigenvalues associated with the strain rate tensor.

   (d) Find the viscous part of the stress tensor assuming a Newtonian fluid.

4. Appealing to standard classical mechanics, write separately the gravitational and inertial components of $\mathbf{g}$ in equation (1.14) in terms of

the actual gravitational field, the rotation vector of the earth, and the radius vector $\mathbf{r}$ relative to the center of the earth. Use this to show that if the rotation rate of the earth changes, then the definition of "horizontal" changes as well. For the purposes of computing the gravitational field, assume a spherical earth, even though this is not strictly correct.

5. Suppose for a fluid $\partial v_x/\partial t = 0$, but $v_x = Cx$, where C is a constant. Find the x component of the total force per unit mass on the fluid. Assuming further that $v_y = v_z = 0$, find $d \ln \rho/dt$.

6. Suppose an ideal gas is at rest in a uniform gravitational field.

   (a) If the gas has constant entropy with height, determine how the pressure varies with height.

   (b) Determine how the pressure and entropy vary with height if the temperature is constant with height.

7. Suppose instead that the gravitational field obeys $g = g_0(r_0/r)^2$, where g_0 is the gravitational field at radius r_0. Determine the distribution of pressure with radius r for an ideal gas atmosphere of constant temperature. Does the pressure go to zero as $r \to \infty$?

8. Solve for ρ as a function of θ and Π for an ideal gas.