Introduction to Continuum Mechanics

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Chapter 1 -- Introduction

Continuum mechanics is a theory of the kinematics and dynamics of material bodies in the limit in which matter can be assumed to be infinitely subdividable. Scientists have long struggled with the question as to whether matter consisted ultimately of an aggregate of indivisible “atoms”, or whether any small parcel of material could be subdivided indefinitely into smaller and smaller pieces. As we all now realize, ordinary matter does indeed consist of atoms. However, far from being indivisible, these atoms split into a staggering array of other particles under sufficient application of energy -- indeed, much of modern physics is the study of the structure of atoms and their constituent particles.

Previous to the advent of quantum mechanics and the associated experimental techniques for studying atoms, physicists tried to understand every aspect of the behavior of matter and energy in terms of continuum mechanics. For instance, attempts were made to characterize electromagnetic waves as mechanical vibrations in an unseen medium called the “luminiferous ether”, just as sound waves were known to be vibrations in ordinary matter. We now know that such attempts were misguided. However, the mathematical and physical techniques that were developed over the years to deal with continuous distributions of matter have proven immensely useful in the solution of many practical problems on the macroscopic scale. Such techniques typically work when the scale of a phenomenon is much greater than the separation between the constituent atoms of the material under consideration. They are therefore of great interest to geophysicists, astrophysicists, and other types of applied physicists, as well as to applied mathematicians and engineers. Indeed, the modern development of the subject has been largely taken over by mathematicians and engineers.

This textbook develops the subject of continuum mechanics from the point of view of an applied physicist with interests in geophysics and astrophysics. The subject of continuum mechanics is a vast one, and the above interests have guided the selection of material. However, the basic subjects covered, i. e., elastic bodies and Newtonian fluids, transcend the author’s particular interests, and are central to the full spectrum of applications of continuum mechanics.

The key mathematical concept in continuum mechanics is the tensor -- in no other area of physics do tensors appear so naturally and ubiquitously. The main problem for the student is to connect the rather abstract mathematical notion of a tensor to the physics of continuous media. To this end, the properties of tensors are developed in parallel with the physical notions of stress and strain.

Certain mathematical preparation beyond elementary calculus is needed to master continuum mechanics. The student should be familiar with vector analysis, including the laws of Gauss and Stokes, and should have some understanding of matrix operations. In addition, experience with the solution of elementary differential equations, such as the harmonic oscillator equation, is essential.

Chapter 2 -- The Notion of Stress

Atoms and molecules in liquids and solids are subject to two types of forces, namely long range forces such as gravity, and short range, molecular bonding forces. In this chapter we consider how short range forces are treated in continuum mechanics. This gives rise to the notion of stress, a concept that is central to the subject. In order to understand stress, we further need to develop the mathematical idea of a tensor. This we believe is best done in coincidence with the development of the physical concepts.

Conceptual Model from Atomic Physics

Let us first consider a simple conceptual model of a crystalline solid in two dimensions. Imagine a regular array of atoms or molecules tied together by springs as illustrated in figure 2.1. The springs simulate the intermolecular forces, and a state of equilibrium exists when none of the springs are stretched or compressed from their equilibrium lengths. We are interested in the force acting across the line AB, which is just the vector sum of the spring forces for those springs that cross AB. The nature of this force is most easily appreciated by concentrating on just those springs attached to a single molecule, indicated by the square in figure 2.1. Six springs, a, b, c, d, e, and f are attached to this molecule, but only two of those, a and b, cross AB, and are therefore of current interest.

Figure 2.2 shows what happens when the molecule is displaced small amounts in various directions with no displacements allowed in connecting molecules. If it is displaced parallel to AB, spring a is compressed and spring b is stretched, and the net force is such as to push the molecule back to its original position. This is called a shear displacement. Similar restoring forces occur when the subject molecule is moved toward (compression) or away from (extension) AB.

The point to be recognized here is that the direction of the restoring force is related to the direction in which the molecule is displaced, and is unrelated to the orientation of the line AB except insofar as the choice of AB determines which springs need to be considered in the calculation.

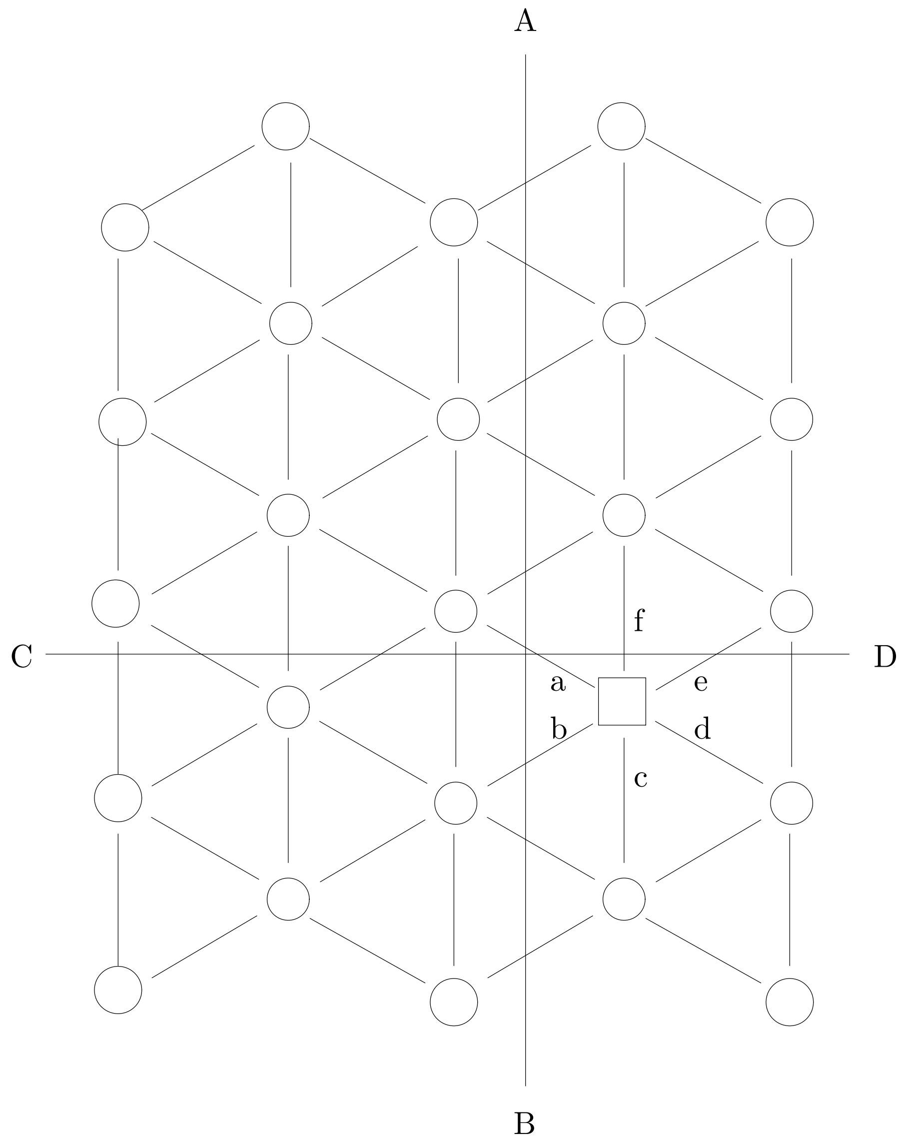

Figure 2.1. Conceptual model of a crystalline solid.

The restoring force is, of course, only part of the force acting on the molecule, because we have not included the forces due to the other springs. Indeed, if we compute instead the force across the line CD in figure 2.1, we may get quite different values for the partial restoring force associated with a given displacement of the molecule, because now springs a, e, and f must be considered. In fact, the net force acting on the molecule across the line AB is almost completely independent of the force acting across CD if displacements of the connected molecules are allowed. The meaning of "almost" in this case will be

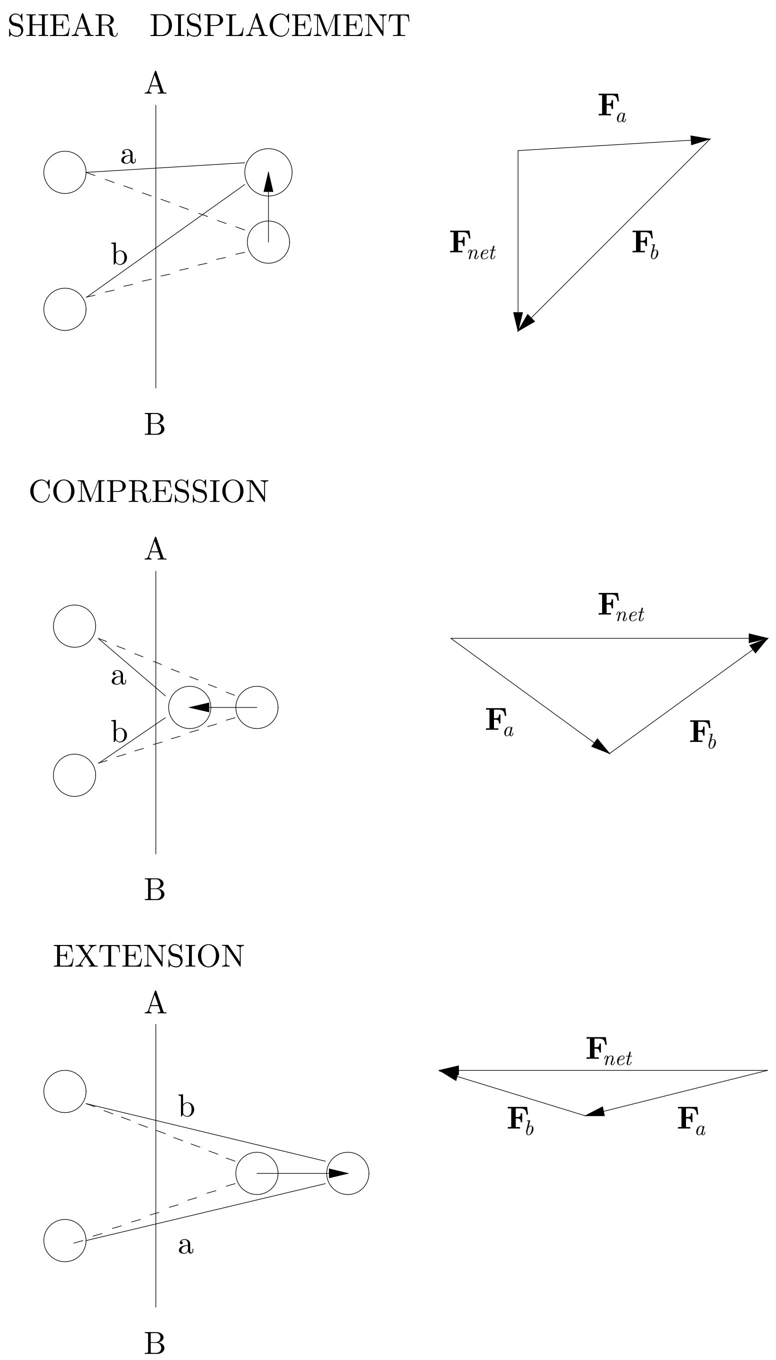

Figure 2.2. Relation between displacement and force on a molecule.

explored more fully later in this chapter.

The sum of all the spring forces acting across AB is called the *stress force* across that line. In three dimensions one would consider the stress force acting across a surface. In continuum mechanics we are interested in the collective behavior of many atoms or molecules, and consider the stress force across a surface to be the integral over the surface of a stress force per unit area, or a *traction*, rather than a sum over discrete molecular bonds. The traction may vary with position on the surface, but this only makes sense if the distance over which significant variation takes place is large compared to molecular spacings. This is because the traction at a particular point is actually the sum of all the spring forces through the surface within some distance of that point, divided by the area on the surface encompassed by this sampling distance. The sampling distance must be much larger than the molecular spacing for this averaging process to make sense, yet much smaller than the distance over which traction varies significantly. It is this assumed scale separation that makes continuum mechanics a significant simplification over explicitly computing the motion of every molecule in a complex system.

Traction Across Arbitrary Planes

We now drop our conceptual crutch of a crystalline solid, and think of matter as being continuously distributed in space. We know, of course, that this is an approximation based on an assumed separation of scales between molecular structure and the phenomenon of interest. The traction, or stress force per unit area across a surface, becomes the central focus of our attention, irrespective of how it is related to phenomena at the molecular level.

We now introduce a convention that is universal to modern continuum mechanics, but is perhaps somewhat counterintuitive. Imagine a plane surface separating two regions, labeled 1 and 2 in figure 2.3. The orientation of the surface is defined by a unit normal vector $\mathbf{n}$, shown as pointing into region 2 in the figure. However, a unit vector pointing in the opposite direction could just as well have defined the orientation of the surface. We take advantage of this ambiguity to ascribe additional significance to the direction of $\mathbf{n}$: If $\mathbf{n}$ pierces region 2, then the traction across the surface (illustrated by the vector $\mathbf{t}$ in the figure) is considered to be the force per unit area exerted *by* region 2 *on* region 1. Thus, the pierced region does the acting.

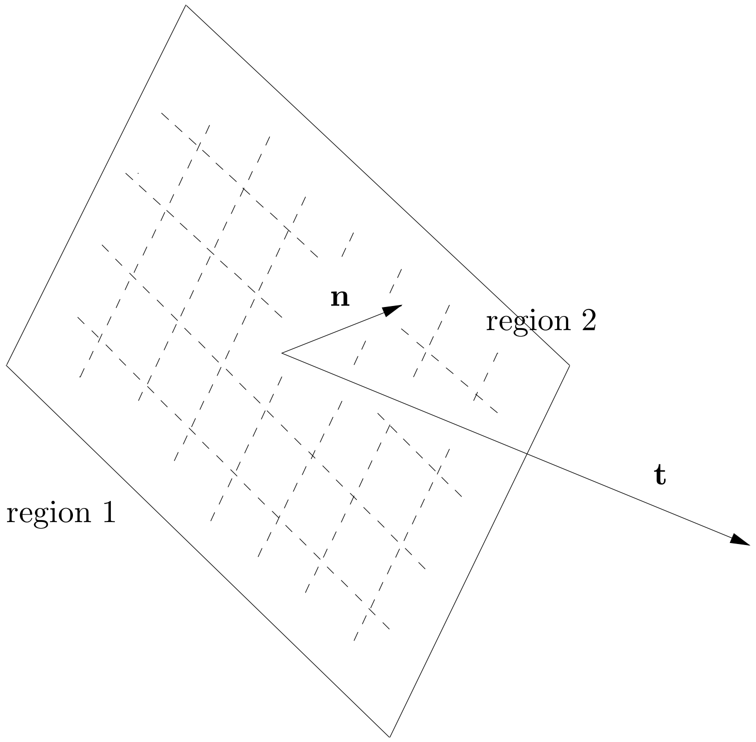

Figure 2.3. Illustration of the traction (**t**) and unit normal (**n**) vectors relative to a surface cutting a continuous medium. The traction is the force per unit area of region 2 acting on region 1.

The above arguments indicate that the traction vector generally varies even at a single point as the orientation of the dividing surface is varied. Thus, an infinite number of different tractions are possible at a single point, depending on the orientation of the surface. However, it seems implausible that all these different tractions could be independent, and in fact it is not true. It turns out that once the traction is specified at a particular point across three mutually perpendicular surfaces (in three dimensions), the traction across any other surface that passes through that point can be computed. This computation leads naturally to the definition of a mathematical entity called a tensor -- the stress tensor in this case.

To prove this point, we turn to Newton's second law. Imagine a chunk of matter in the form of a tetrahedron obtained by cutting off the corner of a cube, as shown in figure 2.4. The Cartesian axes coincide with the edges of the cube, and outward unit normal vectors $-\mathbf{i}$, $-\mathbf{j}$, $-\mathbf{k}$, and $\mathbf{n}$ are shown for each of the

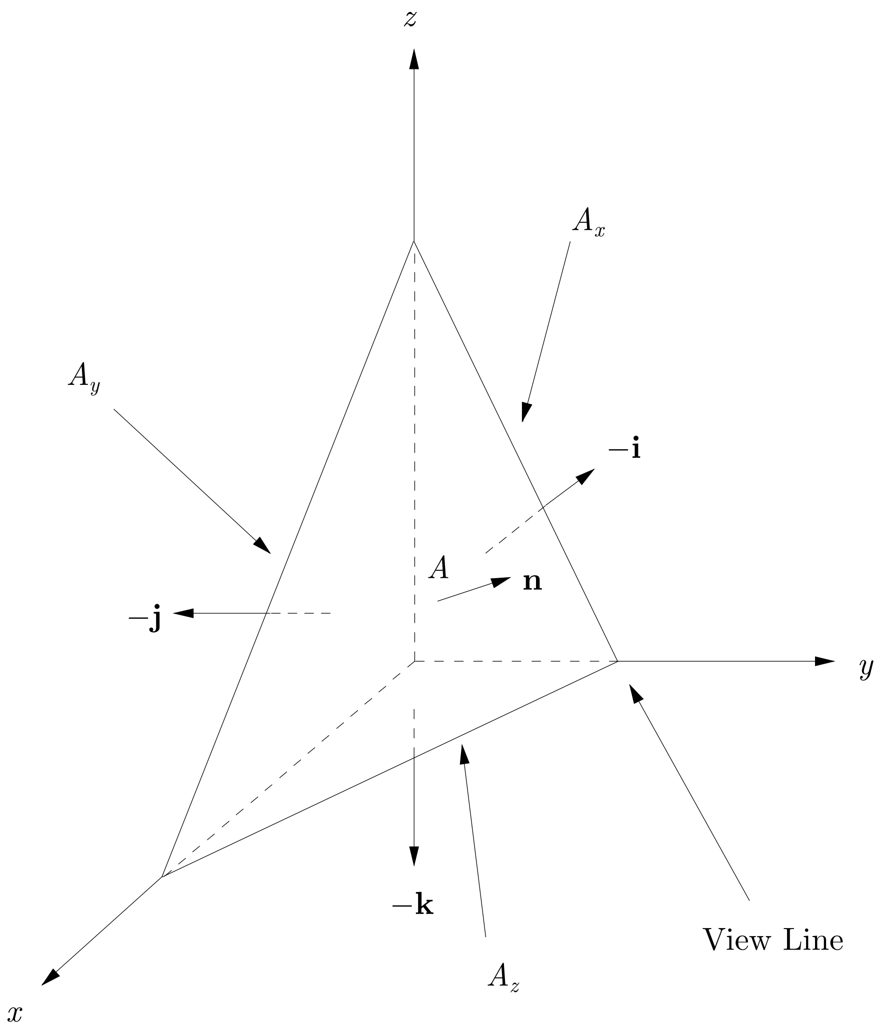

Figure 2.4. Definition sketch of tetrahedron used to derive the traction across an arbitrary surface from the tractions across three mutually perpendicular surfaces.

four surfaces, along with their respective areas, A_x, A_y, A_z, and A. If we assume that the tetrahedron is at rest and ignore long range forces, then the total stress force on the body, which is the sum of the stress forces acting across each surface must be zero:

$$\mathbf{t}_x A_x + \mathbf{t}_y A_y + \mathbf{t}_z A_z + \mathbf{t} A = 0, \tag{2.1}$$

where the traction vectors $\mathbf{t}_x$, etc., are labeled by the surfaces on which they act. The x surface is that surface normal to the x axis, etc. (Note in particular that the subscripts *do not* indicate components of the traction vector in this case!) In setting the stress force across each surface to the product of the traction vector and the area, we have assumed that the traction varies insignificantly over the surface. As we will ultimately let the dimensions of the tetrahedron approach zero, this is not a limiting assumption.

The areas of each face of the tetrahedron are related to the respective unit normals. This may be appreciated by viewing the tetrahedron along one of its oblique edges, as illustrated in figure 2.5. Here we see

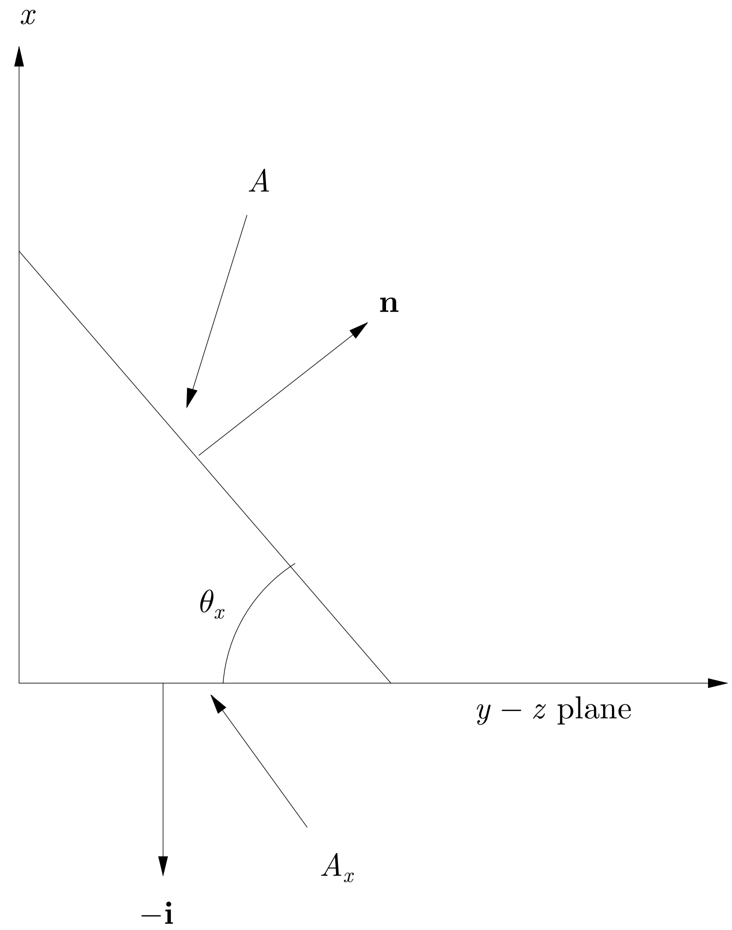

Figure 2.5. View of tetrahedron of figure 2.4 along view line.

the tetrahedron from a point on a line defined by the intersection of the $y - z$ plane and the oblique surface. The x and oblique faces thus appear edge-on at an angle θ_x to each other. Since the area A_x is just the projection of the area A onto the $y - z$ plane, we have $A_x/A = \cos\theta_x = \mathbf{i} \cdot \mathbf{n}$. Similar relationships hold for the y and z surfaces. Solving equation (2.1) for the traction $\mathbf{t}$ across the oblique face of the tetrahedron and eliminating the areas yields

$$\mathbf{t} = -\,\mathbf{t}_x(\mathbf{i}\cdot\mathbf{n}) - \mathbf{t}_y(\mathbf{j}\cdot\mathbf{n}) - \mathbf{t}_z(\mathbf{k}\cdot\mathbf{n}). \tag{2.2}$$

This is precisely the desired result, as it shows how to compute the traction across an arbitrarily oriented surface, assuming that the tractions are known across the three, mutually perpendicular coordinate planes. This oblique traction is defined across a surface that is not precisely collocated with the intersection of the coordinate planes, but since we have assumed that tractions don't vary much with position, this is not a problem.

We have derived equation (2.2) with the restrictions that no long range forces are acting and that the tetrahedron is in static equilibrium. If the tetrahedron is allowed to be very small, it turns out that even these restrictions can be lifted. This can be seen by estimating the relative importance of various terms in the full expression of Newton's second law applied to the tetrahedron:

$$\sum \mathbf{F}_{stress} + \mathbf{F}_{body} = m\mathbf{a}. \tag{2.3}$$

The first term is everything included in equation (2.1), and for fixed values of the tractions, *scales* as L^2, where L is a typical linear dimension of the tetrahedron, such as its diameter. What we mean here is that irrespective of the actual value of this term in the equation, if the tetrahedron is reduced in linear dimension by a factor of 2, the value of the term is reduced by a factor of $2^2 = 4$. This is because the stress term contains areas, which are typically the products of two lengths. If the diameter of the tetrahedron is reduced by a factor of two without changing its shape, then these lengths will also be reduced by this factor.

The acceleration term on the right side of equation (2.3) contains the mass m of the tetrahedron, which is the average mass density times the volume. Assuming that the mass density varies smoothly (if at all) through the material medium, we can see that this term scales with L^3, due to the presence of the volume. Thus, as L is allowed to become very small, the ratio of the stress to the acceleration term goes as *something*$/L$. Irrespective of what "something" is, this ratio will eventually become much larger than unity as L gets smaller. Therefore, for very small L, the acceleration term can be safely ignored relative to the stress term in this calculation.

A similar argument can be made about long range forces, symbolized here as $\mathbf{F}_{body}$. This is because such forces typically also scale with the mass of the body in question. Thus, for very small tetrahedrons, the previously imposed limitations are no longer applicable, and equation (2.2) holds even in the presence of long range forces and accelerations. A side effect of letting L become very small is that spatial variations in tractions are then allowed as long as the variation is reasonably smooth.

The above analysis is valid whether the tetrahedron is a real object or simply part of a larger material body set off by imaginary planes defining the tetrahedron's faces. In the former case, the tractions may be thought of as externally applied to the body by, say, some type of laboratory apparatus. In the latter case, the tractions represent internal forces in which one part of a material body acts on another. In this case it is profitable to think of the *state of stress* of the body as encompassing the values of the tractions on three mutually perpendicular surfaces and their variations from place to place in the body.

A Mathematical Diversion

This book will not present the formal theory of Cartesian tensors. Instead, it will try to show the physical motivation behind the mathematical concept, and give some notions as to how tensors are used in derivations and computations. In order to ease the way, we start with the notion of a dyadic product. This relates tensors back to the more familiar concept of vectors.

Examination of equation (2.2) tempts one to rewrite it in a more efficient manner by factoring out the common unit normal $\mathbf{n}$:

$$\mathbf{t} = (- \mathbf{t}_x\mathbf{i} - \mathbf{t}_y\mathbf{j} - \mathbf{t}_z\mathbf{k}) \cdot \mathbf{n}. \tag{2.4}$$

The resulting combinations such as $\mathbf{t}_x\mathbf{i}$ are called *dyadic products* of vectors. They are distinguished from dot and cross products by the absence of their respective operators, $(\cdot)$ and $(\times)$. The meaning of an arbitrary dyadic product $\mathbf{ab}$ of two vectors, $\mathbf{a}$ and $\mathbf{b}$, only emerges when the dot product is taken with another vector, $\mathbf{c}$:

$$(\mathbf{ab}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}); \quad \mathbf{c} \cdot (\mathbf{ab}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}. \tag{2.5}$$

In other words, the dyadic **ab** yields a number times the vector **a** when dotted on the right by another vector, and a number times the vector **b** when dotted on the left. Notice that the results of dotting from the left and the right are different. Moreover, **ab** is not the same as **ba** because the results of taking dot products from each side are different.

A dyadic is a special case of a *tensor*. Sums of dyadics are also tensors. The quantity in parentheses in equation (2.4) is called the *stress tensor*, and we denote it in this book as **T**. Thus, a shorthand way of representing equation (2.4) is

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n}, \tag{2.6}$$

which means, "if you dot the stress tensor **T** on the right with a unit vector **n**, you get the traction across the surface normal to **n**". Note that $\mathbf{t}_x$, $\mathbf{t}_y$, and $\mathbf{t}_z$ are respectively recovered by substituting $-\,\mathbf{i}$, $-\,\mathbf{j}$, and $-\,\mathbf{k}$ for **n** in equation (2.6).

Since **T** (or any other tensor) is a sum of dyadics, the most general **T** may be obtained by expanding all the tractions forming the individual dyadics into component form, e. g., $\mathbf{t}_x = t_{xx}\mathbf{i} + t_{yx}\mathbf{j} + t_{zx}\mathbf{k}$, where the first and second subscripts of each t respectively represent the Cartesian component of the traction vector and the surface across which the traction acts. Thus, t_{yz} is the y component of the traction across the z surface, i. e., that surface defined by the $x - y$ plane. Therefore,

$$\mathbf{T} = T_{xx}\mathbf{ii} + T_{xy}\mathbf{ij} \cdots, \tag{2.7}$$

where $T_{xx} = -\,t_{xx}$, etc. We infer that the most general tensor in three dimensions has three coordinate planes times three components each, or nine independent components.

Equation (2.7) has a structure reminiscent of the component representation of a vector, e. g., $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Just as the vector is the sum of the products of components a_x, a_y, and a_z with their respective unit vectors **i**, **j**, and **k**, the stress tensor is the sum of the products of the components T_{xx}, T_{xy}, $\cdots$, with the unit dyadics **ii**, **ij** $\cdots$.

There are two advantages in inventing the notion of a tensor and rewriting equation (2.2) as (2.6). First, factoring out **n** separates elements related to the definition of the surface across which the traction **t** is defined from those independent of this particular surface. The latter elements make up the stress tensor, which may be thought of as representing the state of stress of the material. Second, even though $\mathbf{T}$ is constructed from tractions defined across particular coordinate surfaces, it correctly suggests that tensors, like vectors, can be thought of as entities that have meaning independent of one's choice of coordinate system. Thus, $\mathbf{T}$ may be resolved into components in *any* coordinate system, and furthermore, the resulting components are the components of the corresponding tractions across the coordinate surfaces of that coordinate system. This is easily verified by applying equation (2.6) with $\mathbf{n}$ set respectively to the basis vectors of the new system.

Equations like (2.6) may be represented in two alternate forms of notation, namely component notation and matrices. Each type of notation has its value. Dyadic notation is compact and independent of coordinate system, component notation is somewhat more general, and matrix notation facilitates computations. Therefore, all forms, as well as ways of converting between them must be mastered.

In component notation, equations like (2.6) are expressed as sets of component equations expressed in compact form. Referring back to its original form, given by equation (2.2), we see that it can be represented on a component by component basis as the three equations

$$\begin{aligned} t_x &= T_{xx} n_x + T_{xy} n_y + T_{xz} n_z \\ t_y &= T_{yx} n_x + T_{yy} n_y + T_{yz} n_z \\ t_z &= T_{zx} n_x + T_{zy} n_y + T_{zz} n_z \end{aligned} \tag{2.8}$$

where t_{xx} has been replaced by $-T_{xx}$, etc. (Don't confuse t_x, t_y, and t_z, which are the components of $\mathbf{t}$, the traction across the oblique plane, with $\mathbf{t}_x$, etc., which are the traction vectors across the coordinate axis planes in figure 2.4.) Replacing x, y, and z in the subscripts by 1, 2, and 3, the above three equations can be represented as

$$t_i = \sum_{j=0}^{3} T_{ij} n_j, \quad i = 1, 2, 3. \tag{2.9}$$

Notice that the index j occurs twice on the right side of the above equation. This is a general characteristic of this kind of equation, and arises from the fact that operations involving sums are invariably dot products, which are the sums of the products of the components of two vector-like objects. On the other hand, the free index i only occurs once in each term. This gives us a way to distinguish whether a given index is summed, and therefore allows us to simplify the notation by omitting the summation sign:

$$t_i = T_{ij} n_j. \tag{2.10}$$

This is generally called the *Einstein convention*, and is only broken a few places in continuum mechanics. Such exceptions will be explicitly noted so as to avoid confusion.

It is important to remember that equations like (2.10) are *scalar* equations, so that $T_{ij} n_j = n_j T_{ij}$. This is unlike dyadic notation, where in general $\mathbf{T} \cdot \mathbf{n} \neq \mathbf{n} \cdot \mathbf{T}$. The latter dyadic expression would correspond to $n_j T_{ji} (= T_{ji} n_j)$ in component notation. The trick to converting rapidly between the two forms of notation is to order the variables in component notation so that like summed indices are adjacent. Thus, the product of two tensors in component notation, written as $S_{ij} T_{ki}$ could be rewritten as $T_{ki} S_{ij}$, since order doesn't matter in component expressions. It is then clear that this is equivalent to $\mathbf{T} \cdot \mathbf{S}$ in dyadic notation.

Expressions like $S_{ij} T_{ik}$ present a problem here, as no reordering will bring the two instances of the summed index i adjacent to each other. We solve this problem by introducing the notion of the *transpose* of a tensor:

$$T^t_{ij} \equiv T_{ji}. \tag{2.11}$$

The transpose involves nothing more than interchanging T_{xy} with T_{yx}, etc. Thus, the above troublesome expression can be rewritten as $S^t_{ji} T_{ik}$, which is equivalent to the dyadic $\mathbf{S}^t \cdot \mathbf{T}$.

With component notation, more complicated expressions than discussed above can easily be handled. For instance, one might imagine something like $A_{ijk} = B_i C_{jk}$ or $R_{ijkl} = A_{ij} B_{kl} + C_{ik} D_{jl}$. Notice that there are no implied summations in either of these expressions. Quantities like A_{ijk} and R_{ijkl} with three or more indices are also called tensors, but are distinguished from each other by the notion of *order*, which is simply the number of indices. Thus, A_{ijk} is a third order tensor and R_{ijkl} is a fourth order tensor. The stress tensor T_{ij} is a second order tensor. By extension we can call vectors first order tensors and scalars zeroth order tensors. The dot product of two vectors a_i and b_i in component notation is simply $a_i b_i$. Similarly, the dyadic product is $a_i b_j$. Notice that in the first expression there are no free indices, as is to be expected of a scalar. The second has two, since it is a second order tensor.

The *trace* of a second order tensor is simply the scalar obtained by summing the diagonal components, i. e., $T_{ii} = T_{xx} + T_{yy} + T_{zz}$. In terms of dyadic notation, the trace operation corresponds to turning dyadic products into dot products, i. e., $\mathrm{Tr}(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{b}$. As an example, the trace of the stress tensor is related to the *pressure*: $p = -\,T_{ii}/3$. This corresponds to the common definition of pressure in a fluid at rest of the outward normal force per unit area exerted by the fluid on its surroundings. In this case no shear stresses exist and all three components of the normal stress are equal. The minus sign occurs because a positive pressure corresponds to a state of compressional stress. Though defined originally for fluids, the notion of pressure, as defined above, has uses in other areas of continuum mechanics as well.

The unit tensor of second order, $\mathbf{I}$, is equivalent to the Kronecker delta, δ_{ij}, when expressed in component notation. It takes on the value unity when $i = j$ and is zero otherwise. The Kronecker delta has the property that summation over any index simply replaces that index with the other index of the Kronecker delta in the expression, e. g., $T_{ij}\delta_{jk} = T_{ik}$. In dyadic notation, $\mathbf{I} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{I} = \mathbf{T}$.

Symmetry is an important notion for tensors. This refers to how a tensor is changed upon the interchange of two indices. For instance, if $T_{ij} = T_{ji}$, the tensor T_{ij} is said to be symmetric. If, on the other hand, $T_{ij} = -T_{ji}$, then T_{ij} is antisymmetric. If neither of these relations holds, then T_{ij} has no definite symmetry. The notion of symmetry obviously doesn't apply for tensors of order less than two. For higher order tensors, the two indices interchanged need to be specified. For instance, we might have $A_{ijk} = A_{jik} = -A_{ikj}$, which means that A_{ijk} is symmetric with respect to interchange of the first and second indices, but antisymmetric relative to interchange of the second and the third.

The most important third order tensor is the unit antisymmetric tensor of third order, ϵ_{ijk}. This tensor has the values $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$ and $\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$. All components with any two indices the same are zero. It is easy to verify that ϵ_{ijk} is antisymmetric under the interchange of any two indices. Notice also that ϵ_{ijk} doesn't change when the indices are cyclically permuted, i. e., $i \to j$, $j \to k$, and $k \to i$.

The main use for ϵ_{ijk} is to represent the cross product of two vectors in component notation:

$$c_i = \epsilon_{ijk} a_j b_k \tag{2.12}$$

is equivalent to $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. The identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{2.13}$$

is useful in the proof of a number of vector relations involving cross products.

The matrix form of equations like (2.6) may also be deduced from equation (2.8):

$$\begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}. \tag{2.14}$$

Second order tensors are equivalent to square matrices, while vectors are represented by either row or column matrices. The dot product of two vectors, $\mathbf{a} \cdot \mathbf{b}$, is represented by

$$\begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}, \tag{2.15}$$

while the dyadic product $\mathbf{ab}$ is

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \begin{pmatrix} b_x & b_y & b_z \end{pmatrix}. \tag{2.16}$$

Like dyadic notation, matrix notation is limited to representing tensors of second order or less. However, within this limitation, matrices provide an excellent way to organize numerical computations.

We end our mathematical diversion by showing how to obtain the components of a tensor in a new coordinate system that is rotated relative to the initial system. The easiest way to proceed is by returning to dyadic notation, with unit vectors in the old and new coordinate systems renamed $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$. Thus, a vector may be represented in terms of its components in either system as $\mathbf{a} = a_i\mathbf{e}_i = a_i'\mathbf{e}_i'$, where the Einstein summation convention has been employed. Dotting $\mathbf{a}$ by $\mathbf{e}_i'$ yields the ith component of $\mathbf{a}$ in the primed coordinate system. Applying this to the unprimed representation yields

$$\mathbf{e}_i' \cdot \mathbf{a} = a_i' = (\mathbf{e}_i' \cdot \mathbf{e}_j)a_j = q_{ij}a_j. \tag{2.17}$$

The quantity $q_{ij} = \mathbf{e}_i' \cdot \mathbf{e}_j$ is the matrix of direction cosines between unit vectors of the old and new coordinate systems, and is called the *transformation matrix*. Note that in spite of its representation as a square matrix, q_{ij} is not a tensor. A tensor is a physical quantity with different representations in different coordinate systems, whereas the transformation matrix is a tool for converting vector and tensor components between such systems.

An expression similar to equation (2.17) may be obtained for tensors by dotting the tensor from the left and the right with unit vectors of the new coordinate system:

$$\mathbf{e}_i' \cdot \mathbf{T} \cdot \mathbf{e}_j' = T_{ij}' = q_{ik}q_{jl}T_{kl}. \tag{2.18}$$

The generalization to tensors of arbitrary order is obvious, with one transformation matrix for each order. For instance, a fourth order tensor would transform like

$$R_{ijkl}' = q_{im}q_{jn}q_{ko}q_{lp}R_{mnop}. \tag{2.19}$$

In converting component notation to matrix form, one uses the same rule as in converting to dyadic form, namely reorder and transpose until like summed indices are adjacent. Thus, in matrix form, equation (2.18) becomes

$$[T'] = [q][T][q^t], \tag{2.20}$$

where the matrices are not fully written out, but symbolized by the quantities inside the square brackets. Higher order transformations like that in equation (2.19) can't be represented by matrix operations.

Equation (2.17) may be inverted to obtain the transformation matrix from the primed to the unprimed coordinate system. If q_{ij}^{-1} is the matrix inverse of q_{ij}, then

$$a_i = q_{ij}^{-1} a_j{}'. \tag{2.21}$$

However, by definition, $q_{ij}^{-1} = \mathbf{e}_i \cdot \mathbf{e}_j{}' = \mathbf{e}_j{}' \cdot \mathbf{e}_i = q_{ji} = q_{ij}^t$, i. e., the inverse of the transformation matrix is simply its transpose. This type of matrix is called an *orthogonal* matrix.

We note finally, that in constructing the q_{ij} matrix, a simple rule suffices:

$$[q_{ij}] = \begin{pmatrix} \textit{unit vector } 1 \\ \textit{unit vector } 2 \\ \textit{unit vector } 3 \end{pmatrix}, \tag{2.22}$$

where *unit vector* 1 is a row of the matrix consisting of the components in the old coordinate system of the first unit vector of the new coordinate system, etc. Multiplying $[q_{ij}]$ on the right by a column vector is thus equivalent to dotting this vector by each of the three unit vectors of the new coordinate system, the resulting numbers being the three entries of the new column vector. The new vector is thus the old vector resolved in the new system as expected.

The virtue of the above transformation rules is more in their existence than in their actual usage. The point is, given these rules it is possible to show that properly constituted component expressions have the same form in all coordinate systems. For instance, if we have $a_i{}' = B_{ij}{}' c_j{}'$ in the primed reference frame, then this can be written $q_{ik} a_k = q_{ik} B_{kl} q_{jl} q_{jm} c_m$. From equation (2.22) and the fact that the coordinate axis unit vectors are mutually orthogonal, it is easy to show that $q_{jl} q_{jm} = \delta_{jm}$, reducing the right side of the above equation to $q_{ik} B_{kl} c_l$. Finally, multiplication of both sides by q_{ij} and summation over i results in $a_j = B_{jl} c_l$, which shows that (aside from the names of the indices) the equations look the same in both coordinate systems. Thus, if a relationship involving the components of vectors and tensors which is known to be valid in a particular coordinate system can be cast in proper component notation, this component form of the relationship is the same in all coordinate systems.

The gradient, divergence, and curl operations are easily expressed in component notation. For instance, the gradient operation $\mathbf{F} = \nabla\phi$ is expressed

$$F_i = \frac{\partial \phi}{\partial x_i} . \tag{2.23}$$

The rules of free and dummy indices are simply carried over from tensor algebra. Thus, i is a free index in the above expression. The divergence $D = \nabla \cdot \mathbf{F}$ is written

$$D = \frac{\partial F_i}{\partial x_i} , \tag{2.24}$$

while the curl $\mathbf{A} = \nabla \times \mathbf{B}$ is written using ϵ_{ijk} as in the cross product:

$$A_i = \epsilon_{ijk} \frac{\partial B_k}{\partial x_j} . \tag{2.25}$$

The tensor expression

$$A_j = \frac{\partial T_{ij}}{\partial x_i} \tag{2.26}$$

is sometimes written in dyadic notation as $\mathbf{A} = \nabla \cdot \mathbf{T}$, i. e., the divergence of the tensor $\mathbf{T}$. Note that the variation $\partial T_{ij}/\partial x_j$ is difficult to write using dyadic notation, since summation over the second index of T_{ij} implies dotting from the right with the gradient operator, which would then imply that differentiation is applied not to $\mathbf{T}$, but to what follows. This shows the limitations of dyadic notation in more complex expressions.

Symmetry of the Stress Tensor

It turns out that the stress tensor is symmetric. This may be proven by examining the torque imposed on a cube of material by the tractions on its six surfaces. We imagine a small cube of material with edge length l, centered at the origin, as shown in figure 2.6. If the cube is sufficiently small, variations in the stress tensor over the dimensions of the cube will also be small, and we can approximate the tractions on each face of the cube by the appropriate components of the stress tensor evaluated at the center of each face. Thus, the z component of the torque about the center of the cube is

$$\tau_z = 2T_{yx}(l/2)(l^2) - 2T_{xy}(l/2)(l^2) = (T_{yx} - T_{xy})l^3, \tag{2.27}$$

where the torque is computed as the force normal to the moment arm for each

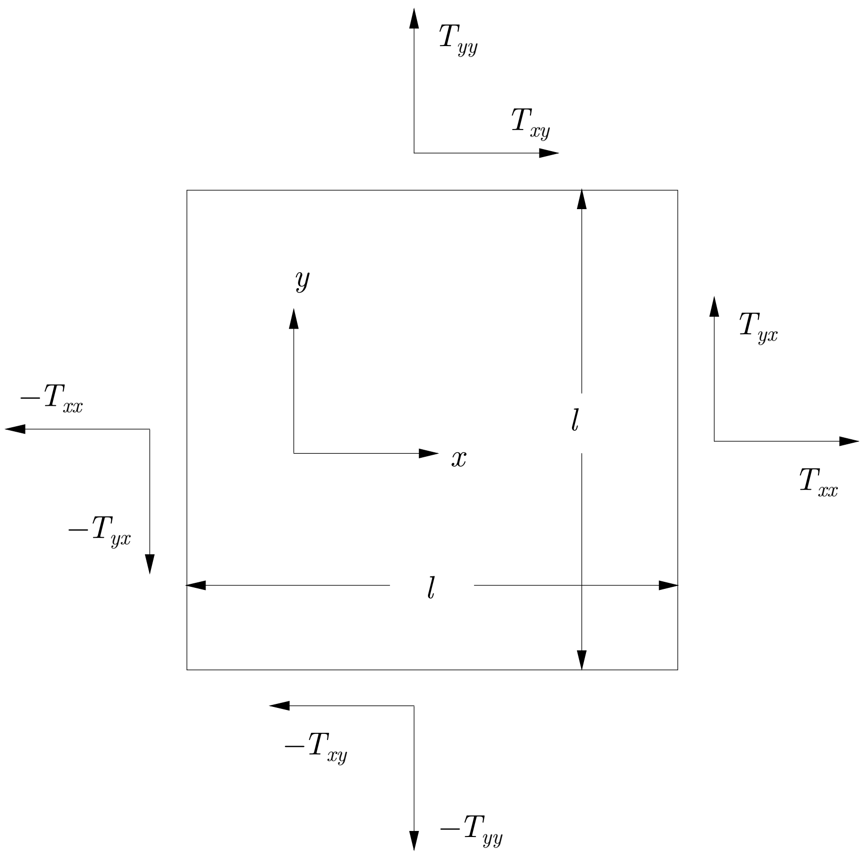

Figure 2.6. Illustration of stress components on faces of a cube of material. In order for angular momentum to be conserved, we must have $T_{xy} = T_{yx}$.

face (e. g., $T_{yx}l^2$) times the moment arm ($l/2$), summed over the appropriate faces. It is clear that if body forces are absent and if the cube is static, the torque must be zero, and $T_{yx} = T_{xy}$. Similar arguments show that $T_{xz} = T_{zx}$ and $T_{zy} = T_{yz}$, proving that the stress tensor is indeed symmetric under these conditions.

When body forces or angular accelerations are present, scaling arguments similar to those invoked in deriving equation (2.2) can be used. The equation relating torque τ and angular momentum $\mathbf{L}$ may be written

$$\tau_{stress} + \tau_{body} = \frac{d\mathbf{L}}{dt}\,. \tag{2.28}$$

From equation (2.27), the first term scales as l^3. Body torques depend on the body force varying over the dimensions of the cube. The difference between the body force per unit volume on one side of the cube and the other should scale as l. Combining this with the moment arm (scales with l) and the computation of body force from body force density (scales with l^3) shows that the body torque scales as l^5. The angular momentum contains the moment of inertia, which scales as a volume (l^3) times the square of a radius of gyration (l^2), and therefore goes as l^5 as well. As l becomes very small, both of these terms become unimportant relative to the torque due to stress, so it is clear that equation (2.28) reduces to $\tau_{stress} = 0$ in the limit of very small l. The above arguments therefore hold, and the stress tensor is symmetric in all circumstances.

It turns out that for any symmetric, second order tensor, there is a coordinate system in which the tensor is diagonal, i. e., all off-diagonal terms in the matrix representation are zero. Physically, what this means for the stress tensor is that all tractions across coordinate plane surfaces are normal, or perpendicular to the surface in this coordinate system. This is easily shown from equation (2.6).

Let us see if we can take advantage of this idea to determine the so-called *principal axis coordinate system.* If we impose the condition that the traction is parallel to the unit normal to a surface, i. e., $\mathbf{t} = \lambda\mathbf{n}$, then the unit normal is a candidate for defining a coordinate axis in the principal axis system. Combining this with equation (2.6) yields

$$\lambda\mathbf{n} = \mathbf{T}\cdot\mathbf{n}, \tag{2.29}$$

or recalling that $\mathbf{I}\cdot\mathbf{n} = \mathbf{n}$,

$$(\mathbf{T} - \lambda\mathbf{I})\cdot\mathbf{n} = 0. \tag{2.30}$$

Writing this in matrix form

$$\begin{pmatrix} T_{xx}-\lambda & T_{xy} & T_{xz} \\ T_{yx} & T_{yy}-\lambda & T_{yz} \\ T_{zx} & T_{zy} & T_{zz}-\lambda \end{pmatrix}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = 0 \tag{2.31}$$

shows that the problem of determining $\mathbf{n}$ reduces to the solution of a set of homogeneous linear equations. As long as the determinant of the square matrix in equation (2.31) is not zero, the only solution is the uninteresting one, $\mathbf{n} = 0$.

However, setting this determinant to zero results in a cubic equation for λ:

$$\lambda^3 + I_1\lambda^2 + I_2\lambda + I_3 = 0, \tag{2.32}$$

where I_1, I_2, and I_3 are combinations of the components of $\mathbf{T}$. This has three solutions, $\lambda^{(1)}$, $\lambda^{(2)}$, and $\lambda^{(3)}$. In general, solutions to a polynomial equation can be either real or complex, but it can be shown that they are all real solutions as long as T_{ij} is symmetric.

Once the three values of λ, called *principal values* or *eigenvalues*, are known, it is possible to solve equation (2.31) for the components of $\mathbf{n}$. A unique solution does not exist because the three equations are no longer linearly independent, but it is usually possible to determine, say, the ratios n_x/n_z and n_y/n_z. Since by definition $\mathbf{n}$ is a unit vector, $\mathbf{n}$ is determined up to an arbitrary sign by these ratios. (If $n_z = 0$, simply use one of the other components in the denominator.)

Imagine now that the *eigenvectors* $\mathbf{n}^{(1)}$, $\mathbf{n}^{(2)}$, and $\mathbf{n}^{(3)}$ have been determined for each eigenvalue. Can these be taken as the unit vectors of a new coordinate system? Only if they are mutually perpendicular! However, this is easily shown as long as T_{ij} is symmetric. The symmetry of the stress tensor insures that

$$\mathbf{n}^{(i)} \cdot \mathbf{T} \cdot \mathbf{n}^{(j)} = \mathbf{n}^{(j)} \cdot \mathbf{T} \cdot \mathbf{n}^{(i)} \tag{2.33}$$

is true for any two eigenvectors $\mathbf{n}^{(i)}$ and $\mathbf{n}^{(j)}$. Using equation (2.29), this may be written

$$(\lambda^{(i)} - \lambda^{(j)})(\mathbf{n}^{(i)} \cdot \mathbf{n}^{(j)}) = 0, \tag{2.34}$$

which shows that any two eigenvectors are mutually perpendicular as long as the corresponding eigenvalues are not equal. Taking this as given for a moment, we see that the eigenvectors do indeed define the axes of a new coordinate system, generally called the *principal axes*. From equation (2.29) it is clear that the eigenvalues are also the diagonal components of the tensor in the principal axis reference frame.

Once the eigenvectors are calculated, it is easy to obtain the transformation matrix from the original reference frame to the principal axis frame. From equation (2.22), we see that the rows of this matrix are simply the components of each eigenvector.

When $\lambda^{(i)} = \lambda^{(j)}$, the eigenvalues are said to be *degenerate*. In this important special case it turns out that *all* vectors in the plane defined by $\mathbf{n}^{(i)}$ and $\mathbf{n}^{(j)}$ are eigenvectors. This is easily shown by substituting $\alpha\mathbf{n}^{(i)} + \beta\mathbf{n}^{(j)}$ for $\mathbf{n}$ in equation (2.29), where α and β are arbitrary constants, since any $\mathbf{n}$ that satisfies this equation is by definition an eigenvector. Since all vectors in the plane are eigenvectors, it is easy to pick out two mutually perpendicular vectors to define principal axes. The choice of principal axes is, of course, not unique as it is in the non-degenerate case.

In the doubly degenerate case in which all three eigenvalues are equal, any vector at all is an eigenvector, and any Cartesian coordinate system is a principal axis system. In this case, the tensor is diagonal in all coordinate systems, with all diagonal components equal, and can be represented as an eigenvalue times the unit tensor, $\mathbf{T} = \lambda\mathbf{I}$.

Two-Dimensional Case

In order to more firmly fix some of the above concepts in our mind, we now explore a number of examples and special cases involving the stress tensor in two dimensions. In this case we need to think of tractions as stress forces per unit length of a line rather than per unit area of a surface.

Let us first look at a two dimensional example in which the stress tensor is diagonal in the original coordinate system:

$$[T] = \begin{pmatrix} T_{xx} & 0 \\ 0 & T_{yy} \end{pmatrix}. \tag{2.35}$$

In this case the tractions on the edges of a square (the two-dimensional analog of a cube) are normal. Figure 2.7a illustrates these tractions for the special case in which $T_{yy} = -T_{xx}$, in which case the tractions are trying to pull the x faces apart while they are trying to push the y faces together. Figure 2.7b shows an alternate case in which $T_{yy} = T_{xx}$.

The matrix corresponding to the transformation to a coordinate system rotated an angle α counter-clockwise relative to the original frame is

$$[q] = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{2.36}$$

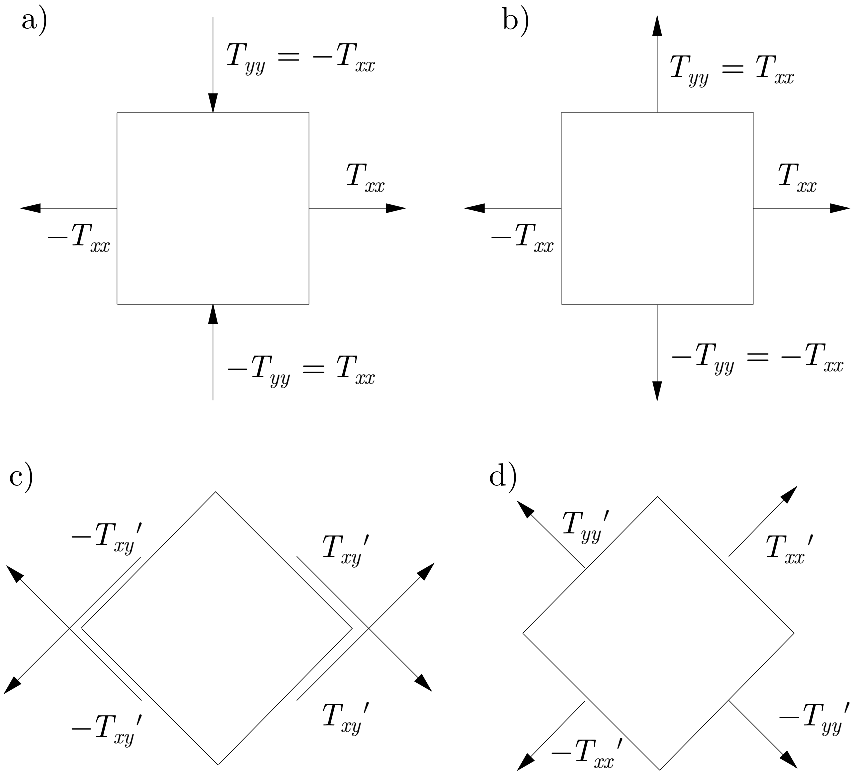

Figure 2.7. Tractions on edges of a square in two different cases, a) $T_{yy} = -T_{xx}$, and b) $T_{yy} = T_{xx}$, with $T_{xy} = T_{yx} = 0$. When the square is rotated by 45°, c) and d) show the transformed tractions.

Computing $T_{ij}' = q_{ik} T_{kl} q^t_{lj}$ yields

$$T_{xx}' = T_{xx} \cos^2 \alpha + T_{yy} \sin^2 \alpha$$

$$T_{yx}' = T_{xy}' = (T_{yy} - T_{xx}) \cos \alpha \sin \alpha \tag{2.37}$$

$$T_{yy}' = T_{xx} \sin^2 \alpha + T_{yy} \cos^2 \alpha \ .$$

Note that maxima occur in $|T_{xy}'|$ for $\alpha = 45°, 135°, 225°, \cdots$, except when $T_{yy} = T_{xx}$. In this case no off-diagonal term occurs for any α, which is to be expected, since this is a completely degenerate case. Note also that when $T_{yy} = -T_{xx}$, the normal stress components T_{xx}' and T_{yy}' disappear when $\alpha = 45° \cdots$. Thus, for a square rotated 45° to the original reference frame in this case, the tractions on the edges of the square are purely tangential, i. e., they are shear tractions. Note, however, that it is incorrect to say that the stress itself is "purely shear" or "purely normal" -- this terminology is only correct for tractions across a particular surface. As figure 2.7 shows, the same stress tensor can generate shear tractions across surfaces with certain orientations and purely normal tractions across others.

The primed stress components in equation (2.37) yield the tractions across surfaces aligned with the coordinate axes of the primed reference frame. For instance, the traction across the x' surface is given by $({T_{xx}}', {T_{xy}}')$. An alternate way to derive these tractions is to apply equation (2.6) to the tensor in the original reference frame. For instance, to get the traction in the above example, take $\mathbf{n} = \mathbf{i}' = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$. We get for the x and y components of the traction $(T_{xx}\cos\alpha\ ,\ T_{yy}\sin\alpha\ )$.

Inspection shows an apparent discrepancy -- the components of the traction vector derived in the two different ways don't agree. However, the discrepancy is resolved when we realize that the first set of vector components are relative to the *primed* frame, whereas the second are relative to the *unprimed* frame. Transformation of the second set to the primed frame yields equivalent results.

Boundary Conditions

So far we have only considered the traction across a surface inside a continuous medium. If the surface is placed so that it is coincident with the boundary of the medium with the unit normal pointing outward, equation (2.6) has a slightly different interpretation. Since the normal points outward, our convention states that the traction is the force per unit area applied by the external world to the surface of the medium. In this case equation (2.6) becomes

$$\mathbf{t}_{applied} = \mathbf{T}\cdot\mathbf{n}\big|_{surface}\ . \tag{2.38}$$

In other words, the stress tensor at the surface is constrained by the value of the applied stress there. Since equation (2.38) represents three conditions, three of the six independent components of stress are fixed at the surface. In the case of a free surface, the applied traction and the corresponding tensor components are zero.

Problems

1. When $\mathbf{n} = \mathbf{i}$ in equation (2.4), the result is $\mathbf{t} = -\,\mathbf{t}_x$. Determine the region on which the traction $\mathbf{t}$ is acting, and explain the origin of the minus sign in the above equation.

2. Write the pair of equations

$$b_1 a_{11} + b_2 a_{21} = 0$$

$$b_1 a_{12} + b_2 a_{22} = 0$$

in matrix form and in component notation.

3. Write

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = 0$$

in component notation and as individual equations.

4. Convert the following component notation expressions to matrix form. (Assume two dimensions.) a) $A_{ij}B_j$. b) $A_{ij}B_i$. c) $B_i A_{ij}$. d) $A_{ij}B_{jk}$. e) $q_{ik}q_{jl}T_{kl}$.

5. Write the matrix corresponding to the following dyadic: $3\mathbf{ii} + 2\mathbf{ij} - 4\mathbf{ji} + \mathbf{kk}$.

6. Referring to figure 2.8, the traction of the $-x$ domain on the $+x$ domain is $5\mathbf{i} + 3\mathbf{j}$, and the traction of the $-y$ domain on the $+y$ domain is $3\mathbf{i} - 2\mathbf{j}$. Find the traction of the unhatched domain on the hatched domain. Be careful of signs.

7. Show that a second order tensor may be split into a symmetric and an anti-symmetric part. For the three dimensional case, indicate how many independent components there are in each part.

8. Consider the unprimed and primed coordinate systems shown in figure 2.9. a) Find the transformation matrix, $q_{ij} = \mathbf{e}_i{}' \cdot \mathbf{e}_j$. b) Using the results of part a, find the primed components of the vector $\mathbf{A}$, where the unprimed components are $A_x = 1$ and $A_y = -1$. c) Using the results of a, find the primed components of the tensor $\mathbf{T}$, whose unprimed components are

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

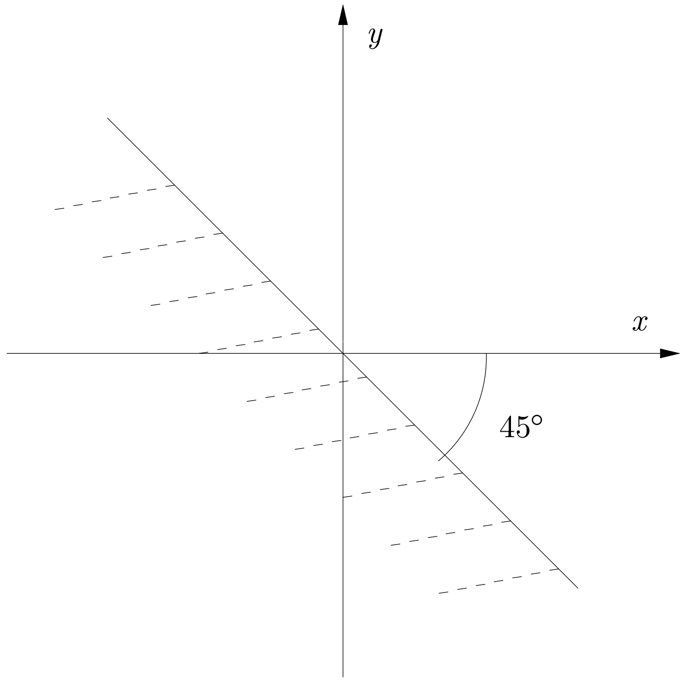

Figure 2.8. See problem 6.

9. Show that $C_i = \epsilon_{ijk} A_j B_k$ represents the cross product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

10. Show that $\epsilon_{ijk} a_i b_j c_k$ equals the determinant

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

11. Reduce the expression $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$ to a simpler form using equation (2.13).

12. In a particular reference frame a tensor has the components

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

a) Find the eigenvalues of this tensor, and write the matrix representing the tensor in the principal axis reference frame. b) Find the eigenvectors. c) Find the transformation matrix from the original reference frame to the new reference frame using the results of b. d) Using the results of c, transform the components

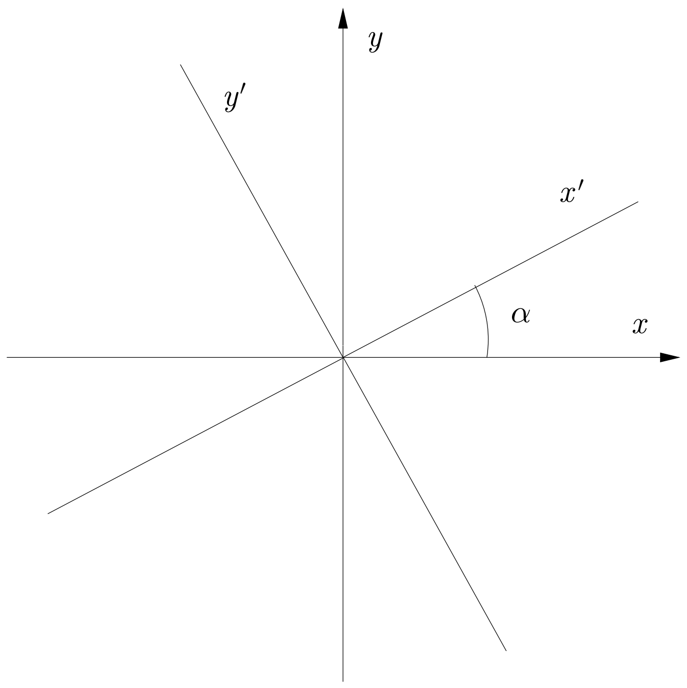

Figure 2.9. See problem 8.

of the tensor from the original reference frame to the principal axis frame, and verify that the results agree with the results of a.

13. Repeat the above problem for the tensor

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

14. Sketch plots of $T_{xx}{}'$, $T_{xy}{}'$, and $T_{yy}{}'$ as a function of α from equation (2.37) for the case when $T_{xx} = 1$ and $T_{yy} = -1$. Note particularly where the maxima and minima occur.

15. A continuous medium is confined to $z < 0$ and the stress in the medium is given by the stress tensor

$$T_{ij} = \begin{pmatrix} \alpha x^2 & \beta xy & \delta z^2 \\ \beta xy & 0 & 0 \\ \delta z^2 & 0 & -\gamma y^2 \end{pmatrix},$$

where α, β, γ, and δ are constants. Find the distribution in the $x - y$ plane of tractions applied to the medium at $z = 0$.

Chapter 3 -- Budgets, Fluxes, and the Equations of Motion

In this chapter we consider how the stress in a medium is related to the motion of the medium. In other words, we develop the continuum mechanics analog to Newton's second law. We do this initially by considering the imbalance between stress forces on the opposing faces of a cube due to the variation of stress with position. It becomes clear from this analysis that net stress forces on parcels of material are only non-zero if the stress varies from place to place. We then introduce the notion of a *budget* of a quantity, starting with the mass budget. The conservation of mass makes this idea particularly easy to understand. The budget for momentum is then discussed, and the budget notion is used to solve some classic problems in mechanics that involve open systems. The momentum budget is then invoked to develop the application of Newton's second law to continuum mechanics in a different way. Finally, we show how this law changes in an accelerated coordinate system and illustrate this with the examples of a rotating frame and the Lagrangian reference frame, in which the coordinate system deforms with the material medium.

Equation of Motion

In the previous chapter we showed that for a small parcel, the stress forces on individual faces of the parcel tend to dominate the contributions from body forces and acceleration in Newton's second law. This fact was used in deriving the notion of a stress tensor. Under these circumstances, it makes sense that if the stress tensor varies smoothly through the material medium, the *imbalance* in the total stress force due to this variation would be of the same order as the acceleration and body forces. Remember that there would be *no* imbalance if the stress tensor were constant.

This notion is verified by recalling that the stress force on individual faces of a parcel goes as l^2, where l is a typical dimension of the parcel, whereas body forces and acceleration go as l^3. The *difference* between tractions across two surfaces of the same orientation, but separated by a distance l should go as l. Multiplying by the surface area of the parcel, which scales as l^2, results in a stress force imbalance that scales with l^3, which is the same as for body forces and acceleration.

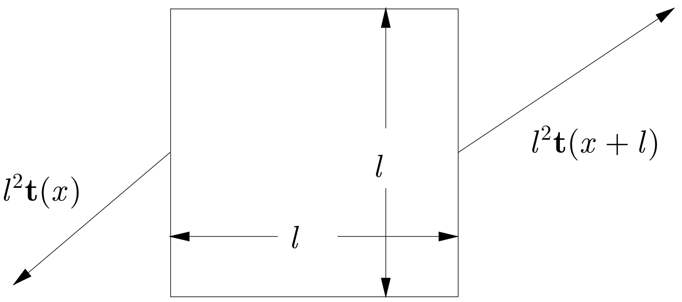

Figure 3.1. Definition sketch for computing the force imbalance on a cube. On the left side of the cube, the force is $l^2\mathbf{t}(x) = -l^2[T_{xx}(x)\mathbf{i} + T_{xy}(x)\mathbf{j} + T_{xz}(x)\mathbf{k}]$, while on the right side, it is $l^2\mathbf{t}(x+l) = l^2[T_{xx}(x+l)\mathbf{i} + T_{xy}(x+l)\mathbf{j} + T_{xz}(x+l)\mathbf{k}]$.

Figure 3.1 shows how to compute the imbalance in the stress forces on a cube of side l. For simplicity, only tractions on the positive and negative x surfaces are shown. The traction on the positive x face is actually $\mathbf{T}\cdot\mathbf{i}$. However, the symmetry of the stress tensor allows this to be written $\mathbf{i}\cdot\mathbf{T} = T_{xx}\mathbf{i} + T_{xy}\mathbf{j} + T_{xz}\mathbf{k}$. Combining this with the traction on the negative face gives the net force on these surfaces:

$$\mathbf{F}_{net\ x} = l^2[(T_{xx}(x+l) - T_{xx}(x))\mathbf{i} + (T_{xy}(x+l) - T_{xy}(x))\mathbf{j} +$$
$$(T_{xz}(x+l) - T_{xz}(x))\mathbf{k}]. \tag{3.1}$$

Multiplying and dividing by l and recognizing that the differences divided by l approximate x derivatives leads to

$$\mathbf{F}_{net\ x} \simeq l^3\left(\frac{\partial T_{xx}}{\partial x}\mathbf{i} + \frac{\partial T_{xy}}{\partial x}\mathbf{j} + \frac{\partial T_{xz}}{\partial x}\mathbf{k}\right) =$$
$$l^3\mathbf{i}\frac{\partial}{\partial x}\cdot(T_{xx}\mathbf{ii} + T_{xy}\mathbf{ij} + T_{xz}\mathbf{ik}) = l^3\mathbf{i}\cdot\frac{\partial\mathbf{T}}{\partial x}, \tag{3.2}$$

where the last step recognizes that terms like $\mathbf{i}\cdot T_{yx}\mathbf{ji}$ are zero.

Similar terms can be derived for the stress forces on the y and z faces, resulting in the net stress force

$$\mathbf{F}_{stress} = l^3 \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z} \right) \cdot \mathbf{T} = l^3 \nabla \cdot \mathbf{T}. \tag{3.3}$$

The approximation in equation (3.2) becomes an equality when l becomes very small. Thus, the net stress force on a small parcel equals the divergence of the stress tensor times the volume of the parcel.

In order to complete our conversion of Newton's second law to continuum form, we note that the mass of the parcel $m = l^3\rho$ where ρ is the mass density. We also write the body force in terms of the body force per unit mass, $\mathbf{B}$, and the density: $\mathbf{F}_{body} = l^3\rho\mathbf{B}$. If $\mathbf{a}$ is the parcel acceleration, then Newton's second law, $m\mathbf{a} = \mathbf{F}_{stress} + \mathbf{F}_{body}$, becomes

$$\rho\mathbf{a} = \nabla \cdot \mathbf{T} + \rho\mathbf{B} \tag{3.4}$$

upon canceling the parcel volume, l^3.

Mass Budget

In continuum mechanics we often deal with open systems, in which mass, momentum, and energy flow freely in and out of the system. The machinery of ordinary particle mechanics is ill-equipped to deal with this kind of situation, and classical analyses of open systems like rockets and conveyor belts have an ad hoc flavor to them. One way of handling these flows is to think in terms of *budgets* for the respective quantities. In other words, we equate the net flow of a quantity into a system plus the net creation rate of the quantity to the time rate of change of the quantity within the system. The momentum budget is perhaps the most crucial to our studies, but we begin with the mass budget because it is simpler and has considerable importance in its own right.

Before considering the mass budget, we need to show how physical variables are represented in continuum mechanics. We first discuss the velocity of a material medium. Imagine that the material forming the continuum of interest is moving with velocity $\mathbf{v}(\mathbf{x}, t)$, where $\mathbf{x}$ is the position vector, and t is time. This functional form implies that the material velocity varies from place to place, and at a given place can vary with time. Furthermore, this type of description loses track of individual parcels of material -- $\mathbf{v}(\mathbf{x}_0, t)$ is the velocity of parcels passing through the point $\mathbf{x} = \mathbf{x}_0$ as a function of time -- different parcels are located at this point at different times. This is called the *Eulerian description* in continuum mechanics.

Note that the trajectory of any given parcel may be recovered by integrating the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t), \tag{3.5}$$

with $\mathbf{x}$ set to the initial position of the parcel at the initial time. Though simple in principle, this is often difficult to do analytically in practice, because both the dependent and independent variables appear on the right side of the equation.

Streamlines are imaginary lines in space that are everywhere tangent to the velocity field. They are sometimes useful for visualizing the flow. When a flow is steady (i. e., $\mathbf{v}$ is not a function of time), parcel trajectories coincide with streamlines. However, when the flow depends on time, this is not necessarily true. The difference between trajectories and streamlines in this case can be quite striking, as shown by problem 3 at the end of this chapter.

Given the Eulerian way of representing variables, we now proceed with our exposition of the mass budget. Since mass is neither created or destroyed in everyday phenomena, we are left with a balance between inflow and increase with time. The trick is to be able to compute the net flow of mass through a surface. To do this we need to understand the *flux* of a quantity. Representing the mass density, $\rho(\mathbf{x}, t)$, in Eulerian form, we define the *mass flux* as $\rho\mathbf{v}$. The meaning of this quantity becomes clear when we dot it with $\mathbf{n}\delta A\delta t$, where δA is the area of a surface element with unit normal $\mathbf{n}$, and δt is a short time interval. As illustrated in figure 3.2, a volume $\mathbf{v} \cdot \mathbf{n}\delta A\delta t$ flows through the surface element in δt. Thus, $\rho\mathbf{v} \cdot \mathbf{n}$ is the mass per unit area per unit time flowing through the surface normal to $\mathbf{n}$. If $\mathbf{v}$ is normal to the surface, then $\mathbf{v}$ and $\mathbf{n}$ are parallel, and $\rho\mathbf{v} \cdot \mathbf{n} = \rho|\mathbf{v}|$. Thus, the magnitude of the mass flux is the mass per unit area per unit time flowing in the direction of the velocity vector.

From the above analysis we compute the rate at which mass flows out of a surface $\partial\Gamma$ that encloses the volume Γ:

$$\textit{Rate of mass outflow} = \int_{\partial\Gamma} \rho\mathbf{v} \cdot \mathbf{n}dA = \int_{\Gamma} \nabla \cdot (\rho\mathbf{v})dV. \tag{3.6}$$

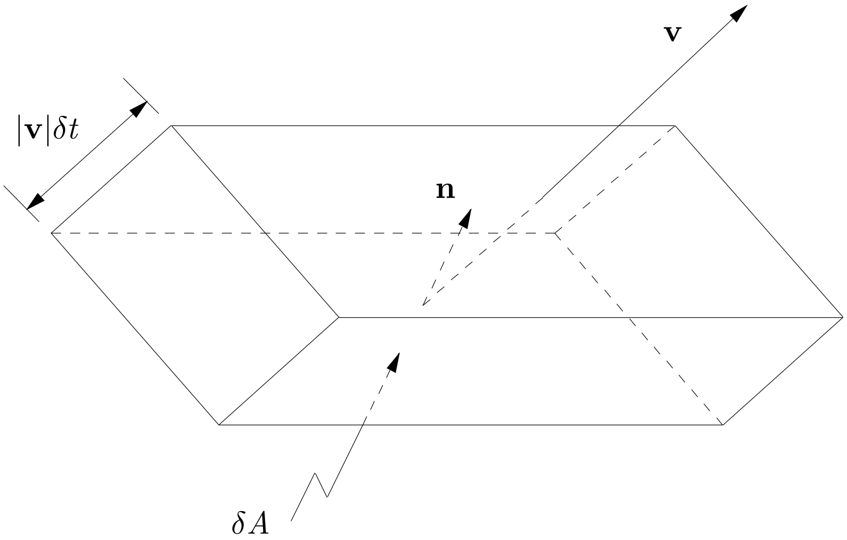

Figure 3.2. Sketch of flow through a surface element δA with unit normal $\mathbf{n}$. The volume of the parallelepiped, $\delta A\mathbf{n}\cdot\mathbf{v}\delta t$, equals the volume of fluid passing through the surface element in time δt. The fluid velocity is $\mathbf{v}$.

The conversion from a surface integral over $\partial\Gamma$ to a volume integral over Γ was performed using Gauss's law. The mass within volume Γ is simply the volume integral of density, so mass conservation can be expressed as

$$\frac{d}{dt}\int_\Gamma \rho dV = \int_\Gamma \frac{\partial\rho}{\partial t}dV = -\int_\Gamma \nabla\cdot(\rho\mathbf{v})dV, \tag{3.7}$$

where the minus sign occurs because the last integral represents outward flow. This balance is expressed in two examples in figure 3.3. In figure 3.3a there is net mass flow into a volume, and mass density must on the average be increasing within the volume. In figure 3.3b there is as much mass flowing out as in, and the average density remains constant.

Placing the two terms in equation (3.7) under a single integral yields

$$\int_\Gamma \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right)dV = 0, \tag{3.8}$$

from which we deduce that the integrand itself must be zero:

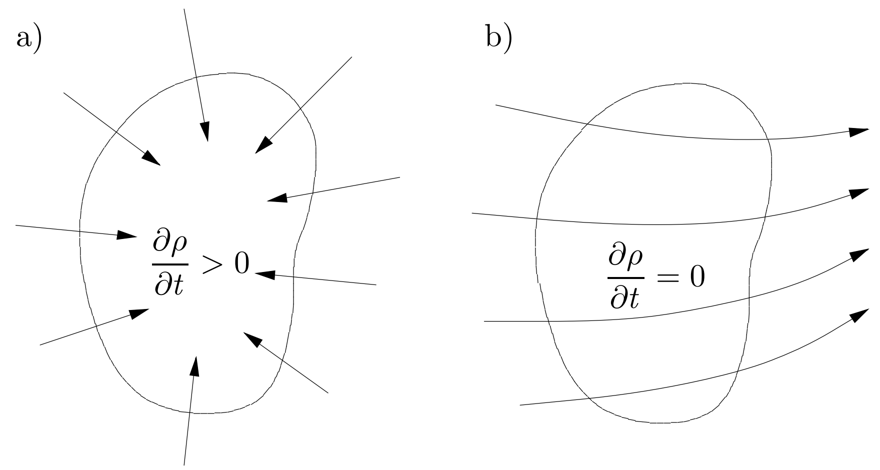

Figure 3.3. Illustration of a) divergent (actually, convergent) and b) nondivergent flow.

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.9}$$

This is because Γ is an arbitrary volume, and may be shrunk down to a tiny sphere over which the integrand doesn't vary much. In this case the integral reduces to the integrand times the volume of the sphere, which may be canceled, resulting in equation (3.9).

Equation (3.9) represents, in Eulerian form, the conservation of mass in the medium of interest. An alternative form may be derived by expanding the second term:

$$\frac{\partial\rho}{\partial t} + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = 0. \tag{3.10}$$

A commonly used shorthand for this representation is

$$\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{v} = 0, \tag{3.11}$$

where the total time derivative is to be interpreted as a derivative following a parcel in the material. The sense of this may be understood by expanding the total derivative using the chain rule:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} + \frac{dy}{dt}\frac{\partial\rho}{\partial y} + \frac{dz}{dt}\frac{\partial\rho}{\partial z} = \frac{\partial\rho}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla\rho. \tag{3.12}$$

The correspondence follows if $d\mathbf{x}/dt = \mathbf{v}$. This serves to emphasize that even though $\mathbf{v}$ is represented as a function of $\mathbf{x}$ and t, it is the velocity of the *parcel* that happens to be at $\mathbf{x}$ at time t.

When $d\rho/dt = 0$ for a material, the material is said to be incompressible. In this case equation (3.11) reduces to $\nabla \cdot \mathbf{v} = 0$. Note that an incompressible medium may have variations in density from parcel to parcel. Imagine, for instance, a river flowing into the ocean. Both the fresh water from the river and the salt water of the ocean are essentially incompressible, but they have different densities. The density field thus varies from place to place in this circumstance, but individual parcels retain their initial density, and the divergence of the velocity field is zero.

Momentum Budget

The budget for momentum is more complicated than that for mass for two reasons. First momentum is a vector, so the flux of momentum is a second order tensor. Second, external forces as well as the material transport of momentum enter the balance. Thus, a verbal statement of the momentum budget of some system is that the rate of change of momentum in the system equals the rate of inflow minus the rate of outflow via mass transport, plus the total external force on the system. Consideration of certain classical physics problems helps us to better understand the notion of a momentum budget. The problem of an accelerating rocket is particularly illuminating.

Figure 3.4 shows a rocket with mass M and speed v_r. Both of these quantities are changing with time. In particular, the mass of the rocket is decreasing at a rate $R = -\,dM/dt$ as a result of the expulsion of exhaust gas. The exhaust velocity of the gas is v_e relative to the rocket, so that in the rest frame the gas is leaving the rocket at speed $v_e - v_r$. The momentum per unit time leaving the dashed box surrounding the rocket is therefore $-\,R(v_e - v_r)$. We assume that all this is occurring in free space so that there are no external forces acting on the rocket. The momentum budget is thus a balance between the rate of increase of momentum in the box, which is mainly the momentum of the rocket, Mv_r, and the rate of flow of momentum out of the box:

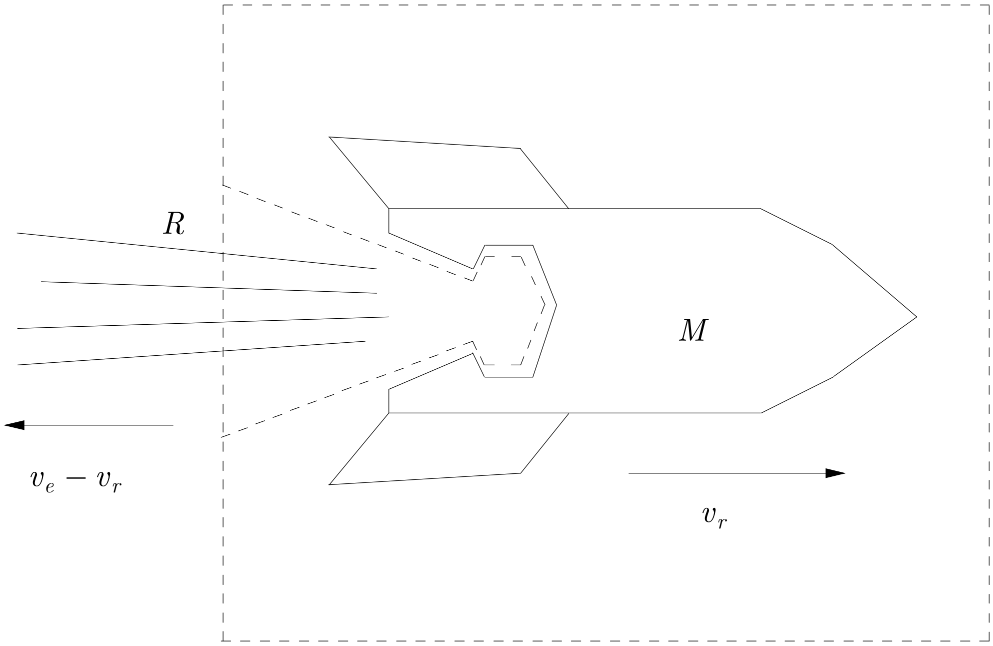

Figure 3.4. Sketch of a rocket moving to the right with speed v_r. The rocket has mass M, and is losing mass at the rate R via an exhaust with exhaust velocity v_e.

$$\frac{d(Mv_r)}{dt} = R(v_e - v_r). \tag{3.13}$$

Expansion of the left side by the product rule leads to the cancelation of the term $-Rv_r$ on both sides, leaving the classical formula

$$M\frac{dv_r}{dt} = Rv_e. \tag{3.14}$$

The right side of the above equation is normally interpreted as the thrust force of the rocket. However, in our interpretation the thrust is totally a consequence of the export of momentum in the exhaust stream -- no external "thrust" force is acting. This duality in the interpretation of the momentum budget in open systems persists in the continuum mechanics description. The applicability of each interpretation depends on exactly how the open system is defined -- i. e., where the dashed line is located in figure 3.4. If this line follows the inside of the rocket's combustion chamber rather than cutting straight through the exhaust stream, as illustrated in figure 3.4, the pressure force of the exhaust gas on the combustion chamber would constitute the external force that delivers the thrust. Furthermore, the fuel and oxidizer entering the chamber would do so with very little momentum in the reference frame of the rocket. Thus, even though this mass is exiting the system (by crossing the dotted line), it contributes little to the momentum budget, and the balance is primarily one between the pressure force on the combustion chamber and the rate of change of the rocket's momentum.

This example illustrates how important it is to carefully define what is inside and what is outside the system to which Newton's second law is to be applied. Minor changes in this definition change the way in which various physical effects are treated. Some ways of defining a system result in easier calculations than others. For instance, it is easier to compute the rocket's thrust in terms of the velocity of the gas passing through the dashed rectangle in figure 3.4 than it is to integrate the detailed pressure distribution of the exhaust gas over the complicated inner surface of the combustion chamber. However, any consistent way of defining a system should lead to a correct description of the phenomenon of interest, such as the rocket thrust in the above example. The danger is in deriving terms of the momentum budget based on an inconsistent view as to what is included in the system. For instance, including both the pressure force on the combustion chamber and the momentum flux of the exhaust gas after it has left the combustion chamber in the momentum budget could result from an inconsistent definition of the system boundary, and would be incorrect.

We now translate the momentum budget to a form applicable to continuous media. The equation for momentum may be developed in the same way as the equation for mass. The flux of mass is the mass density ρ times the fluid velocity $\mathbf{v}$. Similarly, the bulk flux of any quantity is simply its density times the velocity. The density of momentum is $\rho\mathbf{v}$, which coincidentally is also the mass flux. The momentum flux is therefore $\rho\mathbf{v}\mathbf{v}$. Note that this is a second order tensor, because we have taken the product between the momentum density to be the dyadic product. This is the second example of a general rule, which states that the flux of a tensor of order n is a tensor of order $n+1$.

The momentum per unit time being carried out of a volume Γ by the material flow is

$$\int_{\partial\Gamma} \rho\mathbf{v}\mathbf{v}\cdot\mathbf{n}dA = \int_{\Gamma} \nabla\cdot(\rho\mathbf{v}\mathbf{v})dV, \tag{3.15}$$

where Gauss's law is used as in equation (3.6). The time rate of change of momentum in volume Γ is therefore

$$\frac{d}{dt}\int_{\Gamma} \rho\mathbf{v}dV = \int_{\Gamma} \frac{\partial\rho\mathbf{v}}{\partial t}dV = -\int_{\Gamma} \nabla\cdot(\rho\mathbf{v}\mathbf{v})dV + \mathbf{F}, \tag{3.16}$$

where $\mathbf{F}$ is the sum of the stress and body forces on the volume. From equation (2.6), the stress force is

$$\mathbf{F}_{stress} = \int_{\partial\Gamma} \mathbf{T}\cdot\mathbf{n}dA = \int_{\Gamma} \nabla\cdot\mathbf{T}dV, \tag{3.17}$$

where Gauss's law has again been invoked. The body force is simply the volume integral of the body force per unit volume, $\rho\mathbf{B}$:

$$\mathbf{F}_{body} = \int_{\Gamma} \rho\mathbf{B}dV. \tag{3.18}$$

Combining equations (3.16)-(3.18) and applying the logic used in the previous section yields

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \nabla\cdot\mathbf{T} + \rho\mathbf{B}. \tag{3.19}$$

Comparison of this result with equation (3.4) suggests that the left side of equation (3.19) is just the density times the parcel acceleration. Product rule expansion of the left side and slight rearrangement yields

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right)\mathbf{v}. \tag{3.20}$$

Comparison with equation (3.9) shows that the last two terms vanish by virture of mass continuity. Furthermore the first two terms reduce to $\rho d\mathbf{v}/dt$, which is nothing more than the parcel acceleration. The equivalence of equations (3.4) and (3.19) is thus proved, and equation (3.19) can be written

$$\rho\frac{d\mathbf{v}}{dt} = \rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = \nabla\cdot\mathbf{T} + \rho\mathbf{B}. \tag{3.21}$$

The two forms of this equation are useful in different circumstances.

Accelerated and Non-Cartesian Coordinate Systems

Equation (3.21) is nothing more than an expression of Newton's second law, and as such is valid only in an inertial reference frame. Sometimes it is desirable to work in an accelerated reference frame, in which case it is necessary to modify this equation. Two instances of accelerated reference frames are commonly seen. Sometimes it is useful to view the motion of a continuum in a reference frame that is rotating at a uniform rate about a fixed axis. The flow of the atmosphere and the oceans on the rotating earth is one example. Another example occurs when the coordinate system itself is fixed to the material of the continuum, and thus moves, accelerates, and deforms with the material. This is commonly called the *Lagrangian* reference frame in contrast to the *Eulerian* frame, which remains fixed in space.

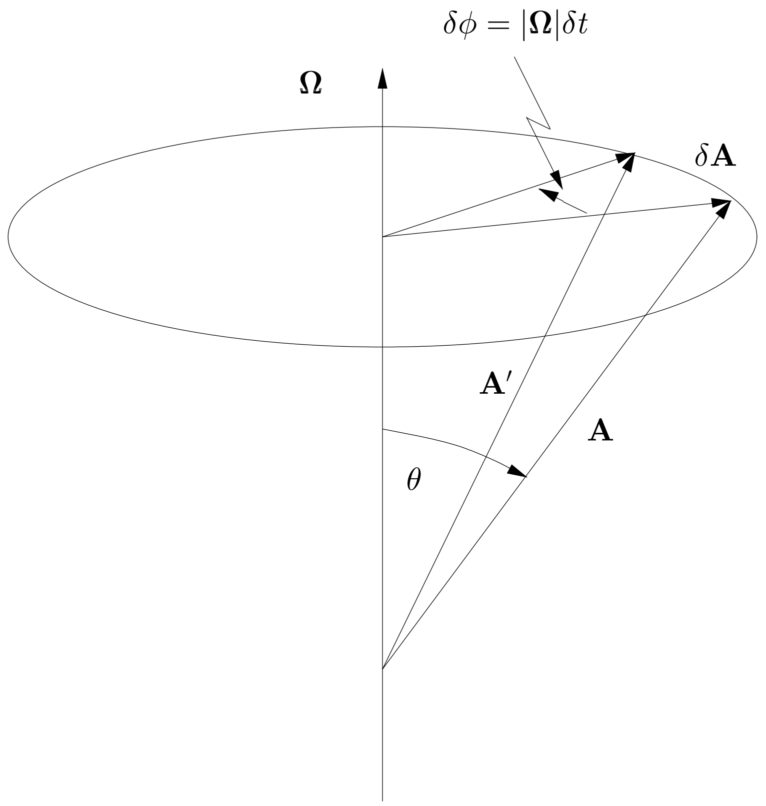

Figure 3.5. Definition sketch for relating the components of a vector in stationary and rotating reference frames.

Figure 3.5 shows how a vector $\mathbf{A}$, at rest in a reference frame which rotates with frequency $|\mathbf{\Omega}|$, moves relative to an external observer. The axis of rotation is defined by the vector $\mathbf{\Omega}$, which makes an angle θ with $\mathbf{A}$. In time δt the component of $\mathbf{A}$ normal to $\mathbf{\Omega}$ rotates through an angle $\delta\phi = \Omega\delta t = \delta A/(A\sin\theta)$. Therefore, $\delta A/\delta t = \Omega A \sin\theta = |\mathbf{\Omega}\times\mathbf{A}|$. Invocation of the right-hand rule shows that $\delta\mathbf{A}$ is in the direction of $\mathbf{\Omega}\times\mathbf{A}$, so the vector law

$$\frac{d\mathbf{A}}{dt} = \mathbf{\Omega}\times\mathbf{A} \tag{3.22}$$

holds. If $\mathbf{A}$ is changing in the *rotating* frame, the effect on $d\mathbf{A}/dt$ is additive, i. e.,

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_r + \mathbf{\Omega}\times\mathbf{A}. \tag{3.23}$$

We now apply equation (3.23) to the acceleration:

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d}{dt}\left[\left(\frac{d\mathbf{x}}{dt}\right)_r + \mathbf{\Omega}\times\mathbf{x}\right] =$$

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_r + 2\mathbf{\Omega}\times\left(\frac{d\mathbf{x}}{dt}\right)_r + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}). \tag{3.24}$$

If we redefine $\mathbf{v}$ as the velocity in the rotating frame, $\mathbf{v} \equiv (d\mathbf{x}/dt)_r$, then equation (3.21) becomes

$$\rho\frac{d\mathbf{v}}{dt} = \nabla\cdot\mathbf{T} + \rho\mathbf{B} - 2\rho\mathbf{\Omega}\times\mathbf{v} - \rho\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}). \tag{3.25}$$

The extra components of the acceleration that result from being in a rotating reference frame have been placed on the right side of equation (3.25) to show that they can be interpreted as body forces. The term $-\,2\mathbf{\Omega}\times\mathbf{v}$ is the *Coriolis force* per unit mass, while $-\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x})$ is the *centrifugal force* per unit mass. These two forces are often called *inertial forces* to distinguish them from such things as gravity and Coulomb attraction which are commonly thought to arise from fundamental physical processes rather than one's choice of reference frame. The Coriolis force in particular plays a fundamental role in atmospheric and oceanic circulations.

We now examine how equation (3.21) is modified when a Lagrangian reference frame is chosen. Imagine a transformation from Cartesian coordinates $\mathbf{x} = (x, y, z)$ to a new coordinate system $\mathbf{X} = (X, Y, Z)$ that deforms with the material. In general, $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$, where t is time. We may imagine that $\mathbf{X}$ is the location of each parcel at time $t = 0$, i. e., $\mathbf{X}(\mathbf{x}, 0) = \mathbf{x}$. Thus, parcels are labeled by their initial position. This vector relationship may in principle be inverted to obtain $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. The total time derivative of $\mathbf{x}$ may then be written in component notation as

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial X_j}\frac{dX_j}{dt}\,. \tag{3.26}$$

However, the second term on the right side of equation (3.26) vanishes because $dX_j/dt = 0$. This follows from the original definition of the (X, Y, Z) coordinate system; since it moves with the material medium, the medium cannot move relative to the coordinate system, and the parcel velocity is zero in this reference frame. A second application of this logic shows that the acceleration simply reduces to $\partial^2\mathbf{x}/\partial t^2$, i. e., the $\mathbf{v} \cdot \nabla\mathbf{v}$ term of equation (3.21) disappears.

Such simplification in one part of equation (3.21) is unfortunately accompanied by additional complications in another part. The complication arises because the stress, $\mathbf{T}$, will generally be defined as a function of $\mathbf{X}$ rather than $\mathbf{x}$. Thus, a change of variables needs to be performed in the term $\nabla \cdot \mathbf{T}$. Furthermore, it is incorrect to simply apply the chain rule to the component form of the divergence of the stress tensor, $\partial T_{ij}/\partial x_j$. This is because in the derivation of the component form from the more fundamental dyadic form, the spatial derivatives of unit vectors were ignored. This is justified in a Cartesian coordinate system in which unit vectors are constants. However, in general a Lagrangian coordinate system will not remain Cartesian, and the spatial variations in unit vectors must be considered.

The theory of tensors in arbitrary coordinate systems is beyond the scope of this book, and readers interested in this subject are referred to the book by McConnell (1957). Many problems using the Lagrangian approach are expressable in terms of coordinate systems that are locally orthogonal, i. e., the coordinate lines at each point are mutually perpendicular. Polar and spherical coordinates are well known examples of orthogonal coordinate systems. Problems in such systems can be approached with somewhat less theoretical work. Batchelor (1967) derives numerous useful relations for such coordinate systems.

In the Lagrangian representation, equation (3.11) is not a useful way to express mass conservation. An alternative method is to consider how the volume of a parcel changes with time. Suppose a parcel is initially a tiny cube of side l, with edges parallel to the coordinate axes. After some time it will in general be deformed into a parallelepiped with edges defined by the vectors $\Delta\mathbf{x}^{(1)}$, $\Delta\mathbf{x}^{(2)}$, and $\Delta\mathbf{x}^{(3)}$. The volume of this parallelepiped will be

$$\Delta V = \Delta\mathbf{x}^{(1)} \cdot \Delta\mathbf{x}^{(2)} \times \Delta\mathbf{x}^{(3)} = \epsilon_{ijk} \Delta x_i^{(1)} \Delta x_j^{(2)} \Delta x_k^{(3)}. \tag{3.27}$$

The edge vectors may be expressed in terms of the original edge vectors $\Delta X_i^{(I)}$ $(I = 1, 2, 3)$ by the transformation

$$\Delta x_i^{(I)} = \frac{\partial x_i}{\partial X_j} \Delta X_j^{(I)}. \tag{3.28}$$

(Recall that $\mathbf{X} = \mathbf{x}$ at time $t = 0$.) Since the edges of the cube are aligned with the coordinate axes at $t = 0$, we have $\Delta X_i^{(1)} = l\delta_{1i}$, etc., and the volume of the parcel at time t is

$$\Delta V = \Delta V_0 \epsilon_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3}, \tag{3.29}$$

where $\Delta V_0 = l^3$ is the initial volume of the parcel.

The terms involving ϵ_{ijk} and the partial derivatives form the determinant of $\partial x_i/\partial X_j$. This determinant is also known as the *Jacobian* of the transformation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. Writing equation (3.29) in terms of the density, $\rho = \Delta M/\Delta V$, and the initial density $\rho_0 = \Delta M/\Delta V_0$, where ΔM is the mass of the parcel, mass conservation in the Lagrangian frame becomes

$$\frac{\rho_0}{\rho} = \det\left(\frac{\partial x_i}{\partial X_j}\right). \tag{3.30}$$

Problems

1. If the tensor $\mathbf{T}$ is symmetric, show that $\mathbf{n}\cdot\mathbf{T}=\mathbf{T}\cdot\mathbf{n}$ for any $\mathbf{n}$.

2. Given the stress tensor

$$T_{ij}=\begin{pmatrix}\alpha x^2 & \beta xy & 0\\ \beta xy & 0 & 0\\ 0 & 0 & -\gamma y^2\end{pmatrix}$$

in a medium of density ρ, where α, β, and γ are constants, find the acceleration vector at each point in the medium. (Assume no body forces.)

3. Given a flow field $\mathbf{v}=(C\cos\omega t, C\sin\omega t, 0)$, where C and ω are constants, and t is time, find the trajectory of a parcel starting at $\mathbf{x}=(x_0, y_0, 0)$. Sketch the flow field at $t=0$ and at $t=\pi/2\omega$. Sketch the trajectory of a parcel starting at the origin at $t=0$.

4. The density of water near the mouth of a river varies in space and time as $\rho=A+B\tanh(\chi/d)$ where $\chi=x+d\cos(\omega t)$ due to periodic tidal effects. The down-river direction is given by positive x, and t is time. A, B, d, and ω are constants. Assuming that the flow is purely upstream and downstream, and that water is incompressible, determine the flow speed in the river as a function of x and t. Hint: Set $d\rho/dt=0$ and solve for v_x.

5. Solve the conveyor belt problem using momentum flux methods (see figure 3.6). In other words, a conveyor belt moves along at a speed v. Mass (say, coal or wheat) falls onto the conveyor belt at a rate R. Determine the force required to keep the conveyor belt moving at speed v.

6. Given the flow field $\mathbf{v}=(Cx/r^3, Cy/r^3, Cz/r^3)$, where $r^2=x^2+y^2+z^2$ and C is a constant, determine the acceleration of parcels at each point in space. Also, determine whether the flow is incompressible.

7. Derive an expression for the divergence of a vector, $\nabla\cdot\mathbf{V}$, in polar coordinates. Proceed by setting $\mathbf{V}=V_r\mathbf{e}_r+V_\theta\mathbf{e}_\theta$, where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are respectively the unit vectors in the r and θ directions and V_r and V_θ are the vector components in these directions. The gradient operator is

$$\nabla=\mathbf{e}_r\frac{\partial}{\partial r}+\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

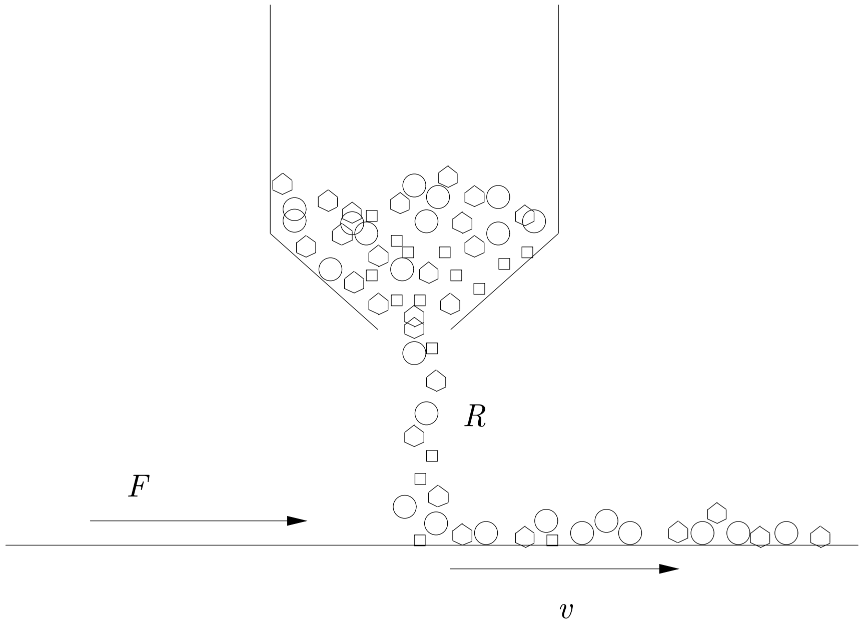

Figure 3.6. See problem 5.

in polar coordinates. Finally, use $\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ and $\mathbf{e}_\theta = -\,\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$ to compute the derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ with respect to r and θ.

8. Consider the Lagrangian representation of a uniformly expanding gas, with the parcel position $\mathbf{x}$ at time t given as $\mathbf{x} = (1 + t/\tau)\mathbf{X}$, where τ is a constant and $\mathbf{x} = \mathbf{X}$ at $t = 0$. Determine the parcel velocity as a function of $\mathbf{X}$ for all parcels. Solve for $\mathbf{X}$ in terms of $\mathbf{x}$, and combine with the above results to obtain the velocity field as a function of $\mathbf{x}$. Finally, if the density of the gas is uniformly ρ_0 at $t = 0$, find its density at later times.

References

Batchelor, G. K., 1967: *An introduction to fluid dynamics.* Cambridge University Press, 615 pp.

McConnell, A. J., 1957: *Applications of tensor analysis.* Dover, 318 pp.

Chapter 4 -- Kinematics in Continuum Mechanics

In this chapter we learn how to describe the motion of continuously distributed matter, independent of what is causing the motion. Central to this discussion is the *displacement vector*, $\mathbf{u}(\mathbf{X},t) \equiv \mathbf{x}(\mathbf{X},t) - \mathbf{X}$. Since $\mathbf{x}$ is the position of a parcel whose position at time $t = 0$ is $\mathbf{X}$, the displacement $\mathbf{u}$ is simply the movement of the parcel since the initial time.

Of particular interest is the *variation* in the displacement of neighboring parcels. If all parcels moved together, all displacements would be the same. This would correspond to a uniform translation of the entire body. The differences in displacement between neighboring parcels are related to more interesting things like rotation and deformation of the body.

In this chapter we first show how small displacements in the neighborhood of some point decompose into a combination of a translation plus a rotation plus a strain. We then show that the components of the so-called strain tensor are subject to certain conditions called the compatibility conditions. Finally, we consider the case of a continuously deforming fluid, in which the rate of displacement of parcels, or the velocity field, is of interest.

Small Displacements

Figure 4.1 illustrates the relationship between two parcels, A and B, as they move to their new positions, A' and B'. The initial positions of the parcels are given by the vectors $\mathbf{X}_A$ and $\mathbf{X}_B$, while their final positions are indicated by $\mathbf{x}_A$ and $\mathbf{x}_B$. The displacement vectors are $\mathbf{u}_A$ and $\mathbf{u}_B$. The position of parcel B relative to parcel A is $\delta\mathbf{X} = \mathbf{X}_B - \mathbf{X}_A$ at the initial time and $\delta\mathbf{x} = \mathbf{x}_B - \mathbf{x}_A$ at the later time. Simple substitution shows that $\delta\mathbf{x} = \delta\mathbf{X} + (\mathbf{u}_B - \mathbf{u}_A)$.

If parcel displacements vary smoothly over some small region, and if $|\mathbf{u}_B - \mathbf{u}_A| << |\delta\mathbf{X}|$ for all pairs of parcels in the region, then a first order Taylor series expansion about some reference parcel yields a good approximation to the variation in $\mathbf{u}$ over the region. Changing now to component notation,

$$u_i(X_j) \simeq u_i(X_{0j}) + \frac{\partial u_i}{\partial X_j}\,\delta X_j, \tag{4.1}$$

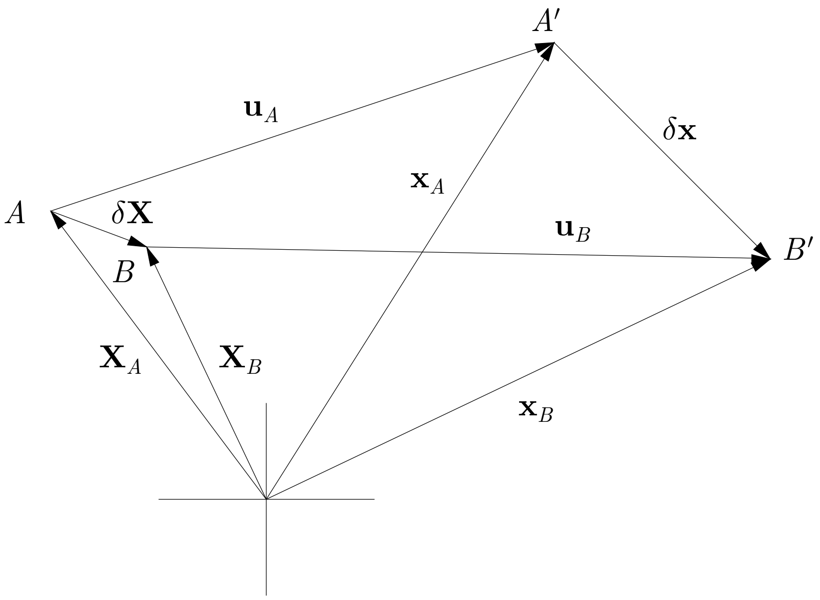

Figure 4.1. Relationship between the displacements of two parcels, A and B.

where now $\delta X_j = X_j - X_{0j}$ is the position of an arbitrary parcel relative to the reference parcel, X_{0j}. The condition $|\mathbf{u}_B - \mathbf{u}_A| << |\delta\mathbf{X}|$ is called the *small deformation approximation*, and is satisfied in many, but not all situations. In this book we will restrict our attention to this case. Spencer (1980) gives an extensive discussion of finite deformations.

The second order tensor $\partial u_i/\partial X_j$ is called the *deformation tensor*. It has no particular symmetry, but can be split into symmetric and antisymmetric parts,

$$\frac{\partial u_i}{\partial X_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \equiv E_{ij} + R_{ij}. \tag{4.2}$$

The symmetric part, E_{ij}, is called the *strain tensor*, while the antisymmetric part, R_{ij}, is called the *rotation tensor*. The displacement in a small neighborhood thus becomes

$$u_i = u_{0i} + E_{ij}\delta X_j + R_{ij}\delta X_j, \tag{4.3}$$

where u_{0i} is shorthand for $u_i(X_{0j})$, the displacement of the reference parcel.

The origin of the name "rotation tensor" is clear once the properties of antisymmetric tensors are understood. Antisymmetric tensors in three dimensions have a close relationship with vectors, since they have three independent components, just as vectors do. If A_{ij} is an arbitrary antisymmetric tensor, its components can be related to those of a vector, a_i, by $A_{ij} = -\epsilon_{ijk} a_k$. (Recall that ϵ_{ijk} is defined in the context of the cross product in chapter 2.) The vector a_i is called the *dual vector* of the antisymmetric tensor A_{ij}. Note that the relation between these two quantities can be inverted, since $\epsilon_{ijl} A_{ij} = -\epsilon_{ijl}\epsilon_{ijk} a_k = -2a_l$. The reduction of $\epsilon_{ijl}\epsilon_{ijk}$ can be computed using equation (2.13), plus the properties of the Kronecker delta.

Imagine now a displacement field with $u_{0i} = 0$ and $E_{ij} = 0$. If θ_i is the dual vector to R_{ij}, then the displacement field becomes

$$u_i = R_{ij}\delta X_j = -\epsilon_{ijk}\theta_k \delta X_j, \tag{4.4}$$

or in vector notation, $\mathbf{u} = \theta \times \delta\mathbf{X}$.

Reference to the section on accelerated coordinate systems in chapter 3, and particularly to figure 3.5, shows that displacements in this case take the form of a small, rigid body rotation. Note especially that equation (3.22) rewritten as $d\mathbf{A} = (\mathbf{\Omega} dt) \times \mathbf{A}$ is analogous to equation (4.4) if $d\mathbf{A} = \mathbf{u}$, $\mathbf{\Omega} dt = \theta$, and $\mathbf{A} = \delta\mathbf{X}$. Thus, the axis of rotation, which passes through the reference parcel, is defined by θ, and the angle through which the body rotates is $|\theta|$.

The meaning of the strain tensor, E_{ij} is best understood by reference to its effect on the dot product between two vectors embedded in the material medium. Figure 4.2 illustrates three points, A, B, and the reference point, O. The two vectors of interest are the respective displacements of the points A and B from O. The change in the dot product of these two vectors is

$$\begin{aligned}
\delta\mathbf{x}_A \cdot \delta\mathbf{x}_B - \delta\mathbf{X}_A \cdot \delta\mathbf{X}_B &= (\delta\mathbf{X}_A + \mathbf{u}_A - \mathbf{u}_O) \cdot (\delta\mathbf{X}_B + \mathbf{u}_B - \mathbf{u}_O) - \delta\mathbf{X}_A \cdot \delta\mathbf{X}_B \\
&= \left(\delta X_{Ai} + \frac{\partial u_i}{\partial X_j}\delta X_{Aj}\right)\left(\delta X_{Bi} + \frac{\partial u_i}{\partial X_k}\delta X_{Bk}\right) - \delta X_{Ai}\delta X_{Bi} \\
&= \delta X_{Ai}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)\delta X_{Bj} \\
&\simeq 2\delta X_{Ai} E_{ij} \delta X_{Bj},
\end{aligned} \tag{4.5}$$

where the approximation of leaving out the nonlinear term in the third line is

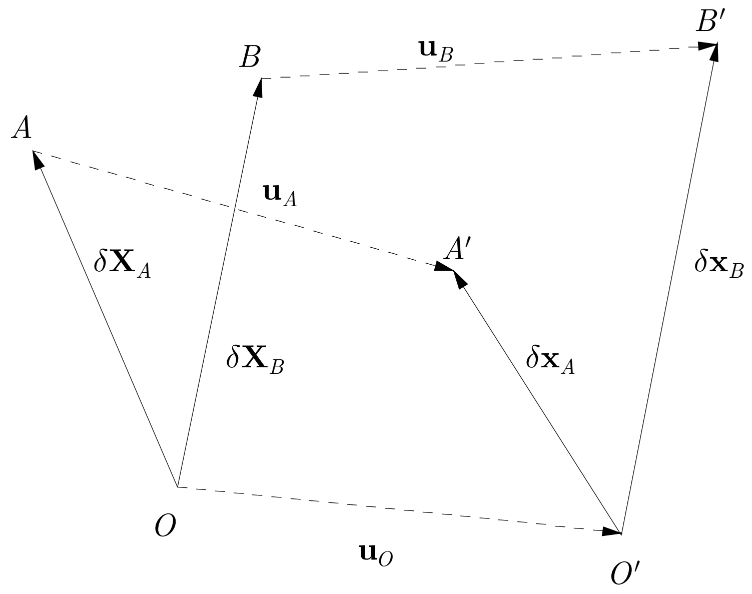

Figure 4.2. Displacement of three points, A, B, and O.

justified by the small deformation assumption. In dyadic notation this becomes

$$\delta\mathbf{x}_A \cdot \delta\mathbf{x}_B = \delta\mathbf{X}_A \cdot \delta\mathbf{X}_B + 2\delta\mathbf{X}_A \cdot \mathbf{E} \cdot \delta\mathbf{X}_B. \tag{4.6}$$

Two examples serve to illustrate the meaning of this relationship. If $\delta\mathbf{X}_A = \delta\mathbf{X}_B = l\mathbf{n}$ and $\delta\mathbf{x}_A = \delta\mathbf{x}_B = l'\mathbf{n}'$, where $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors, and where $l' = l + \delta l$, then equation (4.6) becomes

$$l'^2 = l^2(1 + 2\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}). \tag{4.7}$$

By the small deformation approximation, l and l' won't differ by very much, so $l'^2 \simeq l^2 + 2l\delta l$. Therefore, equation (4.7) reduces to

$$\frac{\delta l}{l} = \mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}. \tag{4.8}$$

In other words, $\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}$ is the fractional change in length of a line segment embedded in the material medium with initial orientation defined by $\mathbf{n}$. This fractional change in length is refered to as the *unit extension* in the $\mathbf{n}$ direction. In the special case in which, for instance, $\mathbf{n} = \mathbf{i}$, the combination $\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}$ reduces to E_{xx}. Thus, the diagonal components of $\mathbf{E}$ are the unit extensions along the respective coordinate axes.

The second example in the use of equation (4.6) is when $\delta \mathbf{X}_A$ and $\delta \mathbf{X}_B$ are normal to each other. If $|\delta \mathbf{X}_A| = |\delta \mathbf{X}_B| = l$, and $|\delta \mathbf{x}_A| = l + \delta l_A$ while $|\delta \mathbf{x}_B| = l + \delta l_B$, then equation (4.6) becomes

$$(l + \delta l_A)(l + \delta l_B) \cos(\pi/2 - \alpha) = 2l^2 \mathbf{n}_A \cdot \mathbf{E} \cdot \mathbf{n}_B, \tag{4.9}$$

where the angle α measures the deviation of $\delta \mathbf{x}_A$ and $\delta \mathbf{x}_B$ from orthogonality, and where $\mathbf{n}_A$ and $\mathbf{n}_B$ are unit vectors in the directions of the two vectors. By the small deformation approximation, the changes in the vector lengths are fractionally small, and the change in the angle between the vectors is small as well. Thus, $|\alpha| << 1$, and the small angle approximation yields $\cos(\pi/2 - \alpha) \simeq \alpha$. Equation (4.9) therefore becomes

$$\alpha = 2\mathbf{n}_A \cdot \mathbf{E} \cdot \mathbf{n}_B. \tag{4.10}$$

The angle α is the *decrease* from $\pi/2$ of the angle between the two vectors under the influence of the deformation. If, for instance, $\mathbf{n}_A = \mathbf{i}$ and $\mathbf{n}_B = \mathbf{j}$, then $\alpha = 2E_{xy}$. The off-diagonal terms of the strain tensor are thus related to changes in the angles between vectors initially aligned with the coordinate axes.

We now examine the change in volume of a small cubical parcel of material under the influence of a deformation field. The rotation part of the deformation does not influence the volume, since solid body rotations do not change the size and shape of objects. The effect of the strain tensor is to change the lengths of the edges of the cube and to skew the cube slightly into the shape of a parallelepiped.

If the initial diameter of the cube is l, the vectors representing three nearly perpendicular edges of the resultant parallelepiped can be written $\delta \mathbf{x}_A = (l + \delta l_A)\mathbf{n}_A$, $\delta \mathbf{x}_B = (l + \delta l_B)\mathbf{n}_B$, and $\delta \mathbf{x}_C = (l + \delta l_C)\mathbf{n}_C$. The volume of the parallelepiped is

$$V = (\delta \mathbf{x}_A \times \delta \mathbf{x}_B) \cdot \delta \mathbf{x}_C =$$

$$(l + \delta l_A)(l + \delta l_B)(l + \delta l_C)(\mathbf{n}_A \times \mathbf{n}_B) \cdot \mathbf{n}_C. \tag{4.11}$$

To first order in small quantities, this reduces to $V = l^3 + l^2(\delta l_A + \delta l_B + \delta l_C)$. The triple product of unit vectors remains unity to first order, since all of the deviations from unity occur in the form of $1 - \cos(\beta) \simeq \beta^2/2$, where β is a small angle. The fractional change in volume of the cube is thus $\delta V/V = (\delta l_A + \delta l_B + \delta l_C)/l$.

If the cube were initially aligned with the coordinate axes, the fractional change in volume could be written

$$\frac{\delta V}{V} = E_{xx} + E_{yy} + E_{zz} = E_{ii}, \tag{4.12}$$

since $E_{xx} = \delta l_A/l$, etc. For an arbitrarily oriented cube, one could define a coordinate with appropriately aligned axes, and we would have $\delta V/V = E_{ii}'$ in this coordinate system as well. However, as we showed in chapter 2, components of a tensor in different coordinate systems are related by $E_{ij}' = q_{ik}q_{jl}E_{kl}$, where q_{ij} is the transformation matrix between the coordinate systems. Thus, $E_{ii}' = q_{ik}q_{il}E_{kl} = E_{kk}$. The last equality is by virtue of the relationship $q_{ik}q_{il} = \delta_{kl}$, which follows directly from equation (2.22). Hence, E_{ii} has the same value in any coordinate system, and is equal to the fractional change in volume of a small, arbitrarily oriented cubical parcel. As indicated in chapter 2, the sum of the diagonal components of a second order tensor is called the trace of the tensor, so

$$\frac{\delta V}{V} = \mathrm{tr}(\mathbf{E}). \tag{4.13}$$

The part of the displacement field associated with the strain tensor is simply called the *strain*. If the strain tensor is diagonal in a particular coordinate system, a cube with edges aligned with the coordinate axes becomes a rectangular parallelepiped after being strained, i. e., the angles between edges remain 90°. Only the lengths of edges are changed. This is called *normal strain*. However, if off-diagonal terms of the strain tensor are non-zero, then not only the lengths of edges are changed, but also the angles between them. When angles change, the cube is said to have been subject to *shear strain*.

Note that the concepts of normal and shear strain are dependent on the choice of coordinate system. Figure 4.3 shows two cubes rotated 45° to each other, both subject to the same strain tensor, with $E_{xx} = E_{yy} = 0$ in the unprimed frame and $E_{xy}' = E_{yx}' = 0$ in the primed frame. In the unprimed frame the $x - y$ face of the cube is deformed into a diamond, while the same face of the cube aligned with the primed frame becomes a rectangle. To show that the above

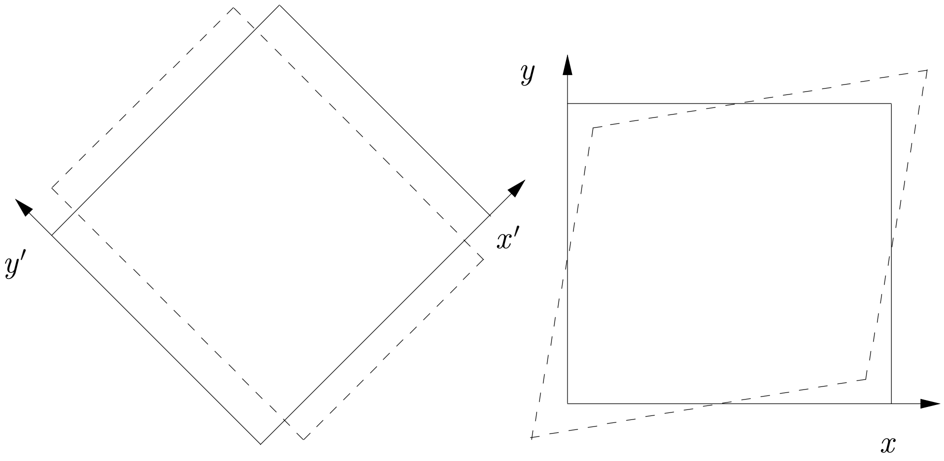

Figure 4.3. Effect of strain on two squares rotated 45° relative to each other.

statements about the unprimed and primed components of **E** are consistent, note that the transformation from the unprimed frame to the primed frame yields (in two dimensions)

$$\begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ -2^{-1/2} & 2^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & E_{xy} \\ E_{xy} & 0 \end{pmatrix} \begin{pmatrix} 2^{-1/2} & -2^{-1/2} \\ 2^{-1/2} & 2^{-1/2} \end{pmatrix} =$$

$$\begin{pmatrix} E_{xy} & 0 \\ 0 & -E_{xy} \end{pmatrix} = \begin{pmatrix} E_{xx}{}' & 0 \\ 0 & E_{yy}{}' \end{pmatrix}, \tag{4.14}$$

showing that $E_{xx}{}' = E_{xy}$ and $E_{yy}{}' = -E_{xy}$. Therefore, it is important to specify the coordinate system when discussing the normal and shear components of strain.

Compatibility Conditions

Given a displacement field, it is easy to compute the strain tensor. However, the inverse process involves solving a set of partial differential equations. Furthermore, not all symmetric tensors can be strain tensors. This is most easily illustrated in the two dimensional case in which

$$E_{xx} = \frac{\partial u_x}{\partial X}, \ E_{yy} = \frac{\partial u_y}{\partial Y}, \ E_{xy} = E_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial Y} + \frac{\partial u_y}{\partial X}\right). \tag{4.15}$$

In this case, three functions of x and y are derived from only two functions, $u_x(X,Y)$ and $u_y(X,Y)$. Therefore, E_{xx}, E_{yy}, and E_{xy} are not independent of each other. It is easily verified that

$$\frac{\partial^2 E_{xy}}{\partial X \partial Y} = \frac{1}{2}\left(\frac{\partial^2 E_{xx}}{\partial Y^2} + \frac{\partial^2 E_{yy}}{\partial X^2}\right). \tag{4.16}$$

This is called a *compatibility condition*, and any candidate strain tensor in two dimensions must satisfy equation (4.16). If it does not, it is impossible to obtain a consistent set of solutions for the displacement field from integrating equations (4.15).

In the three dimensional case, the situation is somewhat more complicated. Six independent strain components (recall that the strain tensor is symmetric) are derived from three displacement components. One would therefore expect three compatibility conditions rather than one, and indeed, satisfaction of equation (4.16) and the two additional equations

$$\frac{\partial^2 E_{xz}}{\partial X \partial Z} = \frac{1}{2}\left(\frac{\partial^2 E_{xx}}{\partial Z^2} + \frac{\partial^2 E_{zz}}{\partial X^2}\right) \tag{4.17}$$

and

$$\frac{\partial^2 E_{yz}}{\partial Y \partial Z} = \frac{1}{2}\left(\frac{\partial^2 E_{yy}}{\partial Z^2} + \frac{\partial^2 E_{zz}}{\partial Y^2}\right) \tag{4.18}$$

are required. However, in addition, the three conditions

$$\frac{\partial^2 E_{xx}}{\partial Y \partial Z} = \frac{\partial}{\partial X}\left(-\frac{\partial E_{yz}}{\partial X} + \frac{\partial E_{zx}}{\partial Y} + \frac{\partial E_{xy}}{\partial Z}\right), \tag{4.19}$$

$$\frac{\partial^2 E_{yy}}{\partial Z \partial X} = \frac{\partial}{\partial Y}\left(-\frac{\partial E_{zx}}{\partial Y} + \frac{\partial E_{xy}}{\partial Z} + \frac{\partial E_{yz}}{\partial X}\right), \tag{4.20}$$

$$\frac{\partial^2 E_{zz}}{\partial X \partial Y} = \frac{\partial}{\partial Z}\left(-\frac{\partial E_{xy}}{\partial Z} + \frac{\partial E_{yz}}{\partial X} + \frac{\partial E_{zx}}{\partial Y}\right) \tag{4.21}$$

need to be satisfied as well. Equations (4.19)-(4.21) are not completely independent of equations (4.16)-(4.18). Note that differentiating equation (4.19) with respect to Y and equation (4.20) with respect to X, and adding, yields the Z derivative of equation (4.16). However, all six equations need to be satisfied in order to insure that a consistent displacement field can be derived from the strain field.

The displacement field derived from a given strain field in general is not unique. However, any two solutions for the displacement field should differ at most by a translation and a rigid rotation. This follows because any other difference would be reflected as a difference in the strain field itself.

One final point needs to be made. With the small deformation approximation, $\partial/\partial x_i \simeq \partial/\partial X_i$. We sometimes find it convenient to replace one by the other when the small deformation approximation is valid. Thus, for instance, we might write the strain tensor as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{4.22}$$

in some situations.

Continuous Motion -- Strain and Rotation Rates

Sometimes, particularly in the case of fluids, it is desirable to work in terms of the rate of displacement rather than the displacement itself. The time rate of change of displacement is simply the parcel velocity field,

$$\mathbf{v}(\mathbf{X}, t) = \lim_{\delta t \to 0}\left(\frac{\mathbf{u}(\mathbf{X}, t + \delta t) - \mathbf{u}(\mathbf{X}, t)}{\delta t}\right) . \tag{4.23}$$

We can arbitrarily set the time origin at t, which makes $\mathbf{u}(\mathbf{X}, t) = 0$, and $\mathbf{u}(\mathbf{X}, t + \delta t)$ arbitrarily small. Thus, the small deformation approximation is automatically satisfied in this case.

The analogs to the strain and rotation tensors in this case are the *strain rate tensor*,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) , \tag{4.24}$$

and the *rotation rate tensor*,

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) . \tag{4.25}$$

The use of lower rather than upper case spatial variables in the above equations is intentional. Since only times very near the reference time, t, are considered, we can assume that $\mathbf{x} = \mathbf{X}$ for the purposes of this derivation. The reference time can, of course, be moved around at will.

The dual vector of Ω_{ij} is

$$-\frac{1}{2}\,\epsilon_{ijk}\Omega_{ij} = -\frac{1}{4}\,\epsilon_{ijk}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = -\frac{1}{2}\,\epsilon_{kij}\,\frac{\partial v_i}{\partial x_j}\,. \tag{4.26}$$

This is simply $(\nabla \times \mathbf{v})/2$. The last step in the derivation of the above equation results from $\epsilon_{kij}\partial v_j/\partial x_i = -\epsilon_{kji}\partial v_j/\partial x_i = -\epsilon_{kij}\partial v_i/\partial x_j$. The *vorticity*, which is defined $\omega = \nabla \times \mathbf{v}$, and plays an important role in fluid mechanics, is thus twice the dual vector of the rotation rate tensor. By analogy with the rotation tensor $\mathbf{R}$, the dual of the rotation rate tensor defines the direction and speed of the rigid body rotation of a small parcel. The magnitude of the vorticity vector is thus twice the local rigid body rotation rate.

Problems

1. For the displacement field $\mathbf{u}(\mathbf{X}) = \beta Y\mathbf{i}$, do the following: a) Compute the rotation tensor and the angle through which small parcels are rotated. b) Compute the strain tensor and the angle which a small square initially aligned with the coordinate axes in the $x - y$ plane is skewed from square. c) Compute the principal axes and eigenvalues of the strain tensor. What angles do the principal axes make with respect to the original coordinate axes? d) What are the unit extensions in the directions of the principal axes? e) What is the fractional change in volume of a small parcel as a result of the deformation? f) What condition is required to make the small deformation approximation valid?

2. Repeat problem 1 for the displacement field $\mathbf{u} = \beta\mathbf{X}$.

3. A rectangular parallelepiped with respective x, y, and z dimensions a, b, and c, is stretched uniformly δa in the x direction, δb in the y direction, and δc in the z direction. a) Determine the strain tensor. b) Determine the exact fractional change in volume assuming that δa is not small compared to a, etc. c) As the unit extensions $\delta a/a$, etc., become small, show that this is well approximated by E_{ii}.

4. For the strain tensor

$$\epsilon\begin{pmatrix}2 & 1 & 0\\ 1 & -3 & 3\\ 0 & 3 & 1\end{pmatrix}$$

where $|\epsilon| << 1$, find the unit extension in the direction defined by the vector $(3\,1-1)$.

5. For the strain tensor defined in problem 4, find the change caused by the strain in the angle between the two vectors with initial directions $(3\,1-1)$ and $(1\,0\,3)$.

6. Show that the strain tensor

$$E_{ij}=\begin{pmatrix}2\alpha Y & \alpha X\\ \alpha X & 2\beta Y\end{pmatrix}$$

satisfies the compatibility conditions, and then integrate equations (4.15) to obtain a displacement field consistent with E_{ij}.

7. Repeat problem 6 for the strain tensor

$$E_{ij}=\begin{pmatrix}0 & \alpha Y+\beta X\\ \alpha Y+\beta X & 0\end{pmatrix}.$$

8. If $\mathbf{v}(\mathbf{x},t)$ represents the velocity field in a fluid, show that $\nabla\cdot\mathbf{v}$ is the fractional time rate of change of the volume of a small fluid element. Show also that it is minus the fractional time rate of change of density and compare with equation (3.11).

References

Spencer, A. J. M., 1980: *Continuum mechanics.* Longman, 183 pp.

Chapter 5 -- Elastic Bodies

Most solid materials behave somewhat like a spring as long as the applied forces are sufficiently small. Figure 5.1 shows how a rectangular parallelepiped

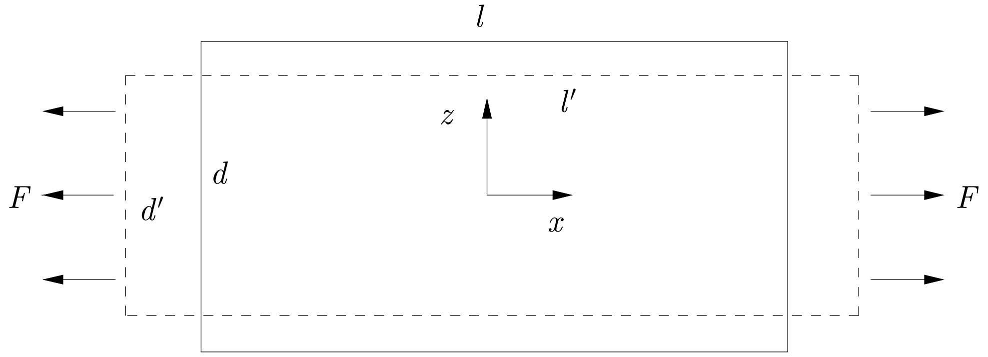

Figure 5.1. Deformation of a rectangular parallelepiped under the influence of uniform normal tractions on the left and right ends.

of length l, width (into the page) w, and depth d is deformed by normal forces F distributed uniformly over the left and right ends. The result is a somewhat longer and skinnier parallelepiped of dimensions $l' \times w' \times d'$, with $l' > l$, $w' < w$, and $d' < d$. As with a spring, Hooke's law holds, i. e., the amount of stretch is proportional to the applied stretching force:

$$F \propto l' - l \equiv \delta l \tag{5.1}$$

In addition the parallelepiped shrinks or stretches in the lateral dimensions in proportion to the applied force:

$$F \propto -(w' - w) \equiv -\delta w \qquad F \propto -(d' - d) \equiv -\delta d\ . \tag{5.2}$$

Equations (5.1) and (5.2) can be represented in a form that is independent of the actual dimensions of the parallelepiped by converting the force to applied traction and the extensions and contractions to strain components. The fractional changes in dimension of most materials are small as long as the forces are such that the material isn't permanently deformed or broken. Thus, the small deformation approximation holds, and $E_{xx} = \delta l/l$, $E_{yy} = \delta w/w$, and $E_{zz} = \delta d/d$. In the computation of the traction applied to the ends of the parallelepiped, it doesn't matter whether the old or new area is used, i. e.,

$T_{xx} = F/(wd) \simeq F/(w'd')$.

Rewriting equations (5.1) and (5.2) in terms of stress and strain,

$$T_{xx} = E_Y E_{xx} \quad \nu T_{xx} = -\, E_Y E_{yy} \quad \nu T_{xx} = -\, E_Y E_{zz} \ , \tag{5.3}$$

we have introduced the proportionality constants dependent upon the material, E_Y, called Young's modulus, and ν, called Poisson's ratio. Poisson's ratio relates the (negative) unit extension in the directions transverse to the applied traction to the unit extension in the direction of the traction:

$$\nu = -\,\frac{E_{yy}}{E_{xx}} = -\,\frac{E_{zz}}{E_{xx}} \ . \tag{5.4}$$

Implicit in this is the assumption that the characteristics of the material medium do not change with direction -- otherwise it is conceivable that E_{yy} might differ from E_{zz}, requiring two different values of ν for the two transverse directions. This assumption of an *isotropic medium* is approximately valid for many, but by no means all materials. Metals tend to be isotropic unless they are specially treated, but wood, for instance, behaves very differently along different axes. Many composite materials that consist of strong, but brittle fibers embedded in a supporting matrix are highly anisotropic by design. Fiberglas is an example of such a material.

Hooke's law implies that a deformed body returns to its original shape when the applied force is released. Beyond a certain limit on the stress, called the *elastic limit*, all real materials begin to deform in an irreversible manner. This is called *plastic deformation.* Beyond plastic deformation, the material eventually breaks. Repeated cycling of applied forces, even within the elastic limit, can cause some materials to break as well. This is the phenomenon of *fatigue.* Other materials *work harden* or *anneal* under repeated cycling, i. e., become more or less brittle. These phenomena are beyond the scope of this book, and we shall consider only the behavior of isotropic solid materials within their elastic limit.

The above example shows the effect of applied normal forces. When tangential forces are applied to a rectangular parallelepiped, as in figure 5.2, the parallelepiped takes on a nonrectangular form, with the angle α defining the deviation from rectangularity. The tangential traction on the top and bottom surfaces is

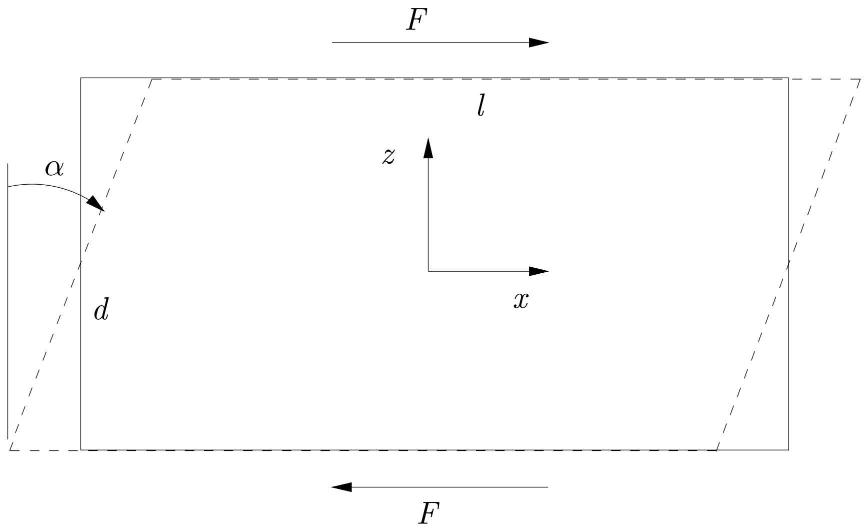

Figure 5.2. Deformation of a rectangular parallelepiped under the influence of shear tractions.

$T_{xz} = F/(lw)$, where w is the dimension of the parallelepiped normal to the page. From equation 4.10, $\alpha = 2E_{xz}$, and within the elastic limit, $T_{xz} \propto E_{xz}$. This is usually written

$$T_{xz} = 2\mu E_{xz} \ , \tag{5.5}$$

where μ is a proportionality constant called the *shear modulus* of the material.

Omitted from figure 5.2 for clarity are the tangential forces on the left and right ends of the parallelepiped. The stress tensor is symmetric, i. e., $T_{xz} = T_{zx}$, and applied tractions are required to match the stress tensor on the ends as well as the top and bottom of the parallelepiped.

The above examples illustrate particular instances of the general stress-strain relationship for isotropic elastic media, which we develop in the next section. Following that, we derive formulae for the potential and kinetic energies in an isotropic elastic medium.

Stress-Strain Relationship for an Isotropic Medium

Hooke's law behavior implies that there is a linear relationship between the components of stress and the components of strain. The most general linear, homogeneous relationship between stress and strain components consists of six equations, one for each of the stress components. (There are six rather than nine in three dimensions due to the symmetry of the stress tensor, which makes three of the possible nine equations redundant.) Each of the stress equations in turn contains linear contributions from each of the six independent strain components. (Recall that the strain tensor is also symmetric.) The result is 36 independent coefficients characterizing the material.

In reality, even the most anisotropic elastic medium can be characterized by fewer than 36 constants, and in an isotropic medium the number of independent constants reduces to two. We now see how this comes about. First, for an isotropic medium, the principal axes of the stress and strain tensors will be coincident. This is equivalent to saying that a rectangular parallelepiped of material subject to applied forces normal to its surfaces retains its rectangular shape, as illustrated in figure 5.1 -- in other words, normal forces don't result in the type of shear deformation shown in figure 5.2.

Since the principal axes of stress and strain coincide, and since an isotropic material has the same characteristics in all directions, no generality is lost by writing the stress-strain relationship in the principal axis coordinate system. In these coordinates all off-diagonal terms are zero, leaving only the three diagonal components of stress and strain. This reduces the 36 independent coefficients to nine, as shown below:

$$\begin{aligned} T_{xx} &= AE_{xx} + BE_{yy} + CE_{zz} \\ T_{yy} &= DE_{xx} + FE_{yy} + GE_{zz} \\ T_{zz} &= HE_{xx} + IE_{yy} + JE_{zz} \end{aligned} \tag{5.6}$$

Further reductions follow from ideas of symmetry. Since the material is isotropic, the relationship between normal stress and strain in the same direction should be the same in all directions, which implies that $A = F = J$. Furthermore, relationship between a normal stress in one direction and normal stress in another direction should also be independent of the two directions, which results in $B = C = D = G = H = I$. Equation (5.6) thus becomes

$$T_{xx} = AE_{xx} + BE_{yy} + BE_{zz}$$

$$T_{yy} = BE_{xx} + AE_{yy} + BE_{zz} \tag{5.7}$$

$$T_{zz} = BE_{xx} + BE_{yy} + AE_{zz}$$

We rewrite equation (5.7) as

$$T_{xx} = (A - B)E_{xx} + B(E_{xx} + E_{yy} + E_{zz})$$

$$T_{yy} = (A - B)E_{yy} + B(E_{xx} + E_{yy} + E_{zz}) \tag{5.8}$$

$$T_{zz} = (A - B)E_{zz} + B(E_{xx} + E_{yy} + E_{zz})$$

because this is easily recognized to correspond to the component notation form of the tensor equation

$$T_{ij} = (A - B)E_{ij} + B\delta_{ij}E_{kk}. \tag{5.9}$$

Since equation (5.9) is valid in the principal axis coordinate system, and since it is a tensor equation, it is equally valid in any coordinate system. Thus, in a non-principal axis system in which off-diagonal terms are non-zero, we see that, for instance, $T_{xz} = (A - B)E_{xz}$, which by comparison with equation (5.5) shows that $(A - B) = 2\mu$. We conventionally write equation (5.9) in terms of the so-called Lamé constants, λ, and the previously defined shear modulus, μ:

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}. \tag{5.10}$$

Given the strain tensor, $\mathbf{E}$, equation (5.10) shows how to compute the stress tensor, $\mathbf{T}$. Sometimes we need the reverse relationship in which the strain tensor is computed in terms of the stress tensor. In order to invert equation (5.10), we need to calculate the trace of the strain tensor, E_{kk}, in terms of the stress tensor. This is easily done by taking the trace of the entire equation,

$$T_{kk} = (3\lambda + 2\mu)E_{kk}, \tag{5.11}$$

and solving for E_{kk}. Combining this and equation (5.10) results in

$$E_{ij} = \frac{1}{2\mu}T_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}T_{kk}. \tag{5.12}$$

This is conventionally written in terms of the Poisson ratio and Young's modulus as

$$E_{ij} = -\frac{\nu}{E_Y}\,\delta_{ij}T_{kk} + \frac{(1+\nu)}{E_Y}\,T_{ij}, \tag{5.13}$$

which shows that the Lamé constants are related to the Poisson ratio and Young's modulus by

$$\frac{\nu}{E_Y} = \frac{\lambda}{2\mu(3\lambda + 2\mu)} \tag{5.14}$$

and

$$\frac{(1+\nu)}{E_Y} = \frac{1}{2\mu}\,. \tag{5.15}$$

The characteristics of an isotropic, elastic medium may be expressed either in terms of λ and μ or in terms of ν and E_Y.

For the special case shown in figure 5.1, in which T_{xx} is the only non-zero stress component, equation (5.13) reduces to

$$E_Y E_{xx} = T_{xx} \tag{5.16}$$

and

$$E_Y E_{yy} = E_Y E_{zz} = -\,\nu T_{xx}, \tag{5.17}$$

which agrees with the analysis of this case in the beginning of this chapter.

It is important to distinguish between this situation, in which the y and z faces of the parallelepiped are allowed to move in response to normal tractions applied to the x faces, and the one in which the lateral faces are constrained not to move. In the latter case, $E_{yy} = E_{zz} = 0$, but the corresponding normal stresses, T_{yy} and T_{zz} are nonzero. This is because additional tractions must be applied to the lateral faces to keep them from moving in response to applied tension or compression in the x direction.

When the lateral faces are constrained, equation (5.10) shows that

$$T_{xx} = (\lambda + 2\mu)E_{xx} \tag{5.18}$$

$$T_{yy} = T_{zz} = \lambda E_{xx} = \frac{\lambda}{\lambda + 2\mu}\,T_{xx}. \tag{5.19}$$

In the unconstrained case, the "stiffness" of the material, i. e., amount of stress required per unit strain is simply E_Y, while in the constrained case it is $\lambda + 2\mu$. By manipulating equations (5.14) and (5.15) it is possible to show that

$$\lambda + 2\mu = \frac{E_Y(1-\nu)}{(1+\nu)(1-2\nu)} =$$

$$\frac{E_Y}{1-\nu^2}\left(1 + \frac{\nu^2}{1-2\nu}\right) > E_Y, \quad \nu \leq 1/2\ . \tag{5.20}$$

Thus, when the parallelepiped is laterally constrained, it is stiffer under stretching and compression than when the lateral faces are free.

The limit $\nu = 1/2$ is interesting to explore. Taking the trace of equation (5.13) results in

$$E_Y E_{kk} = (1-2\nu)T_{kk}. \tag{5.21}$$

Recall that E_{kk} is the fractional change in volume of a small parcel of material. This is normally expected to be positive under tensional stresses, i. e., $T_{kk} > 0$, and negative under compressional stresses. Thus, $\nu < 1/2$. The limit $\nu = 1/2$ implies no change in volume regardless of the stress, and therefore corresponds to the case of an incompressible material. Note that in the above example an incompressible material has infinite stiffness to normal stresses if the lateral faces are constrained.

From chapter 2, the pressure is defined as $p = -T_{ii}/3$. From equations (5.18) and (5.21) it is clear that the pressure is linearly related to the fractional change in volume of an isotropic elastic medium by

$$p = -\ kE_{ii}, \tag{5.22}$$

where the *bulk modulus*, k, can be related to the Poisson ratio and Young's modulus:

$$k = E_Y/[3(1-2\nu)]. \tag{5.23}$$

Energy in Isotropic Elastic Media

As shown in elementary mechanics texts, a spring with spring constant κ exhibits restoring force $F = -\kappa x$ when stretched x from equilibrium, and stores potential energy $U = \kappa x^2/2$. A similar quadratic dependence on deformation exists for elastic media. In this section we develop expressions for both the kinetic and elastic potential energies of an isotropic elastic medium.

The expression for kinetic energy of an elastic medium is particularly simple in the small deformation approximation. In this case the velocity of parcels in the Eulerian representation, $\mathbf{v} = d\mathbf{u}/dt$, may be approximated by $\partial \mathbf{u}/\partial t$, since parcels don't move very far from their initial position. The vector $\mathbf{u}$ is the parcel displacement from its initial position, and the approximation amounts to ignoring $\mathbf{v} \cdot \nabla \mathbf{u}$, which is quadratic in $\mathbf{u}$, and thus negligible as long as $|\mathbf{u}|$ is sufficiently small. In this case the kinetic energy is simply the volume integral of the kinetic energy density:

$$K = \int \frac{\rho}{2} |\mathbf{v}|^2 dV. \tag{5.24}$$

The potential energy takes somewhat more effort to derive. We begin by noting that the work due to external forces on a body can be divided into a contribution due to body forces and to tractions on the body. The incremental work due to body forces associated with a small change in the displacement δu_i is simply

$$\delta W_{body} = \int \delta u_i \; \rho \; B_i \; dV, \tag{5.25}$$

where the integral is over the volume of the body and B_i is the body force per unit mass. The incremental work due to the applied tractions is in the form of an integral over the surface of the body:

$$\delta W_{trac} = \int \delta u_i \; T_{ij} n_j \; dA, \tag{5.26}$$

where $T_{ij} n_j = t_i$ is the applied traction. Equation (5.26) may be converted to a volume integral over the material using the divergence theorem:

$$\delta W_{trac} = \int \frac{\partial}{\partial x_j} \left(\delta u_i T_{ij} \right) dV = \int \left(\frac{\partial \delta u_i}{\partial x_j} T_{ij} + \delta u_i \frac{\partial T_{ij}}{\partial x_j} \right) dV. \tag{5.27}$$

Combining the work from the two contributions results in

$$\delta W = \delta W_{body} + \delta W_{trac} = \int \left[\frac{\partial \delta u_i}{\partial x_j} T_{ij} + \delta u_i \left(\frac{\partial T_{ij}}{\partial x_j} + \rho B_i \right) \right] dV. \tag{5.28}$$

The terms within parentheses are simply the right side of equation (3.21), which is the equation of motion, and therefore equal $\rho(dv_i/dt)$. Setting this to $\rho(\partial v_i/\partial t)$ by the small deformation approximation, and relating the change in the displacement to a the velocity field over time interval δt, $\delta u_i = v_i \delta t$, equation (5.28) becomes

$$\delta W = \int \left[\frac{\partial \delta u_i}{\partial x_j} T_{ij} + \frac{\rho}{2} \frac{\partial |\mathbf{v}|^2}{\partial t} \delta t \right] dV, \tag{5.29}$$

where $v_i v_i = |\mathbf{v}|^2$.

Assuming that the density is constant, the second term on the right side of equation (5.29) reduces to $\delta(\rho|\mathbf{v}|^2/2)$, or the change in the kinetic energy density. In the first term the δ and the differentiation may be interchanged. From equation (4.2), $\partial u_i/\partial x_j = E_{ij} + R_{ij}$, and since $R_{ij}T_{ij} = 0$ due to the antisymmetry of R_{ij}, the first term may be written $\delta E_{ij}\ T_{ij}$. Note that we have differentiated the displacement u_i with respect to x_j rather than X_j, because with the small deformation approximation, the two differ only negligibly.

The first term may be further simplified using the stress-strain relationship, equation (5.10):

$$\delta E_{ij}\ T_{ij} = \delta E_{ij}\ (\lambda \delta_{ij} E_{kk} + 2\mu E_{ij}) =$$

$$\lambda\ \delta E_{ii}\ E_{kk} + 2\mu \delta E_{ij}\ E_{ij} =$$

$$\delta(\lambda E_{kk}^2 + 2\mu E_{ij} E_{ij})/2 =$$

$$\delta(E_{ij} \lambda E_{kk} \delta_{ij} + 2\mu E_{ij} E_{ij})/2 = \delta(E_{ij} T_{ij})/2. \tag{5.30}$$

Combining all this, and interchanging the δ and the integration (possible because the integration volume is assumed to remain constant), the incremental work due to external forces may be written

$$\delta W = \delta \int (E_{ij} T_{ij} + \rho |\mathbf{v}|^2)/2\, dV. \tag{5.31}$$

Since it was possible to extract the δ from the the integral, equation (5.31) is a perfect differential, and the *total* work, made up of a sum of increments of work, can be expressed in terms of the stress, strain, and velocity fields, independent of of how this state was reached:

$$W = \int (E_{ij}T_{ij} + \rho|\mathbf{v}|^2)/2 \, dV. \tag{5.32}$$

By conservation of energy, work done on a parcel of material must be reflected in an increase in the sum of kinetic, potential, and internal energies of the material. The second term is simply the kinetic energy as defined in equation (5.24). We identify the first term as the elastic potential energy,

$$U = \int E_{ij}T_{ij}/2 \, dV, \tag{5.33}$$

and note that conversion to internal energy doesn't occur in the idealization of an elastic body.

We now apply equation (5.33) to a few simple cases of uniform stress and strain. For example, when normal tractions are applied to the ends of a parallelepiped of length l, width w, and height h, $T_{xx} = E_Y E_{xx}$ from equation (5.3). If the other surfaces of the cube are free, all other components of the stress tensor are zero, and other components of the strain tensor don't enter the expression for energy, which is simply

$$U = lwhE_Y E_{xx}^2/2 = E_Y wh\delta l^2/2l, \tag{5.34}$$

where $\delta l = lE_{xx}$ is the amount the parallelepiped is stretched in the x direction. The "spring constant" of the parallelepiped is thus $\kappa = E_Y wh/l$.

For a parallepiped with the lateral faces constrained, T_{yy} and T_{zz} are nonzero, but all strain components except E_{xx} are zero. Equation (5.18) gives $T_{xx} = (\lambda + 2\mu)E_{xx}$ in this case, and

$$U = lwh(\lambda + 2\mu)E_{xx}^2/2 = (\lambda + 2\mu)wh\delta l^2/2l, \tag{5.35}$$

and the new spring constant is $\kappa = (\lambda + 2\mu)wh/l$, which is greater than the above case in which the lateral faces are unconstrained.

When a cube of side l is subjected to shearing tractions that result in shear strain $E_{xy} = E_{yx}$, the corresponding components of stress are $T_{xy} = T_{yx} = 2\mu E_{xy}$. If all other stress and strain components are zero, the potential energy is

$$U = 2\mu l^3(E_{xy}^2 + E_{yx}^2)/2 = 2\mu l^3 E_{xy}^2 = \mu l^3 \alpha^2/2, \tag{5.36}$$

where α, defined in equation (4.10), is the deviation in the angle between the x and y faces from $\pi/2$.

Problems

1. Given

$$\frac{\nu}{E_Y} = \frac{\lambda}{2\mu(3\lambda + 2\mu)}, \qquad \frac{(1+\nu)}{E_Y} = \frac{1}{2\mu},$$

solve for λ and μ in terms of E_Y and ν, and vice versa.

2. For a rectangular parallelepiped aligned with the coordinate axes, assume that the y faces are constrained (i. e., $E_{yy} = 0$) and the z faces are free ($T_{zz} = 0$). For a uniform normal stress, T_{xx}, find E_{xx}.

3. Using the results of problem 2, show that T_{xx}/E_{xx} is intermediate in this case between the xx stress-strain ratio for total lateral constraint (equation (5.20)) and no lateral constraint (equation (5.3)).

Chapter 6 -- Waves in an Elastic Medium

As our first example of elastic body mechanics we consider waves in an isotropic elastic medium. We limit our discussion to small amplitude waves so that the small deformation approximation is valid. For an isotropic elastic medium we will find two types of waves, *longitudinal waves*, in which the direction of material oscillation is parallel to the direction of wave motion for plane waves, and *transverse waves*, in which the material oscillates in a plane normal to the wave propagation direction. This is in contradistinction to the case of sound waves in a liquid or a gas, in which only one type of wave exists, the longitudinal wave.

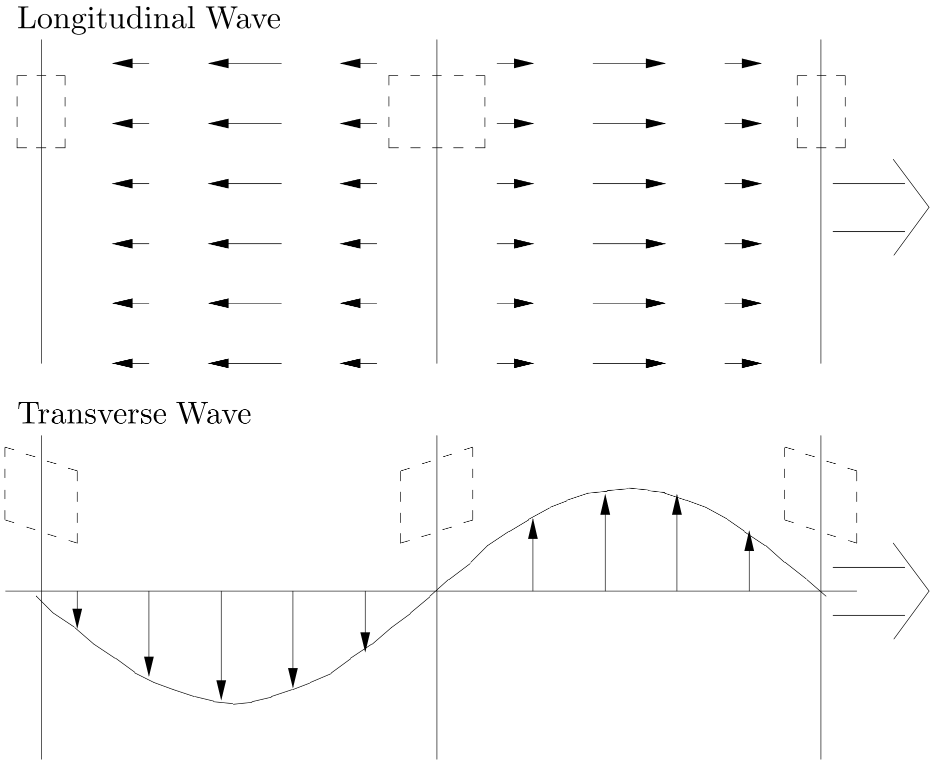

Figure 6.1. Displacement fields for longitudinal and transverse waves moving to the right. The dashed boxes show how an initial cube of material is deformed.

Figure 6.1 shows the displacement fields in these two types of waves.

Seismologists call longitudinal waves *P waves* and transverse waves *S waves*, which respectively originate from the Latin terms *undae primae* and *undae secundae*, or *first waves* and *second waves.* This terminology arose from the fact that longitudinal waves travel faster than transverse waves. P waves therefore arrive first after a distant earthquake, followed by the S waves. See Leet (1950) for an elementary description of these waves.

In this chapter we first discuss waves in an unbounded medium, investigating initially the case of plane waves. We then extend the distinction between the two types of waves to non-plane case in which P waves are not always strictly longitudinal and S waves are not necessarily purely transverse. We discover that more descriptive terms for the two types are respectively *irrotational* and *equivoluminal.* We finally examine the behavior of waves near a free boundary to the elastic medium, such as might be presented by the surface of the earth. We find that elastic waves reflect off the boundary and see that reflection can convert P waves to S waves and vice versa.

Waves in an Unbounded Medium

For problems involving elastic waves it is generally most productive to write the governing equations entirely in terms of the displacement field **u**. Ignoring body forces, which play no central role in elastic waves, Newton's second law in continuum form, represented by equation (3.4), may be written in component notation as

$$\rho a_i = \frac{\partial T_{ij}}{\partial x_j} \, . \tag{6.1}$$

The stress-strain relationship for an isotropic elastic medium is given by equation (5.10). Eliminating the strain in favor of the displacement field using equation (4.2) results in

$$T_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) , \tag{6.2}$$

which may be substituted in equation (6.1) to yield

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \, . \tag{6.3}$$

The small deformation approximation makes the acceleration equal $\partial^2 u_i/\partial t^2$.

Equation (6.3) may be rewritten in vector notation as follows:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u}. \tag{6.4}$$

If the first term on the right side of equation (6.4) were dropped, it would be the classic wave equation, and the wave propagation speed would be $c = (\mu/\rho)^{1/2}$. It turns out that even with this term, equation (6.4) represents waves. The additional complexity serves to distinguish the two types of elastic waves discussed above.

In order to demonstrate this, let us assume a plane wave moving in the plus x direction. We provisionally assume that the material oscillations associated with the wave are partially longitudinal and partially transverse. We will find that such a combination is impossible, and that a plane wave must be purely one or the other, with each component moving at a distinct speed. The assumed form for $\mathbf{u}$ is

$$\mathbf{u} = (\mathbf{U}_t + U_l\mathbf{i})\exp[i(kx - \omega t)], \tag{6.5}$$

where $\mathbf{U}_t$ is the transverse component, i. e., $\mathbf{U}_t \cdot \mathbf{i} = 0$, U_l is the longitudinal component, and k and ω are respectively the assumed wavenumber and frequency of the wave.

We note that $\nabla^2\mathbf{u} = -k^2\mathbf{u}$ in this case. However, $\nabla \cdot \mathbf{u} = ikU_l\exp[i(kx - \omega t)]$, so the first term on the right side of equation (6.4), $\nabla(\nabla \cdot \mathbf{u}) = -k^2U_l\exp[i(kx - \omega t)]\mathbf{i}$, involves only the longitudinal oscillation. Substituting equation (6.5) in equation (6.4) and canceling the common exponential factor results in

$$-\rho\omega^2(\mathbf{U}_t + U_l\mathbf{i}) = -(\lambda + \mu)k^2U_l\mathbf{i} - \mu k^2(\mathbf{U}_t + U_l\mathbf{i}), \tag{6.6}$$

which can be written

$$[\rho\omega^2 - \mu k^2]\mathbf{U}_t + [\rho\omega^2 - (\lambda + 2\mu)k^2]U_l\mathbf{i} = 0. \tag{6.7}$$

Since $\mathbf{U}_t$ is normal to $\mathbf{i}$, both terms in the above equation have to be zero. This is only possible if one or the other of $\mathbf{U}_t$ or U_l is zero, since the two terms in square brackets can't be zero simultaneously. Thus, plane waves must be either purely transverse or purely longitudinal. In the transverse case ($U_l = 0$) we have a phase speed for the waves of

$$c_t = \frac{\omega}{k} = \left(\frac{\mu}{\rho}\right)^{1/2}, \quad \text{transverse}, \tag{6.8}$$

while longitudinal waves move with a speed

$$c_l = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \quad \text{longitudinal}. \tag{6.9}$$

It is clear that $c_l > c_t$, as asserted at the beginning of this chapter, since $\lambda + 2\mu > \mu$.

Examination of figure (6.1) suggests that longitudinal and transverse waves respectively have certain important properties. The dashed boxes show how a tiny cube of material is deformed at different locations in each type of wave. At the 0° phase line in the longitudinal wave, the cube is stretched in the direction of wave propagation, while at the±180° lines, the cube is compressed in this direction. Thus, volume changes take place in small material elements as a longitudinal wave passes. However, no rotation of volume elements occurs. In other words, the displacement field is *irrotational.*

On the contrary, small cubes of material maintain constant volume when a transverse wave passes, which means that the displacement field is *equivoluminal.* However, the cubes are alternately sheared one way and then the other in the direction normal to wave propagation. The displacement field for oscillations in the $\pm y$ direction is given by equation (6.5) as

$$\mathbf{u} = U_t \mathbf{j} \exp[i(kx - \omega t)]. \tag{6.10}$$

From equation (4.2), $R_{yx} = -R_{xy} = ikU_t\mathbf{j}\exp[i(kx - \omega t)]/2$. Thus, the rotation tensor is non-zero, and small parcels are alternately rotated one way and the other as a transverse wave passes. (Note that since $E_{yx} = R_{yx}$ in this case, they are also subject to alternating strains.)

The irrotational nature of plane longitudinal waves and the equivoluminal nature of transverse waves suggests a way to generalize to the non-plane wave case. Any vector field may be represented as the sum of the gradient of a scalar and the curl of a vector:

$$\mathbf{u} = \nabla\phi + \nabla\times\mathbf{a}. \tag{6.11}$$

Since the curl of a gradient is zero, $\nabla\phi$ is irrotational. Likewise, since the divergence of a curl is zero, $\nabla\times\mathbf{a}$ is equivoluminal. Substituting equation (6.11) into equation (6.4) results in

$$\rho\,\frac{\partial^2\nabla\phi}{\partial t^2} + \rho\,\frac{\partial^2\nabla\times\mathbf{a}}{\partial t^2} = \mu\nabla^2\nabla\phi + \mu\nabla^2\nabla\times\mathbf{a} + (\lambda+\mu)\nabla\nabla^2\phi, \tag{6.12}$$

where we have recognized that $\nabla\cdot(\nabla\times\mathbf{a}) = 0$.

Taking the divergence of equation (6.12) yields a wave equation

$$\rho\,\frac{\partial^2\chi}{\partial t^2} = (\lambda+2\mu)\nabla^2\chi, \tag{6.13}$$

where

$$\chi = \nabla^2\phi. \tag{6.14}$$

These equations describe purely irrotational waves that reduce to longitudinal waves in the plane wave case. Similarly, taking the curl of equation (6.12) yields

$$\rho\,\frac{\partial^2\mathbf{b}}{\partial t^2} = \mu\nabla^2\mathbf{b}, \tag{6.15}$$

where

$$\mathbf{b} = \nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}. \tag{6.16}$$

Since only the rotational part of $\mathbf{a}$ yields a component of the displacement field, we can set $\nabla\cdot\mathbf{a} = 0$ without loss of generality. Thus, equation (6.16) simplifies to $\mathbf{b} = -\nabla^2\mathbf{a}$ as long as $\mathbf{a}$ is constrained to be divergence-free.

As an example of a non-plane wave, imagine an irrotational wave with the structure

$$\phi = \Phi\exp[i(kx-\omega t) + mz], \tag{6.17}$$

where Φ is a constant. Substituting this into equations (6.13) and (6.14) yields the relationship

$$\omega^2 = c_l^2(k^2 - m^2) \tag{6.18}$$

and the displacement components

$$(u_x \;\; u_z) = (ik\phi \;\; m\phi). \tag{6.19}$$

Since this represents a wave moving in the x direction with amplitude that increases with z, it is definitely not a longitudinal wave, even though it is irrotational. Waves of this type occur when the elastic medium is bounded by a free surface at $z = 0$. They are called surface waves due to the fact that they have their maximum amplitude at the surface, decaying toward the interior with an e-folding length of m^{-1}. Real surface waves actually are made up of a combination of irrotational and equivoluminal surface waves that march in lockstep. This is impossible for plane waves because these two wave types move with different speeds. However, the speed $c = \omega/k = c_l(1 - m^2/k^2)^{1/2}$ of a surface wave can be adjusted by changing the value of m. The combination is such that zero traction exists on the bounding surface. Surface waves are important in seismology.

The elastic potential energy density is $E_{ij}T_{ij}/2$ and the kinetic energy density is $\rho|\mathbf{v}|^2/2$. It turns out that these two densities are everywhere equal for elastic plane waves. We now demonstrate this for longitudinal plane waves. For a longitudinal plane wave moving in the x direction, the only non-zero component of the strain tensor is $E_{xx} = \partial u_x/\partial x$. Assuming that the physical displacement is given by the real part of equation (6.5). For a longitudinal wave $u = U_l\mathbf{i}\cos(kx - \omega t)$, so $E_{xx} = -\,kU_l\sin(kx - \omega t)$. From equation (5.10), $T_{xx} = (\lambda + 2\mu)E_{xx}$, so the elastic energy density is $(\lambda + 2\mu)k^2U_l^2\sin^2(kx - \omega t)/2$. On the other hand, the material velocity is $\mathbf{v} = \partial\mathbf{u}/\partial t$ in the small deformation approximation, so the kinetic energy density is $\rho\omega^2U_l^2\sin^2(kx - \omega t)/2$. Comparing these two densities shows that they are equal since $\rho\omega^2 = (\lambda + 2\mu)k^2$.

Figure 6.2 illustrates the relationship between the energy densities and the wave displacement. Both the potential and the kinetic energy densities are maximum where the displacement is zero, because both the strain and the material velocity are maximal there.

Reflection of Seismic Waves at the Earth's Surface

Consider now a longitudinal or P wave incident on an unconstrained boundary to the elastic material. An example of this phenomenon is the approach of a P wave to the earth's surface. Since the surface is unconstrained, the applied traction must be zero. Under these circumstances, there are two rather than one

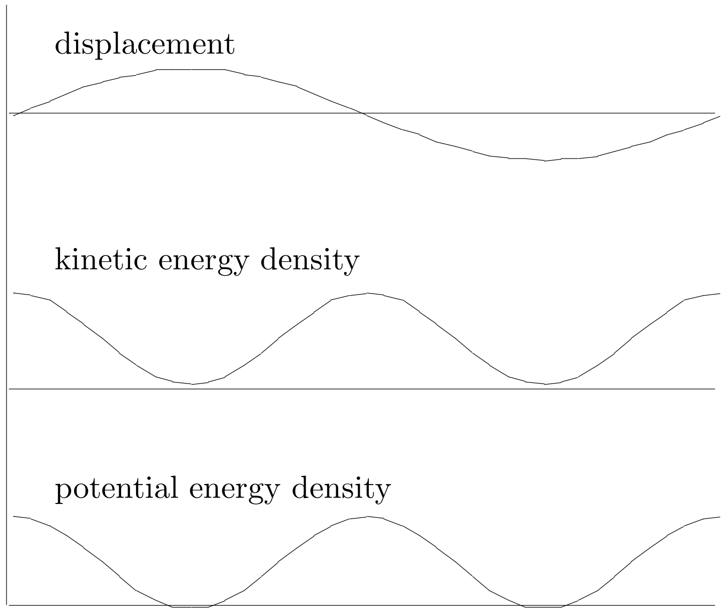

Figure 6.2. Relationship between wave displacement kinetic and potential energy densities in an elastic wave.

reflected waves, a P wave, with an angle of reflection θ_p equal to the angle of incidence of the incoming P wave, and an S wave, with a smaller angle of reflection, θ_s. Figure 6.3 illustrates the situation.

The smaller angle of reflection of the S wave is a consequence of the smaller speed of the S wave relative to the P wave. If a plane wave has the space and time dependence $\exp[i(kx + mz - \omega t)]$, the *phase speed* of the wave is $c = \omega/(k^2 + m^2)^{1/2}$. Since all three wave components must march in phase at $z = 0$, k and ω must be the same for all. A smaller phase speed with k and ω held constant implies a larger value of m. Since the incident or reflected angle is $\theta = \tan^{-1}(k/m)$, a larger m implies a smaller θ. Since $\sin\theta = k/(k^2 + m^2)^{1/2} = kc/\omega$, the relationship between θ_p and θ_s can be written

$$\frac{\sin\theta_p}{\sin\theta_s} = \frac{c_l}{c_t} \equiv r \ . \tag{6.20}$$

This is nothing more than a version of the Snell's law of optics.

We now attempt to compute the amplitudes of the two reflected waves relative to the amplitude of the incident wave. To do this we assume a superposition

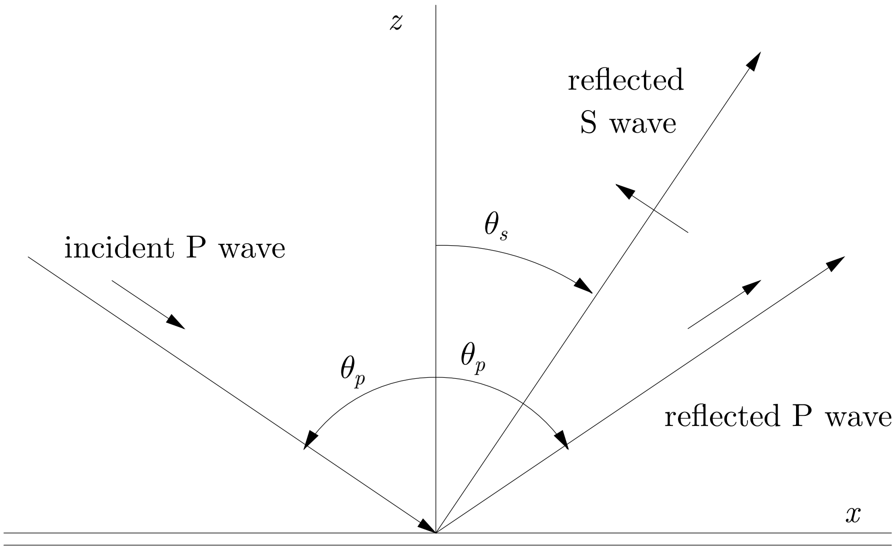

Figure 6.3. Illustration of the reflection of a P wave from a stress-free boundary at $z = 0$. Reflected P and S waves are created.

of the three plane waves, the transverse oscillation of the S wave being in the $x - z$ plane:

$$(u_x,\, u_z) = I(\sin\theta_p,\, -\cos\theta_p)\exp[i(kx - m_p z - \omega t)] +$$

$$P(\sin\theta_p,\ \cos\theta_p)\exp[i(kx + m_p z - \omega t)] +$$

$$S(-\cos\theta_s,\ \sin\theta_s)\exp[i(kx + m_s z - \omega t)], \tag{6.21}$$

where I, P, and S are respectively the amplitudes of the incident P wave, the reflected P wave, and the reflected S wave, and m_p and m_s are respectively the wave numbers in the z direction for P and S waves. The arrows in figure 6.3 show what are considered to be positive directions for I, P, and S.

The condition of no traction at $z = 0$ is simply that $\mathbf{T} \cdot \mathbf{k} = 0$ there. Since there is nothing with any dependence on y in the problem, this reduces to $T_{xz} = T_{zz} = 0$. Since $T_{xz} = 2\mu E_{xz}$ and $T_{zz} = \lambda(E_{xx} + E_{zz}) + 2\mu E_{zz}$, the zero traction condition reduces to the two equations

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0 \tag{6.22}$$

and

$$\lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu) \frac{\partial u_z}{\partial z} = 0. \tag{6.23}$$

Substituting equation (6.21) in these, setting $z = 0$, and canceling the common factor $\exp[i(kx - \omega t)]$ results in

$$P + \frac{2\sin^2\theta_s - 1}{2\cos\theta_p \sin\theta_s} S = I \tag{6.24}$$

and

$$P + \frac{(r^2 - 1)\sin\theta_p \cos\theta_s}{\sin^2\theta_p + r^2\cos^2\theta_p} S = -I, \tag{6.25}$$

where we have used $r = \sin\theta_p / \sin\theta_s$, $m_p = k/\tan\theta_p$, and $m_s = k/\tan\theta_s$.

These equations can be solved for P and S in terms of I, θ_p, and r, but the solution is quite messy. Instead, we shall explore various limits to the equations as a means to understand what they are telling us.

For normal incidence, $\theta_s = \theta_p = 0$. The only way equation (6.24) can be satisfied is to have $S = 0$. In this limit equation (6.25) tells us that $P = -I$. Thus, (given the sense of the arrows in figure 6.3), the free surface oscillates up and down with an amplitude that is twice the amplitude of the incident wave.

For nearly grazing incidence of the incoming P wave, $\theta_p \to \pi/2$. In this limit equation (6.24) shows that $S \to 0$ in this case as well. Thus, from equation (6.25) we again have $P \to -I$. The net oscillation of the free surface is thus zero, and we find that a free surface doesn't move at all in the limit of grazing incidence reflection of a P wave.

In the intermediate case the free surface undergoes a complex oscillatory motion in response to a reflecting P wave.

Problems

1. Show that in a transverse plane seismic wave the principal axes of strains are rotated 45° from the direction of wave propagation.

2. Look up the appropriate constants for steel to find the propagation speeds of plane P and S waves in this medium.

3. Consider a thin plate of elastic material of density ρ, with its faces at $z = 0$ and $z = d$ stress-free. For longitudinal elastic waves moving in the x direction in the plate with wavelength $\lambda >> d$, one can assume that $T_{xz} = T_{yz} = T_{xy} = T_{zz} = 0$ and $E_{yy} = 0$. This implies a relationship between E_{xx} and T_{xx} for longitudinal waves. Find this relationship and the resulting speed of such waves.

4. Consider longitudinal waves moving *normal* to a plate of thickness d. Since they reflect at the stress-free surfaces of the plate, the general solution is a superposition of waves moving in opposite directions through the plate, which together constitute a standing wave. Find the frequencies of such standing modes in terms of the plate thickness, its density, ρ, and the Lamé constants.

5. Show that the elastic potential and kinetic energy densities are equal for transverse elastic waves.

6. Show that surface irrotational waves move more slowly than plane irrotational waves.

7. Compute the reflected P and S wave amplitudes when a P wave encounters a *fixed* plane surface at angle of incidence θ_p. Is there a value of θ_p for which the amplitude of the reflected P wave is zero?

References

Leet, L. D., 1950: *Earth waves.* Harvard University Press, Cambridge, Mass.

Chapter 7 -- Statics of Elastic Media

In the statics of elastic media we find it more convenient to work with the stress and the strain tensors than with the displacement field. This is because boundary conditions on statics problems are usually expressed as applied surface tractions, which impose constraints on the stress tensor. In such problems there are three elements that must be considered. First, Newton's second law must be satisfied for the static case, i. e.,

$$\nabla \cdot \mathbf{T} + \rho \mathbf{B} = 0. \tag{7.1}$$

In the usual situation in which the body force per unit mass is derivable from a scalar potential, i. e., $\mathbf{B} = -\nabla U$, and in which the mass density ρ is constant, the above equation can be written

$$\nabla \cdot (\mathbf{T} - \rho U \mathbf{I}) = 0. \tag{7.2}$$

In many cases of interest $\rho U \mathbf{I}$ is much smaller in magnitude than $\mathbf{T}$, and can be ignored.

The second element is that boundary conditions on the stress must be satisfied. For problems in which one seeks the response of a body to a distribution of applied tractions, one must be sure that $\mathbf{T} \cdot \mathbf{n}$ equals the applied tractions at every point on the surface of the body.

The third element is that the compatibility conditions on the strain tensor must be satisfied. These translate, via equation (5.13), into conditions on the stress tensor. In three dimensions, the three independent compatibility conditions plus the three relationships implied by Newton's second law are sufficient to uniquely determine the six independent stress components as long as the boundary conditions are properly applied. In two dimensions, the single compatibility condition and the two components of Newton's second law similarly determine the three independent stress components.

Note that the strain tensor as a whole must satisfy the compatibility conditions. In particular, that part of the stress tensor resulting from body forces, $\rho U \mathbf{I}$, may not by itself generate a compatible strain tensor. (See, for example, problem 2.) In this case, the above separation of $\mathbf{T}$ into inhomogeneous and homogeneous parts may or may not be useful.

In this chapter we examine a number of simple problems in the statics of isotropic elastic media. We first examine torsion in a cylindrical bar, obtaining the macroscopic torsional spring constant in terms of the shear modulus and the dimensions of the bar. We then explore pure bending of a beam. An approach to two dimensional problems is then outlined, and we finally look into the use of energy methods to understand buckling of thin beams under compressional loads.

Torsion on a Cylinder

Imagine a right circular cylinder of radius R and length l, as

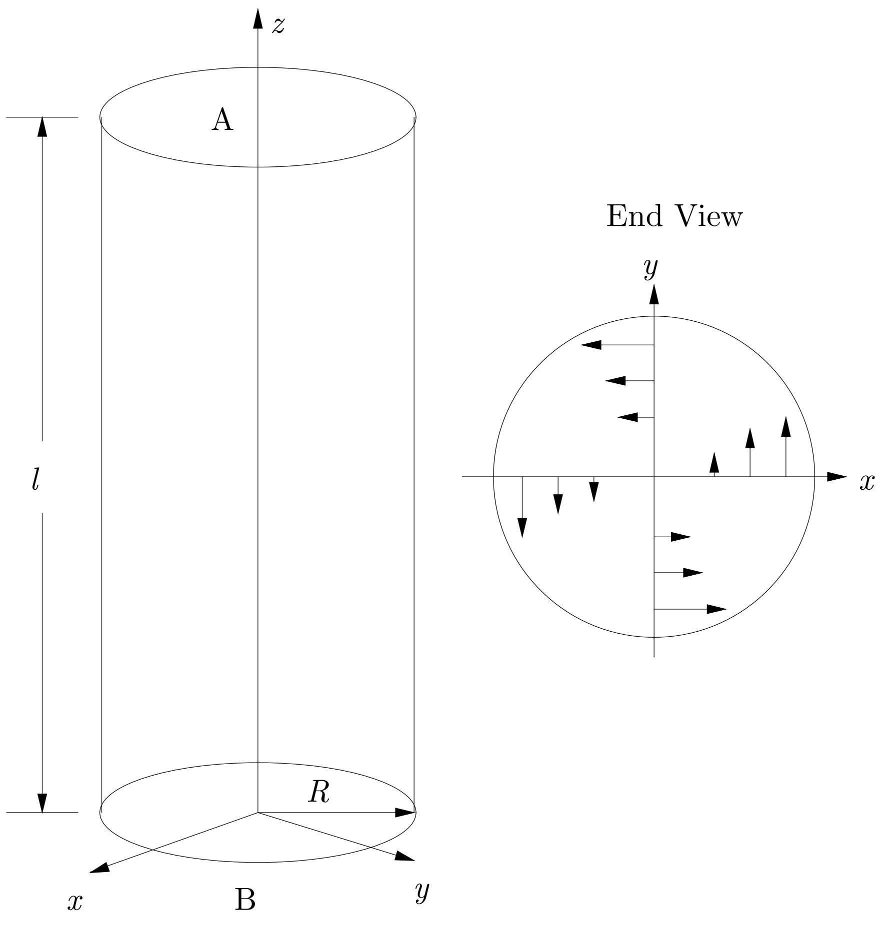

Figure 7.1. Definition sketch for torsion applied to ends of a cylinder.

illustrated in figure 7.1. The ends of the cylinder are subjected to tangential tractions as shown in the end view in figure 7.1. The tractions on the two ends of the cylinder take the opposite sense, so the the cylinder is in torque balance, and the sides of the cylinder are assumed to be traction-free.

We assume a traction of the form $\mathbf{t} = (-Cy \;\; Cx \;\; 0)$ applied to end A of the cylinder, where C is a constant, with minus this applied to end B. A stress tensor that matches these tractions is

$$T_{ij} = \begin{pmatrix} 0 & 0 & -Cy \\ 0 & 0 & Cx \\ -Cy & Cx & 0 \end{pmatrix}, \tag{7.3}$$

as is easily shown by dotting $\mathbf{T}$ with $\pm\,\mathbf{k}$. The zero stress condition is also satisfied on the sides of the cylinder. We show this by noting that the unit outward normal on the sides of the cylinder is $\mathbf{n} = (x/r \;\; y/r \;\; 0)$, where $r = (x^2 + y^2)^{1/2}$. $\mathbf{T} \cdot \mathbf{n} = 0$ in this case.

We now compute the torque on each end of the cylinder resulting from the applied tractions. The torque is given by the formula

$$\mathbf{N} = \int \mathbf{x} \times \mathbf{t}\; dA, \tag{7.4}$$

where $\mathbf{t}\; dA$ is the force applied to the surface element dA and $\mathbf{x}$ is the moment arm from the pivot point, taken here as the origin. The integration adds up the contribution to the torque from all the surface elements, and the integration is taken to be over one or the other of the cylinder ends. The cross product in equation (7.4) takes the form $\mathbf{x} \times \mathbf{t} = -\,C(\mathbf{i}x + \mathbf{j}y)z + \mathbf{k}Cr^2$. The first two terms cancel by symmetry in the integration, and the z component of the torque, which is the only surviving component, is

$$N_z = \int_0^R Cr^2\, 2\pi r\; dr = \frac{\pi C R^4}{2}\,. \tag{7.5}$$

Thus, the constant $C = 2N_z/(\pi R^4)$ in terms of the applied torque and the radius of the cylinder.

We now check the other two elements to the solution of this problem. It is easy to verify that $\nabla \cdot \mathbf{T} = 0$ in this case, and it remains only to obtain the components of the strain tensor and check to see that the compatibility conditions are satisfied. Since the diagonal components of the stress tensor are zero in this case, equation (5.10) can be used to show that

$$E_{ij} = \frac{1}{2\mu} T_{ij} = \frac{1}{2\mu} \begin{pmatrix} 0 & 0 & -Cy \\ 0 & 0 & Cx \\ -Cy & Cx & 0 \end{pmatrix} . \tag{7.6}$$

This strain tensor is easily shown to satisfy the compatibility conditions, equations (4.16) - (4.21). Therefore, the stress tensor postulated above is indeed a solution to the problem of torsional tractions applied to the ends of a cylinder.

Let us now determine how much the cylinder is twisted by the applied torques. To do this, we must obtain the displacement field **u** from **E**. From equation (7.6), we see that since $\partial u_x/\partial x = E_{xx} = 0$, $u_x = f(y,z)$, where $f(y)$ is an arbitrary function of y and z. Similarly, $u_y = g(x,z)$, and $u_z = h(x,y)$, where $g(x,z)$ is an arbitrary function of x and z and $h(x,y)$ is an arbitrary function of x and y. Now,

$$E_{xz} = E_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial z} + \frac{\partial h}{\partial x}\right) = -\frac{Cy}{2\mu} , \tag{7.7}$$

while

$$E_{yz} = E_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial g}{\partial z} + \frac{\partial h}{\partial y}\right) = \frac{Cx}{2\mu} . \tag{7.8}$$

There is not a unique displacement field corresponding to this strain tensor. In particular, displacement fields that differ by a translation or a rigid rotation yield the same strain tensor. However, displacements and rigid rotations don't interest us here, as we seek only the amount that the cylinder has been twisted. We therefore look for a simple particular solution from which we can deduce this twist. Such a solution is

$$\mathbf{u} = (- Cyz/\mu \;\; Cxz/\mu \;\; 0). \tag{7.9}$$

A quick check shows that $E_{xy} = 0$ with this choice of **u**, as is necessary for consistency.

Examination of equation (7.9) shows that the displacement is in the form of a rotation about the z axis through an angle Cz/μ, as long as the angle of rotation is small. Thus, the twisting of the cylinder increases linearly along the cylinder, and the rotation angle of one end of the cylinder relative to the other is $\alpha = Cl/\mu = 2lN_z/(\pi R^4 \mu)$. Thus, the torque required to twist the cylinder through an angle α is

$$N_z = \left(\frac{\pi R^4 \mu}{2l}\right)\alpha = \kappa\alpha \ , \qquad (7.10)$$

where $\kappa = \pi R^4 \mu/(2l)$ is the torsional spring constant for the cylinder.

Technically, this result is only valid if $|\alpha| << 1$. However, when phrased in terms of twist angle rather than displacements, the result is valid for arbitrary twist angles as long as the twist angle for a cylinder segment of length comparable to the cylinder diameter is small. This may be understood by thinking of a long, skinny cylinder as a sequence of shorter cylinders, applying the theory rigorously to each segment, and adding up the accumulated twist.

Bending of a Beam

The solution to the problem of the pure bending of a beam forms the basis for the engineering theory of beams. It turns out that when a beam is much longer than it is wide, the bending part of any general deformation is the most important part. Thus, the solution to this problem has broad application.

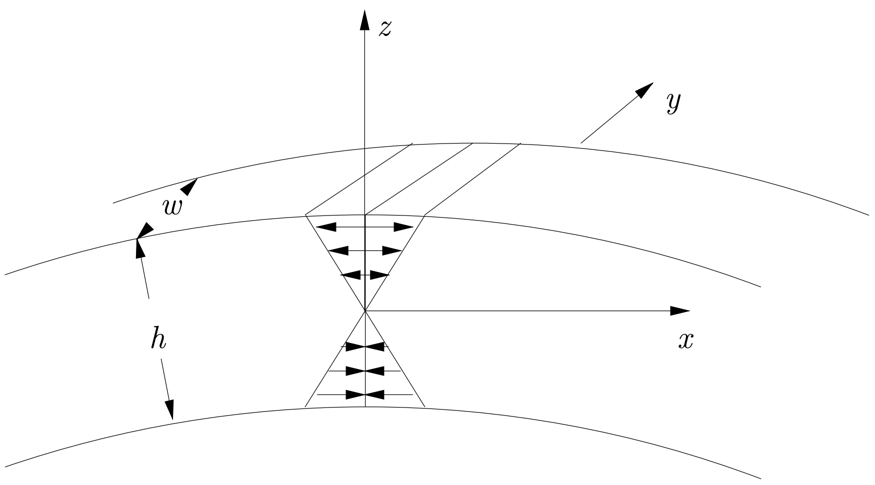

Figure 7.2. Section of a bent beam. The upper part of the beam is in tension, the lower part, compression.

Figure 7.2 shows a segment of a bent beam. The arrows indicate the variation in traction across a surface cutting the beam, with tensional stress occurring on the convex side of the beam and compressional stress on the concave side. These stresses are caused by the stretching and compression in the x direction associated with the bending. It therefore makes sense to postulate a trial stress tensor for the beam material that takes the form

$$T_{ij} = \begin{pmatrix} Cz & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.11}$$

where C is a constant. This exhibits the necessary normal stress in the x direction, and furthermore satisfies the stress-free boundary conditions on the sides of the beam. It is also easy to show that $\nabla \cdot \mathbf{T} = 0$, so Newton's second law is satisfied.

Equation (5.13) shows that

$$E_{xx} = T_{xx}/E_Y = Cz/E_Y, \tag{7.12}$$

while

$$E_{yy} = E_{zz} = -\,\nu T_{xx}/E_Y = -\,\nu Cz/E_Y. \tag{7.13}$$

All off-diagonal components of the strain tensor are zero in this reference frame. This strain tensor satisfies the compatibility conditions. Integrating to obtain the displacement field, we find that

$$u_x = \frac{Cxz}{E_Y} + f_x(y, z), \tag{7.14}$$

$$u_y = -\,\frac{\nu Cyz}{E_Y} + f_y(x, z), \tag{7.15}$$

$$u_z = -\,\frac{\nu Cz^2}{2E_Y} + f_z(x, y), \tag{7.16}$$

where f_x, f_y, and f_z are arbitrary functions of the indicated variables. Setting $E_{xy} = E_{yz} = E_{zx} = 0$ results in

$$\frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} = 0, \tag{7.17}$$

$$\frac{\partial f_y}{\partial z} + \frac{\partial f_z}{\partial y} - \frac{\nu Cy}{E_Y} = 0, \tag{7.18}$$

$$\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}+\frac{Cx}{E_y}=0. \tag{7.19}$$

A consistent solution to this problem occurs if we set $f_x=f_y=0$ and $f_z=-\,C(x^2-\nu y^2)/(2E_Y)$, whence

$$u_x=\frac{Cxz}{E_Y}, \quad u_y=-\,\frac{\nu Cyz}{E_Y}, \quad u_z=-\,\frac{C}{2E_Y}\left[x^2+\nu(z^2-y^2)\right]. \tag{7.20}$$

The assumed stress pattern thus results in a rather complex pattern of deformation. However, the most important aspect of the solution is the vertical deviation of the centerline of the beam from it initial position. This is given by

$$u_{0z}\equiv u_z(x,0,0)=-\,\frac{Cx^2}{2E_Y}, \tag{7.21}$$

which shows that the beam centerline bends into the form of a parabola to the extent that the approximations invoked are valid. Actually, a beam with a transverse stress distribution of the type discussed here which is uniform along its length should bend into a segment of a circle rather than a parabola. However, the small deformation approximation limits the solution to relatively short beam segments over which it is difficult to tell the difference between a parabola and the segment of a circle. The radius of curvature of the beam centerline is also the radius of the circle, and is given by

$$R=\left|\frac{\partial^2 u_{0z}}{\partial x^2}\right|^{-1}=\frac{E_Y}{C} \tag{7.22}$$

Thus, we have related the radius of curvature of the beam, R, to the gradient of normal x stress, C, across the beam.

We now relate C to torques applied to the ends of the beam. Figure 7.3 shows the tractions that need to be applied to the ends of a beam to generate the stress tensor given by equation (7.11). (The tractions shown are those occurring on the left end of the beam and those acting across an imaginary surface that cuts the beam normal to its centerline.) Each set of tractions sums independently to zero net applied force. However, the net applied torque is not zero, but represents a twist in the $x-z$ plane. The non-zero component of torque is thus the y component, N_y, in the coordinate system of figure 7.3. Since the torque $\mathbf{N}=\sum\mathbf{x}_i\times\mathbf{F}_i$, where $\mathbf{F}_i$ is the ith force and $\mathbf{x}_i$ is its associated moment arm,

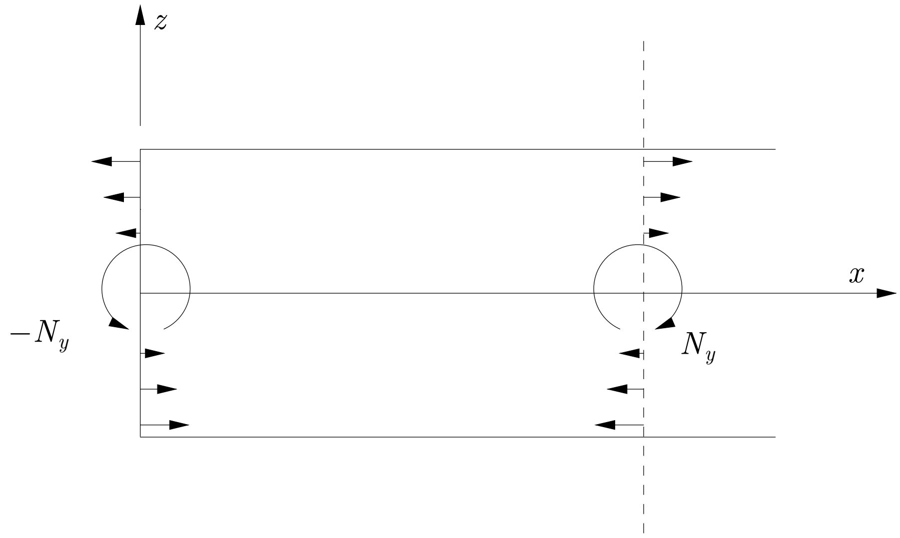

Figure 7.3. Tractions occurring at the ends and across a surface normal to a beam that is undergoing bending. The tractions result in a torque (or bending moment) in the y direction.

$N_y = \sum(z_i F_{ix} - x_i F_{iz})$. $F_{ix} = -\, C z_i\; dx\; dy$ and $F_{iz} = 0$, so the torque is computed as

$$N_y = -\, C \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} z^2 \; dy \; dz = -\, \frac{Cwh^3}{12} \,, \tag{7.23}$$

where the sum has been replaced by an area integral over the end of the beam. Dropping the minus sign and eliminating C between equations (7.22) and (7.23) results in a relationship between the torque applied to the ends of a beam and the radius of curvature of the beam:

$$R = \frac{E_Y}{N_y} \left(\frac{wh^3}{12} \right) = \frac{E_Y I}{N_y} \,, \tag{7.24}$$

where $I = wh^3/12$ is related to the moment of inertia of a small section of beam about the y axis. The torque, N_y, is often called the *bending moment* applied to the beam, and the solution is called the solution for pure bending.

Two Dimensional Problems

For two dimensional problems in which the displacements are all in the $x - z$ plane and are independent of y, and in which body forces are ignored, elastic body mechanics takes on a simple form. In this case $E_{xy} = E_{yy} = E_{yz} = 0$, and a single function $\phi(x, z)$, called the Airy stress function, provides all the non-zero components of stress as follows:

$$T_{xx} = \frac{\partial^2 \phi}{\partial z^2} \qquad T_{zz} = \frac{\partial^2 \phi}{\partial x^2} \qquad T_{xz} = - \frac{\partial^2 \phi}{\partial x \partial z} \ . \tag{7.25}$$

(Do not confuse the Airy stress function with the scalar potential of chapter 6.) Since $E_Y E_{yy} = \ - \nu(T_{xx} + T_{yy} + T_{zz}) + (1+\nu)T_{yy} = 0$, we have $T_{yy} = \nu(T_{xx} + T_{zz})$, and the stress tensor becomes

$$T_{ij} = \begin{pmatrix} \dfrac{\partial^2 \phi}{\partial z^2} & 0 & -\dfrac{\partial^2 \phi}{\partial x \partial z} \\ 0 & T_{yy}(x, z) & 0 \\ -\dfrac{\partial^2 \phi}{\partial x \partial z} & 0 & \dfrac{\partial^2 \phi}{\partial x^2} \end{pmatrix} . \tag{7.26}$$

It is easily verified that $\nabla \cdot \mathbf{T} = 0$, and Newton's second law is thus satisfied in the absence of body forces. The only compatibility condition that is not trivially satisfied is equation (4.17):

$$2 \frac{\partial^2 E_{xz}}{\partial x \partial z} = \frac{\partial^2 E_{xx}}{\partial z^2} + \frac{\partial^2 E_{zz}}{\partial x^2} \ . \tag{7.27}$$

Using the stress-strain relationship for an isotropic elastic medium to represent the strain components in terms of ϕ results in

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial z^2} + \frac{\partial^4 \phi}{\partial z^4} = 0, \tag{7.28}$$

which is called the biharmonic equation.

Buckling of a Thin Beam

The problems in the previous sections always assumed that a given set of externally applied forces result in a unique response. Unfortunately, this is not always the case. You can easily convince yourself of this by trying to compress a thin plastic ruler with your hands, as in figure 7.4. If the ruler were exactly

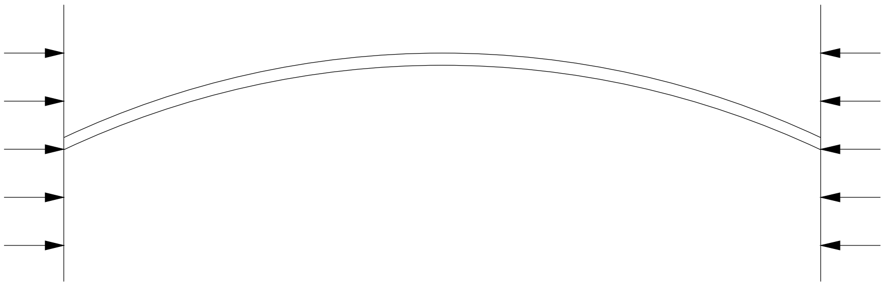

Figure 7.4. Buckling of a thin beam under compression.

straight and if the forces applied by your hands were centered exactly on the axis of the ruler, one would expect from the theory of elastic media that the ruler would remain straight while simply shortening a small amount. What in fact happens is that the ruler bows in one direction or another. Furthermore, the direction in which it bows depends on relatively minor effects, such as precisely how it is held when the force is initially applied. This is an example of the buckling of a thin beam, which we now treat quantitatively.

The approach we use to understand buckling is to compute the elastic potential energies of alternate responses of an elastic body to a given set of applied forces. Generally speaking, the body will take on the lowest energy configuration. For a thin beam in particular, we will compare the elastic potential energy of pure compression with that occurring in a bowed beam. To a good approximation, the bowed beam can be thought of as being in a state of pure bending.

Imagine a beam of rectangular cross section w by h, and with unstressed length l, as shown in figure 7.5. If the beam is compressed by forces F applied to the ends such that the length after compression is $l - \delta$, then the xx component of the stress (assuming that the x axis lies along the axis of the beam) is $T_{xx} = -\, F/(wh)$ and the corresponding strain component is $E_{xx} = -\, \delta/l$. Since the sides of the beam are assumed to be stress-free, we have $T_{xx} = E_Y E_{xx}$, so

$$\delta = \frac{lF}{whE_Y} \,, \tag{7.29}$$

and the potential energy of compression, U_c, is

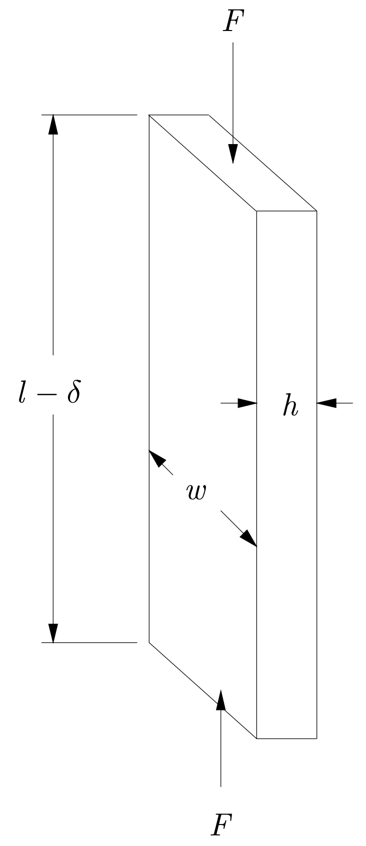

Figure 7.5. Definition sketch for buckling calculation.

$$U_c = \frac{E_{xx}T_{xx}\ lwh}{2} = \frac{whE_Y\delta^2}{2l}\,. \tag{7.30}$$

In the alternate configuration in which the beam is bowed, as shown in figure 7.6, we assume that the arc length of the bowed beam is unchanged from its unstressed value of l, while the cord length across the arc is $l-\delta$. If the arc is in the form of a segment of a circle of radius R, then from figure 7.6, $\theta = l/(2R)$. Furthermore, $\sin\theta = (l-\delta)/(2R)$. Assuming only a slight bow and expanding $\sin\theta \simeq \theta - \theta^3/6$, we find upon eliminating θ that $\delta = l^3/(24R^2)$. However, equation (7.22) relates the radius of curvature of a bent beam to the transverse gradient of longitudinal normal stress, C, by $R = E_Y/C$. Eliminating R and solving for C^2 yields $C^2 = 24E_Y^2\delta/l^3$.

We now compute the elastic potential energy in bowed state, U_b. If z is the coordinate across the beam in the plane of the bending, the potential energy is

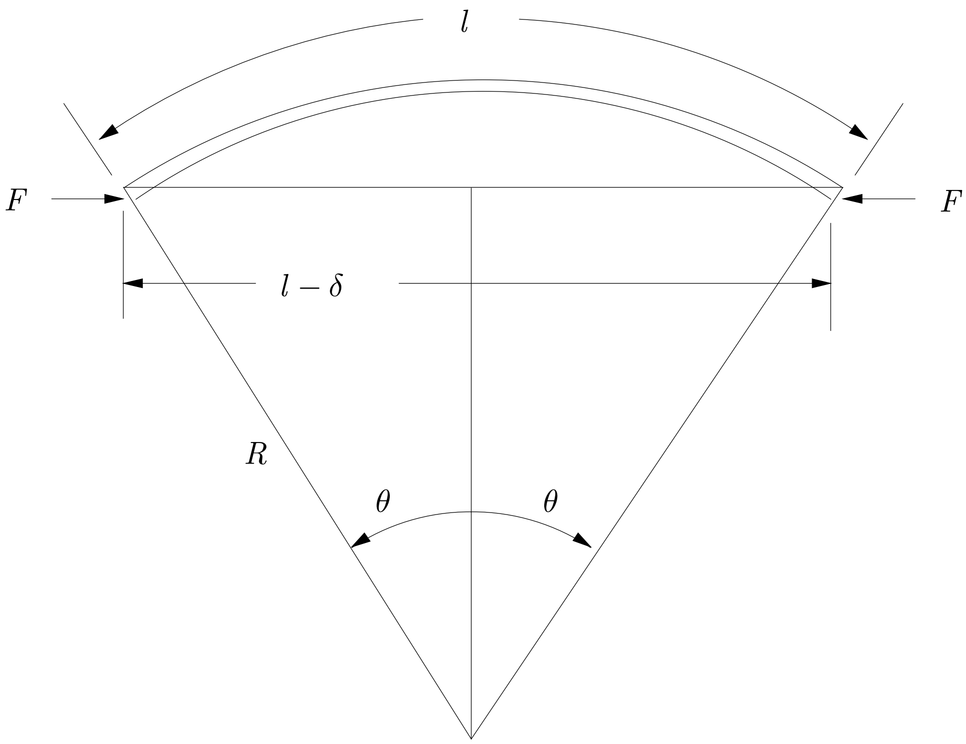

Figure 7.6. Sketch of relationship between the effective shortening, δ, of a beam undergoing buckling, and the radius of curvature, R, of the beam.

$$U_b = wl \int_{-h/2}^{h/2} (E_{xx} T_{xx}/2) dz = \frac{wl}{2E_Y} \int_{-h/2}^{h/2} T_{xx}^2 dz =$$

$$\frac{wlh^3 C^2}{24 E_Y} = \frac{wh^3 E_Y \delta}{l^2}\,, \tag{7.31}$$

where we have used $T_{xx} = Cz$.

Figure 7.7 summarizes these results. The potential energy of compression is quadratic in δ, and therefore starts out being less than the potential energy of bowing, which is linear in δ. However, as the applied force, and hence δ, increases, the compressional potential energy exceeds that for bowing at some point. Defining $\delta = \delta^*$ where the two are equal, we find that

$$\delta^* = 2h^2/l, \tag{7.32}$$

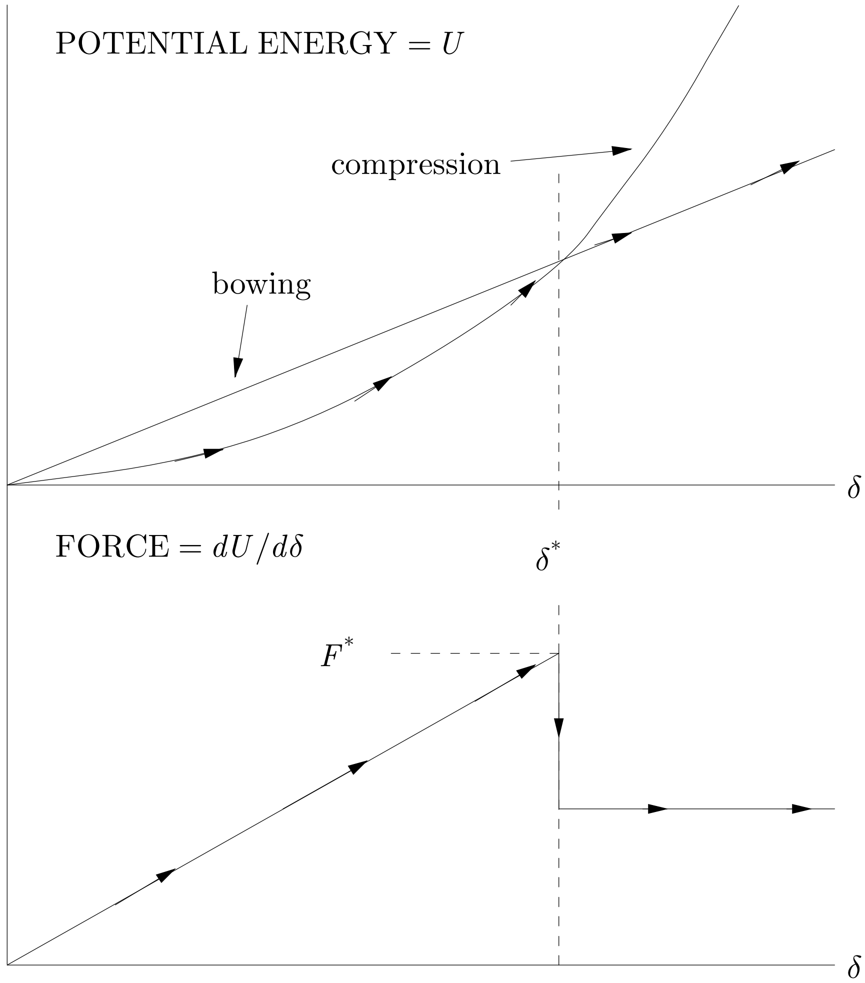

Figure 7.7. Potential energy of a beam shortened by an amount δ under the influence of compressive forces on the ends, and the force required to maintain this reduction in length. Note that at the transition from compression to buckling at $\delta = \delta^*$, the required compressive force drops by a factor of two.

and the corresponding force

$$F^* = 2E_Y wh^3/l^2 \; . \tag{7.33}$$

As the load slowly increases on the beam, the compressional mode is first favored, since it has the lower potential energy. However, as δ exceeds δ^*, the bowing mode becomes favored. Note that this can actually lead to catastrophic failure of the beam. From the work-energy theorem, $F\ d\delta = d(\mathit{total\ energy}) = dU + dK$, where U and K are respectively the kinetic and potential energies. The maximum force the beam can sustain in static equilibrium (i. e., with $K = 0$) is thus $dU/d\delta$, which undegoes an abrupt decrease at $\delta = \delta^*$. If the actual applied force doesn't undergo a corresponding decrease, static equilibrium is no longer possible, and the beam rapidly buckles and collapses. The question of buckling is thus one that needs to be given serious consideration in the design of structures.

Problems

1. Show that the displacement field arising from torsion on a cylinder (equation 7.9) is equivoluminal.

2. Show that the displacement field arising from the bending of a beam (equation 7.20) is neither equivoluminal nor irrotational.

3. In the special case in which the displacement field is irrotational, the displacement can be represented as the gradient of a scalar, i. e., $\mathbf{u} = \nabla\phi$. Show that equation (7.2) reduces to

$$(\lambda + 2\mu)\nabla^2\phi - \rho U = const$$

in this case.

4. A cubical block of material of constant density ρ and height h sits on a table under the influence of gravity. Assuming that the only applied traction is a uniform upward normal traction applied to the bottom of the block by the table, determine the distributions of stress, strain, and displacement field in the block.

5. Compute the elastic potential energy in a cylinder twisted through an angle α. The cylinder has length l radius R, and shear modulus μ.

6. Consider the Airy stress function $\phi = Ax^2z$ in a beam that extends indefinitely in the $\pm y$ directions, where A is a constant. The lateral faces of the beam are at $x = 0, a$ and $z = 0, b$. a) Show that ϕ satisfies the biharmonic equation. b) Compute the stress tensor. c) Compute the tractions that need to be applied to the lateral faces of the beam to match the stresses there.

7. We desire to make as long a beam as possible to support a given compressional force, F. However, we are limited to a mass M of material of density ρ and Young's modulus E_Y. If the beam has a square cross-section, how long can it

be made without danger of buckling?

Chapter 8 -- Newtonian Fluids

We now begin the study of fluid dynamics. At rest, a fluid has a very simple stress tensor, $\mathbf{T} = -\,p\mathbf{I}$. This is a consequence of the inability of a fluid to support a shear stress while at rest. There can be no off-diagonal components of the stress tensor in these circumstances, which is why the stress tensor must take the form of a scalar times the unit tensor. This scalar is minus the pressure, the minus sign being a matter of convention -- a positive pressure corresponds to a compressional normal stress, which is negative.

A Newtonian fluid is an isotropic fluid that has a linear relationship between the rate of strain and that part of the stress related to the motion of the fluid. As for isotropic elastic media, two independent constants are sufficient to characterize this part of the stress-strain relationship. However, unlike the case of ideal elastic media, we also need an equation of state to define the static relationship between pressure and fluid density.

In this chapter we first develop the Navier-Stokes equation, which is the specialization of Newton's second law to the case of a Newtonian fluid. We then investigate the equations of state for two idealized cases, namely an incompressible fluid and an ideal gas. We then learn about the role of energy in a Newtonian fluid, and we study fluid statics, a subject that is much simpler than the statics of elastic media. Finally, we investigate when terms involving viscosity and compressibility are important.

In chapter 3 we derived an equation for mass continuity. Mass continuity plays an important role in fluid mechanics, and so equation (3.11),

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \ , \tag{8.1}$$

must be included in the set of equations to be solved in any fluid dynamics problem.

Navier-Stokes Equation

The starting point for our discussion of the Navier-Stokes equation is equation (3.21),

$$\rho \frac{d\mathbf{v}}{dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{B} \;, \tag{8.2}$$

which results from considerations of continuity of momentum. The problem is then to derive the motion-dependent part of the stress tensor in terms of the fluid velocity.

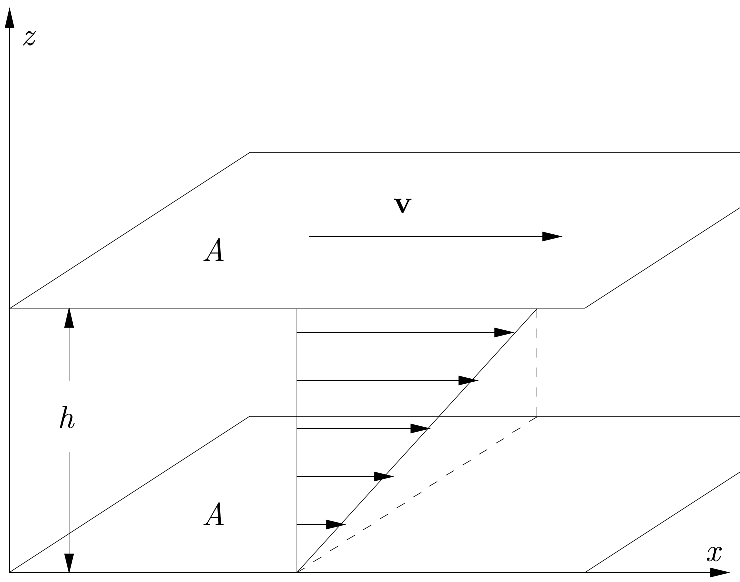

Figure 8.1. Sketch of experiment to determine the resistance of a fluid to shearing deformation.

Figure 8.1 shows a sketch of an experiment to determine the resistance of a fluid to a shearing deformation. Two parallel plates, each of area A and separated by a distance h, are moved relative to each other with a speed v. If the plates are immersed in a fluid, it is found that the fluid generates a drag force of magnitude F between the plates. For most fluids it is found (if the plates are sufficiently close together and not moving too fast relative to each other) that F is directly proportional to v and A, and inversely proportional to h. It is also found that the fluid is linearly sheared between the plates, with fluid immediately adjacent to each plate moving at the speed of the plate.

Two lessons arise out of this experiment. First, it shows that fluids obey what is known as a no-slip boundary condition -- fluid adjacent to a surface moves with the surface. Second, it shows that a linear relationship holds between the shear stress and the fluid shear in unidirectional flow:

$$\frac{F}{A} = T_{xz} = \mu \frac{\partial v_x}{\partial z} \ . \tag{8.3}$$

The constant of proportionality, μ, is called the *coefficient of viscosity*, or simply the *viscosity* for short, and is not to be confused with the shear modulus of elastic body mechanics.

The shear, $\partial v_x/\partial z$, can be related to a component of the rate of strain, since $v_z = 0$ in this case: $\partial v_x/\partial z = 2D_{xz}$. Thus, equation (8.3) can also be written $T_{xz} = 2\mu D_{xz}$.

Generalization of this relationship to arbitrary flow patterns of an isotropic fluid can be accomplished in the same way that we derived the stress-strain relationship for isotropic, elastic solids. In particular, if we assume a general linear relationship between components of the stress tensor and components of the rate of strain tensor, the condition of isotropy forces this into a tensor relationship involving only two arbitrary constants, μ and η:

$$T_{ij} = -\, p\delta_{ij} + (\eta - 2\mu/3)\delta_{ij} D_{kk} + 2\mu D_{ij} \ . \tag{8.4}$$

The first term on the right side of the above equation represents the contribution from the fluid at rest. Furthermore, the xz component of this equation is equivalent to equation (8.3) for the special case of sheared unidirectional flow. However, the equation contains an additional term that doesn't enter into the static case, or in the case in which fluid elements don't change their volume. This term involves the constant η, which is sometimes called the *second coefficient of viscosity*. Since the trace of the rate of strain tensor is the fractional rate of change of the volume of a parcel, which in turn is minus the fractional rate of change of parcel density, taking the trace of equation (8.4) results in

$$-\, T_{ii}/3 = \left(p + \eta \frac{d \ln \rho}{dt} \right) \ . \tag{8.5}$$

The quantity p is the thermodynamic pressure that appears in such things as the ideal gas law. We see that η is a measure of the difference between minus the mean normal stress, $-\, T_{ii}/3$, and the pressure in situations in which a fluid is compressing or expanding. In particular, when a fluid is compressing,

$d \ln \rho / dt > 0$, and minus the mean normal stress is enhanced over p as long as η is positive. This provides additional resistence to compression. On the other hand, when $d \ln \rho / dt < 0$, the fluid is expanding, and the η term opposes the pressure in driving the expansion. The effect of η is therefore to introduce dissipation into expansions and contractions of fluids, just as μ causes dissipation in shearing motions. The quantitative effects of these terms on fluid mechanical energy will be explored later in this chapter. The coefficient of the second term on the right side of equation (8.4) is written as $\eta - 2\mu/3$ simply to isolate the two effects from each other.

Equation (8.4) may be written in terms of the velocities as follows:

$$T_{ij} = - \, p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + (\eta - 2\mu/3) \, \frac{\partial v_k}{\partial x_k} \, \delta_{ij} \; . \qquad (8.6)$$

Substituting this into equation (8.2) yields, after a bit of index manipulation,

$$\rho \, \frac{d\mathbf{v}}{dt} = - \, \nabla[p - (\eta + \mu/3) \nabla \cdot \mathbf{v}] + \mu \nabla^2 \mathbf{v} + \rho \mathbf{B} \; . \qquad (8.7)$$

This is called the Navier-Stokes equation, and forms the basis of fluid dynamics. For a fluid that is incompressible, $\nabla \cdot \mathbf{v} = 0$, and the term involving $\eta + \mu/3$ vanishes.

Equations of State

Strictly speaking, no matter is incompressible. However, it is sometimes a good approximation, particularly for liquids, to assume that incompressibility holds. As noted in chapter 3, the condition of incompressibility is simply

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0 \; , \qquad (8.8)$$

and mass continuity becomes

$$\nabla \cdot \mathbf{v} = 0 \; . \qquad (8.9)$$

Equations (8.7), (8.8), and (8.9) impose five constraints (in three dimensions), and are therefore sufficient to solve for the five unknowns v_x, v_y, v_z, p, and ρ. When the fluid is not only incompressible, but also homogeneous, the density, ρ, is constant, equation (8.8) becomes trivial, and we have four equations in four unknowns. This is the simplest of all fluid dynamical situations, and is applicable in many cases of flowing liquids.

When the fluid is an ideal gas, we must have recourse to thermodynamics. The ideal gas law for a volume V of gas is

$$pV = nRT , \tag{8.10}$$

where p is the pressure, T is the temperature, n is the number of moles of gas, and R is the gas constant. This can be written more conveniently for fluid dynamical applications by dividing both sides by the mass M of the parcel of gas:

$$\frac{p}{\rho} = \frac{RT}{m} , \tag{8.11}$$

where M/V is recognized as the density, ρ, and $m = M/n$ is the mass per mole of gas, or the molecular weight. Equation (8.11) introduces one additional constraint on the solution, but unfortunately also introduces a new variable, the temperature. Since the temperature itself can vary in a compressible fluid flow, yet another constraint is needed.

The first and second laws of thermodynamics provide an approximate answer that is valid in many situations. The first law relates the heat added to a parcel, dQ, and the work done by it, dW, to the change in internal energy, dU:

$$dU = dQ - dW . \tag{8.12}$$

For reversible changes, the work done by a parcel of gas is simply $dW = p\ dV$, where dV is the change in parcel volume, while the heat added is related to the change in entropy, dS, of the parcel by $dQ = T\ dS$. The change in internal energy of an ideal gas is related to the temperature change by $dU = nC_v\ dT$, where C_v is the specific heat of the gas at constant volume per mole and n is again the number of moles in the parcel.

If $ds = dS/n$ is the entropy change per mole, or *specific entropy*, equation can be solved for ds and written in the form

$$ds = C_v \frac{dT}{T} - \frac{R}{m}\frac{d\rho}{\rho} \tag{8.13}$$

where equation (8.11) has been invoked. This is in the form of an exact differential, and can be integrated to

$$s = s_0 + C_v \ln(T/T_0) - \frac{R}{m} \ln(\rho/\rho_0) , \tag{8.14}$$

where s_0 is the specific entropy at temperature T_0 and density ρ_0. The temperature can further be eliminated between equations (8.11) and (8.14) resulting in

$$s = s_0 + C_v \ln(p/p_0) - C_p \ln(\rho/\rho_0) , \tag{8.15}$$

where $p_0 = RT_0\rho_0/m$ and $C_p = C_v + R/m$ is the specific heat at constant pressure.

The second law of thermodynamics states that the entropy of a closed system never decreases. Furthermore, if parcel transformations are close to being reversible adiabatic, and if very little heat flows into or out of parcels, entropy is nearly conserved. Thus, we can often use the approximate equation

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s \simeq 0 \tag{8.16}$$

to describe the evolution of the field of specific entropy. In this case equations (8.1), (8.7), (8.15), and (8.16) represent six constraints on the six unknowns v_x, v_y, v_z, p, ρ, and s.

Kinetic Energy in Fluids

In elastic body mechanics the only sources of energy are the applied tractions. Stress forces simply redistribute energy within the elastic material, and in the elastic idealization there is no dissipation or interaction with internal (i. e., thermal) energy. The situation is very different in fluid mechanics. The pressure part of the stress redistributes energy as in elastic body mechanics, but the viscous parts cause dissipation of mechanical energy into heat. In addition, internal energy can be converted into mechanical energy via the pressure part of the stress. Indeed, if this weren't so, heat engines such as steam turbines and automobile engines wouldn't work. Accounting for energy flows is thus more complicated in fluid dynamics than it is in elastic body mechanics.

We are able to learn a great deal about energy flows in Newtonian fluids by developing an equation for the budget of kinetic energy. We do this by dotting equation (8.2) with the velocity and then manipulating it into a useful form with the use of equation (8.4) and (4.24). Two tricks are used in this derivation. First of all, $\mathbf{v}\cdot\rho(d\mathbf{v}/dt)$ can be written $\rho d(\mathbf{v}\cdot\mathbf{v}/2)/dt = \rho d(v^2/2)/dt$. However, since mass continuity states that $d\rho/dt + \rho\nabla\cdot\mathbf{v} = 0$, we can multiply the left side of this equation by $v^2/2$ and add it to the above term without changing its value:

$$\rho\,\frac{dv^2/2}{dt} + (v^2/2)\,\frac{d\rho}{dt} + (v^2/2)\rho\nabla\cdot\mathbf{v} =$$

$$\frac{d(\rho v^2/2)}{dt} + (v^2/2)\rho\nabla\cdot\mathbf{v} =$$

$$\frac{\partial(\rho v^2/2)}{\partial t} + \mathbf{v}\cdot\nabla(\rho v^2/2) + (v^2/2)\rho\nabla\cdot\mathbf{v} =$$

$$\frac{\partial(\rho v^2/2)}{\partial t} + \nabla\cdot[(\rho v^2/2)\mathbf{v}]\ . \tag{8.17}$$

We recognize $\rho v^2/2$ as the kinetic energy density and $(\rho v^2/2)\mathbf{v}$ as the flux of kinetic energy via mass transport. It is therefore clear that this process is going to lead to a continuity equation for kinetic energy.

The second kind of trick is used on terms like $\mathbf{v}\cdot\nabla p$. Since $\nabla\cdot\mathbf{v}p = \mathbf{v}\cdot\nabla p + p\nabla\cdot\mathbf{v}$, we can write

$$\mathbf{v}\cdot\nabla p = \nabla\cdot\mathbf{v}p - p\nabla\cdot\mathbf{v}\ . \tag{8.18}$$

Similar tricks can be performed on the viscosity terms, with the final result that

$$\frac{\partial(\rho v^2/2)}{\partial t} + \nabla\cdot[(\rho v^2/2)\mathbf{v} + p\mathbf{v} - 2\mu\mathbf{v}\cdot\mathbf{D} - (\eta - 2\mu/3)(\nabla\cdot\mathbf{v})\mathbf{v}] =$$

$$p(\nabla\cdot\mathbf{v}) - 2\mu D_{ij}D_{ij} - (\eta - 2\mu/3)(\nabla\cdot\mathbf{v})^2 + \rho\mathbf{v}\cdot\mathbf{B}\ . \tag{8.19}$$

Let us now try to understand this equation. The first term on the left side is simply the time rate of change of kinetic energy density at a point. The terms inside the square brackets constitute a kinetic energy flux, the divergence of which leads to the deposition or removal of kinetic energy. The important point is that these terms simply move kinetic energy around without creating or destroying it. This can be shown by integrating equation (8.19) over some volume and applying the divergence theorem:

$$\int (\rho v^2/2)\ dV + \int [\cdots]\cdot\mathbf{n}\ dA = \cdots\ . \tag{8.20}$$

This states that the volume integral of the kinetic energy density, i. e., the total kinetic energy in the volume, is only affected by the contents of the square brackets at the surface bounding the volume. Furthermore, the integrand of the surface integral is simply the component of the kinetic energy flux parallel to the unit normal, $\mathbf{n}$.

The kinetic energy flux is much more complicated than the mass flux because kinetic energy can be converted to and from other types of energy. The first term within the square brackets in equation (8.19) is simply mass transport of kinetic energy. However the second term, involving the pressure, corresponds to transport associated with work done by one part of the fluid on another by the pressure force. The third and fourth terms represent work done on one part of the fluid by another via the viscous stresses.

The right side of equation (8.19) represents ways in which kinetic energy may be added to or subtracted from a volume of fluid other than by flowing through the walls of the volume. Since $\nabla \cdot \mathbf{v}$ is the fractional time rate of change of parcel volume, the first term on the right side is simply work per unit volume done by pressure forces, and represents conversion of internal energy into kinetic energy. An example would be the free expansion of a gas released from a container into a vacuum. The second and third terms involving the two coefficients of viscosity are always negative, and represent dissipation of kinetic energy into heat. The fourth term is simply the work done by external body forces on the fluid.

The energy budget is simpler if the fluid is incompressible. Setting $\nabla \cdot \mathbf{v} = 0$ eliminates all effects of the second coefficient of viscosity. More importantly, it eliminates the possibility of converting internal energy into mechanical energy, since the first term on the right side of equation (8.19) then vanishes. This marks the primary difference between compressible and incompressible flow.

Statics of Fluids

The static behavior of fluids is much simpler than the static behavior of elastic media. This is due to the inability of ordinary fluids to sustain a shear stress in static equilibrium, and the consequent reduction of the stress tensor to a very simple form. In static equilibrium mass continuity is trivially satisfied and the

Navier-Stokes equation reduces to

$$\nabla p = \rho \mathbf{B} \ . \tag{8.21}$$

For the case in which $\mathbf{B}$ is derivable from a potential U, equation (8.21) becomes

$$\nabla p = -\ \rho \nabla U \ . \tag{8.22}$$

For a homogeneous, incompressible fluid in which ρ is a constant, this reduces to

$$p = -\ \rho U + p_0 \qquad \text{(constant density)} \ , \tag{8.23}$$

where p_0 is the pressure where $U = 0$. Thus, the pressure is greater deeper in the potential well, and the pressure gradient force balances the body force. For the case of a constant gravitational field, $U = gz$, equation (8.23) reduces to the usual elementary hydrostatic relationship for a fluid of constant density.

The law of Archimedes specifies the upward *buoyancy force* on a body immersed in a constant density fluid in a uniform gravitational field as the weight of the displaced fluid. This is easy to derive from our knowledge

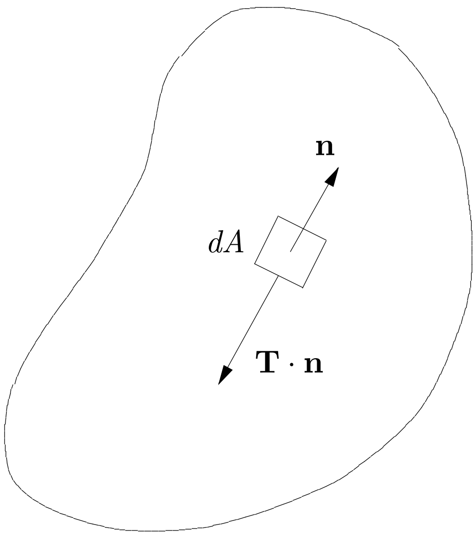

Figure 8.2. Illustration of the traction vector, $\mathbf{T} \cdot \mathbf{n}$, in a fluid at rest. This represents a normal compressional force on the fluid.

of the stress tensor. As figure 8.2 shows, the traction of the fluid on the body is simply $\mathbf{t} = \mathbf{T} \cdot \mathbf{n} = -\ p\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector pointing outward from the surface of the body. The total stress force on the body is thus

$$\mathbf{F} = -\int p\mathbf{n}\, dA = -\int p\mathbf{I}\cdot\mathbf{n}\, dA = \int (\rho g z - p_0)\mathbf{I}\cdot\mathbf{n}\, dA\ . \tag{8.24}$$

Applying the divergence theorem to the right side of this equation yields

$$\mathbf{F} = \int \nabla\cdot[(g\rho z - p_0)\mathbf{I}]dV = g\rho V\mathbf{k}\ , \tag{8.25}$$

where V is the volume of the body. Since $g\rho V$ is the weight of fluid with volume equal to the volume of the body, Archimedes' law is verified.

When a fluid is not of constant density, equation (8.22) is not integrated so easily. However, taking the curl of this equation shows that

$$\nabla\rho \times \nabla U = 0\ . \tag{8.26}$$

In other words, surfaces of constant density and potential must everywhere coincide. From equation (8.22), the pressure gradient and potential gradient point in the same direction, so these surfaces are also surfaces of constant pressure. Thus, pressure and density can be written as functions of potential alone: $p = p(U)$; $\rho = \rho(U)$. Finally, the density may be written as a function of pressure alone as well: $\rho = \rho(p)$.

In many cases, these functions will be single-valued. However, if isolated potential wells exist, than separate pressure-density-potential relationships can exist for the separate wells. For instance, imagine the situation shown in figure 8.3, in which a fluid of variable density is bounded underneath by an undulating surface. The horizontal lines represent surfaces of constant pressure, and their spacing is inversely proportional to the fluid density according to equation (8.22). Above the undulating bottom the spacing of pressure surfaces doesn't vary horizontally. However, within different depressions in the bottom, the spacing of constant pressure surfaces, and the consequent density, is different. For example, if we are talking about the ocean, the right depression in figure 8.3 may contain extremely saline water from a sub-oceanic spring, whereas the left depression contains ordinary ocean water. Density is greater with higher salinity, and there is a greater increase in pressure with depth in the right depression.

For an ideal gas, if density and pressure are constant on surfaces of constant potential, the temperature and entropy must also be constant on these surfaces, since both these quantities are uniquely specified by the pressure and density.

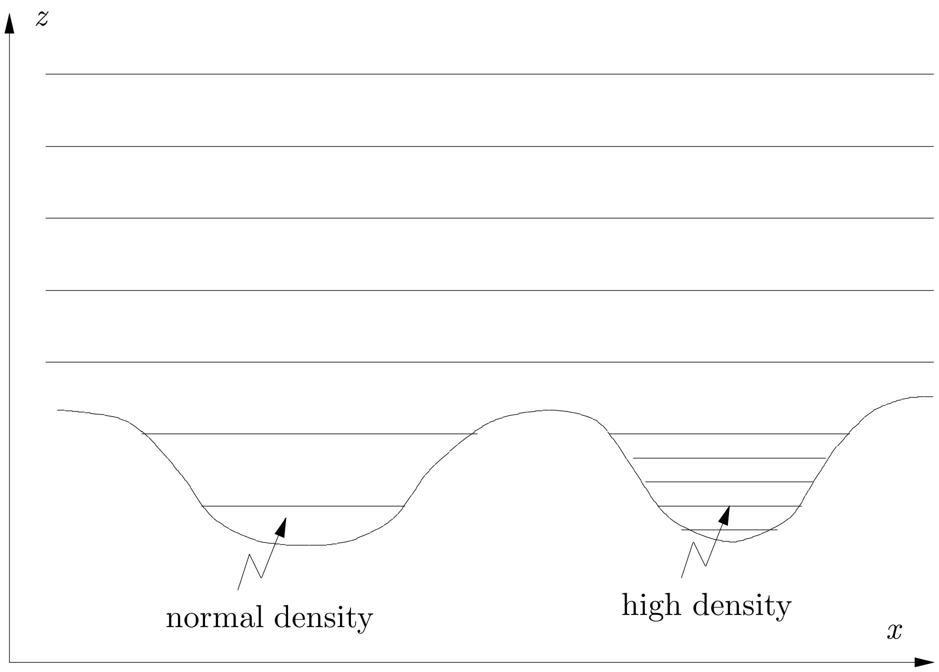

Figure 8.3. Contours of constant density of a fluid at rest in a container with an undulating bottom. Fluid parcels at the same level in "valleys" isolated from each other can have different densities. This cannot occur for parcels above the "ridgeline".

For an ideal gas at constant temperature, the ideal gas law, given by equation (8.11), may be combined with equation (8.22) to yield

$$\nabla p = -\frac{mp}{RT}\nabla U \, , \tag{8.27}$$

which may be integrated to yield

$$p = p_0 \exp\left(-\frac{mU}{RT}\right), \qquad T = constant, \tag{8.28}$$

where p_0 is again the pressure at $U = 0$. If $U = gz$, then the pressure decreases exponentially with height in proportion to $\exp(-z/z_s)$, where $z_s = RT/(mg)$ is called the scale height.

Reynolds Number

Under some circumstances the viscous terms in the Navier-Stokes equations (i. e., those involving μ and η) can be ignored, whereas in other circumstances they dominate the evolution of the flow. The art of deciding when certain terms can be safely discarded in the Navier-Stokes equations is aided by a process of estimation called *scale analysis*. In this process, dependent variables are replaced by estimates of their typical value, and space and time derivatives are replaced by the inverse of estimates of typical space and time scales respectively. The magnitudes of various terms in the equation of interest are then compared.

Let us investigate the magnitude of various terms in the Navier-Stokes equation for a homogeneous, incompressible fluid, i. e., one with $\nabla \cdot \mathbf{v} = 0$ and ρ constant. Ignoring body forces, equation (8.7) becomes

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} \,. \tag{8.29}$$

If velocities have a typical magnitude of V, and the flow structure of interest has a spatial scale L, the time for a parcel to advect through the flow pattern is of order $T = L/V$. If this is the characteristic time scale of the flow pattern, then the ratio of the acceleration and viscous terms in equation (8.29) is

$$\mathrm{R} = \frac{\left(\dfrac{\rho V}{L/V}\right)}{\left(\dfrac{\mu V}{L^2}\right)} = \frac{\rho V L}{\mu} \,. \tag{8.30}$$

R is a dimensionless number called the Reynolds number after Osborne Reynolds, a 19th century British scientist and engineer. If $\mathrm{R} >> 1$, viscous terms can be ignored, whereas if $\mathrm{R} << 1$, the acceleration can be neglected.

Notice all the caveats in this estimation process. The conditions of incompressibility and absence of body forces are easily removed, as is the constant density condition. However, these conditions do indicate that the notion of Reynolds number does apply primarily in those instances in which the essential character of the flow does not depend on compressibility or the existence of external forces. More important is the notion that there is only one length scale in the problem, a condition that is easily and frequently violated. Under these circumstances the appropriate length is some combination of the two or more length scales in the problem. Alternatively, it may be appropriate to define multiple Reynolds numbers. Finally, there is the assumption that the primary source of temporal variation in the velocity of parcels of fluid is movement of the fluid through some established flow pattern. This assumption can be violated in at least one way. If the flow is evolving on a time scale more rapidly than L/V, the ratio of the acceleration and viscous terms will be larger than indicated by the Reynolds number as conventionally defined. An example of this would be any wave phenomenon in which the wave speed, C, is much greater than the material speed, V. In this case, a better estimate of the ratio of the two terms would be $\mathrm{R} = \rho CL/\mu$.

The typical instance in which two length scales are important is where fluid flows by a rigid body. Figure 8.4 shows flow past a thin plate that is aligned with the flow. The flow is parallel far upstream and has a uniform speed V there. Flow near the plate is not uniform because fluid immediately adjacent to the stationary plate must have zero velocity. A sheared flow ensues near the plate, resulting in a tangential viscous stress, which slows the fluid in the vicinity of the plate. The region of thickness D in which the fluid has been significantly slowed as called the *boundary layer*. In order for this to happen, the mean deceleration of parcels as they pass by the plate must be of order $V/T = V/(L/V) = V^2/L$, where L is the distance from the point under consideration to the front of the plate. If this deceleration is caused primarily by the viscous force, then this force must be equated to the deceleration. In other words, the ratio of acceleration to viscous force per unit mass, which is precisely the Reynolds number, must be of order one.

The viscous force per unit mass may be estimated as $\mu V/(\rho D^2)$, since the strongest gradients, which contribute most heavily to ∇^2, are perpendicular to the plate. The Reynolds number therefore becomes

$$\mathrm{R} = \frac{\rho V D^2}{\mu L} \tag{8.31}$$

in this case. Setting $\mathrm{R} = 1$ allows us to use equation (8.31) to estimate the depth of the boundary layer as a function of distance back from the leading edge of the plate:

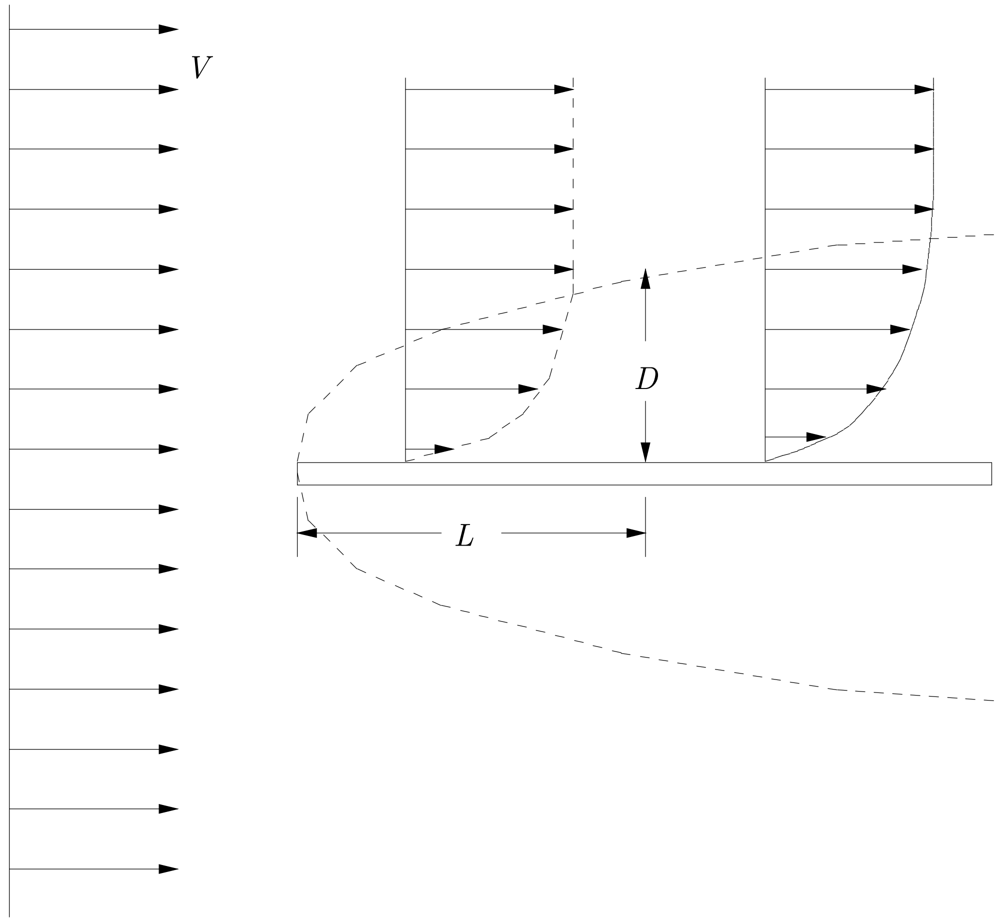

Figure 8.4. Initially uniform flow past a flat plate aligned with the flow. A boundary layer forms next to the plate, in which the fluid velocity is reduced from its initial value. The boundary layer thickens with increasing distance behind the leading edge of the plate.

$$D \simeq \left(\frac{\mu L}{\rho V}\right)^{1/2} . \tag{8.32}$$

The boundary layer thickness increases as the square root of L, as indicated in figure (8.4).

Two lessons can be learned from the above analyses. First, it is not safe to pick length, time, or velocity scales arbitrarily, plug them into equation (8.30), and expect the resulting Reynolds number to mean anything. Some thought has to be given as to how changes in parcel velocity actually come about in order to define a Reynolds number that truly reflects the importance of viscosity. The second lesson is that in estimating the values of variables, don't worry about numerical coefficients. For instance, in the above estimation of the parcel deceleration as it passes the plate, the average velocity would be somewhat less than the free stream velocity, V. The transit time, T, would therefore be somewhat greater than L/V. This defect could in principle be rectified by inserting a numerical coefficient of the proper size in equation (8.31), but it would not be worth the effort. The purpose of scale analysis is to obtain the *functional form* of relationships, for example, the square root dependence of D on L. Obtaining numerical coefficients should be left to computers.

Sound Waves in an Ideal Gas and Mach Number

We now obtain a solution to the Navier-Stokes equation for small amplitude sound waves in a homogeneous ideal gas that is otherwise at rest. This will then lead to a scale analysis that reveals when the compressibility of a gas needs to be considered. For now we ignore the effects of viscosity, which tend to be small in many cases involving gas flow.

A homogeneous ideal gas will have constant entropy. If the entropy is constant, equation (8.15) can be solved for the pressure in terms of the density and a constant K containing the entropy:

$$p = K\rho^{\gamma} \ . \tag{8.33}$$

The constant $\gamma = C_p/C_v$ is the ratio of the specific heats at constant pressure and at constant volume for the gas. It respectively takes on the values 1.67 and 1.4 for ideal monatomic and diatomic gases.

Ignoring viscosity and body forces, the Navier-Stokes equation (8.7) takes the form

$$\frac{d\mathbf{v}}{dt} + \frac{\gamma p}{\rho} \nabla\chi = 0 \tag{8.34}$$

where $\chi = \ln \rho$, and pressure has been eliminated in favor of density using equation (8.33). In deriving this equation, we have used $\nabla p = \gamma\rho^{\gamma-1} K \nabla\rho$ and then eliminated K using equation (8.33) again. Similarly, the mass continuity equation can be written in terms of χ as

$$\frac{d\chi}{dt} + \nabla \cdot \mathbf{v} = 0 \ . \tag{8.35}$$

In both these equations the relationship $d\chi = d \ln \rho = d\rho/\rho$ has been used.

For small amplitude sound waves in a medium at rest, parcels of fluid will not move far from their original positions. Therefore, the total time derivatives can be replaced by partial derivatives. Also, deviations in pressure and density from constant ambient values, p_0 and ρ_0, will not be large, so we can set $\gamma p/\rho = \gamma p_0/\rho_0 \equiv c^2$. The resulting equations

$$\frac{\partial \mathbf{v}}{\partial t} + c^2 \nabla \chi = 0 \tag{8.36}$$

$$\frac{\partial \chi}{\partial t} + \nabla \cdot \mathbf{v} = 0 \tag{8.37}$$

are linear with constant coefficients, and can be combined into a single equation for χ by taking the divergence of equation (8.36) and eliminating $\nabla \cdot \mathbf{v}$ with equation (8.37):

$$\frac{\partial^2 \chi}{\partial t^2} - c^2 \nabla^2 \chi = 0 \ . \tag{8.38}$$

This is simply the wave equation for waves with speed c. Note that using the ideal gas law, $c = (\gamma R T_0/m)^{1/2}$, where T_0 is the ambient temperature of the gas.

Note that from equation (8.36) we find that

$$\mathbf{v} = - \int c^2 \nabla \chi \, dt = - \nabla \int c^2 \chi \, dt \ . \tag{8.39}$$

The flow due to sound waves is therefore irrotational, as it is derived from a scalar potential. Thus, as for irrotational seismic waves, plane sound waves are longitudinal. Furthermore, sound waves are fundamentally dependent on energy transfers due to compression, and therefore cannot exist in an incompressible medium.

The energy equation (8.19) helps us understand the relative importance of compressional effects in fluids. In a fluid with no body forces and negligible viscosity, this equation can be written

$$\frac{\partial(\rho v^2/2)}{\partial t} + \nabla \cdot \mathbf{A} = - p \, \frac{d\chi}{dt} \tag{8.40}$$

with the help of equation (8.35), where $\mathbf{A} = (\rho v^2/2 + p)\mathbf{v}$ represents transfer of kinetic energy from parcel to parcel in the fluid by mass transport and pressure forces. Since this term simply moves energy around, integration over the entire volume of fluid dispenses with it, leaving pressure work as the only source of kinetic energy. If N is an estimate of fluid density, P of pressure, V of velocity, and X of the log density deviation, χ, then equation (8.40) implies

$$\frac{NV^2}{T} = \frac{PX}{T}\,, \tag{8.41}$$

where T is the characteristic time scale of the flow. Thus,

$$X = \frac{NV^2}{P} = \frac{V^2}{c^2} \equiv \mathrm{M}^2\,, \tag{8.42}$$

where we have used the result that the square of the sound speed $c^2 = \gamma p/\rho$. The dimensionless quantity $M = V/c$ is called the Mach number. If $M << 1$, the above equation shows that the fractional compression and expansion occurring in an ideal gas is small. Under these circumstances it is valid to assume that the flow is incompressible even though the fluid is a highly compressible gas, except, of course, when consideration of sound waves is explicitly desired.

Problems

1. Derive equation (8.19) using the hints presented in the text.

2. A sphere of gas in space is initially at rest and has initial radius R, mass M, and pressure p. After it expands 1% in volume, find the total kinetic energy and the root mean squared average fluid velocity. Ignore viscous effects and body forces. (Hint: Integrate the kinetic energy equation over a volume larger than the volume of the sphere, noting that the kinetic energy flux is zero at the boundary of this volume. For a 1% expansion the change in pressure and density can be ignored. For the purposes of this problem you may assume that the pressure is uniform inside the sphere of gas, and zero outside it.)

3. Derive the hydrostatic pressure as a function of height for an ideal gas of constant specific entropy, s, in a constant gravitational field.

4. If a small parcel of gas in an isothermal atmosphere in a constant gravitational field undergoes an adiabatic vertical displacement, show that the resultant of

buoyancy and gravity forces act to return the parcel to its initial level. If these are the only two forces acting, find the frequency of small vertical oscillations of a parcel about its equilibrium level. (Hint: Assume that the pressure in the parcel is the same as that in the surrounding gas at the same level, and note that both the density of the parcel and the surrounding gas change with height.)

5. Assume that a spherical planet of radius a and surface gravity g has an isothermal atmosphere of temperature T and molecular weight m. If the surface pressure is p_s, find the pressure as a function of height. Does the pressure go to zero an infinite distance from the planet?

6. Assume that some fluid flow has characteristic space and time scales L and T, and a characteristic velocity V. (Don't necessarily assume that $V = L/T$.) Do a scale analysis on the two parts of the parcel acceleration,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \ ,$$

and define a dimensionless number A which is the ratio of the time derivative part to the space derivative part. Under what circumstances may one or the other of these terms be neglected?

7. A liquid compresses slightly under the effect of pressure, with the relationship $\rho = \rho_0 + \kappa p$, where κ is a constant, p is the pressure, ρ is the density, and ρ_0 is the density at zero pressure. Find the speed of sound in this liquid. (Hint: You will need to rederive equation (8.34) for the new pressure-density relation.)

Chapter 9 -- Creeping Flow

In the previous chapter we discovered that the term $\rho(d\mathbf{v}/dt)$ can be ignored when the Reynolds number is very much less than unity. In this circumstance, the Navier-Stokes equation reduces to the simple relationship

$$\nabla p = \mu \nabla^2 \mathbf{v} \ , \tag{9.1}$$

where we have ignored body forces and the viscous terms involving $\nabla \cdot \mathbf{v}$. For most cases of creeping flow, this is justified, since the fluid can be considered incompressible, thus making the continuity equation

$$\nabla \cdot \mathbf{v} = 0 \ . \tag{9.2}$$

Low Reynolds number flow is often called creeping flow, since in most every day examples the fluid tends to ooze along, such as when pouring syrup out of a bottle.

Plane Couette Flow

The simplest example of creeping flow occurs when fluid is confined to a channel between two parallel plates, one of which is stationary, the other moving. Figure 9.1 shows the configuration with two plates separated by a distance d and the upper plate is moving at a speed v_0. The flow velocity in this case changes from the velocity of the lower plate (zero) to that of the upper plate as one crosses the channel, due to the no-slip boundary condition, i. e.,

$$\mathbf{v} = v_0(z/d)\mathbf{i} \ . \tag{9.3}$$

In this example, the flow requires no pressure gradient to drive it. Under the condition $\nabla p = 0$ it is easy to verify that equation (9.3) satisfies equations (9.1) and (9.2).

The strain rate tensor associated with this flow is easily calculated from equation (4.24) to be

$$D_{ij} = \begin{pmatrix} 0 & 0 & v_0/2d \\ 0 & 0 & 0 \\ v_0/2d & 0 & 0 \end{pmatrix} . \tag{9.4}$$

Using equation (8.4), we then find the stress tensor in the fluid:

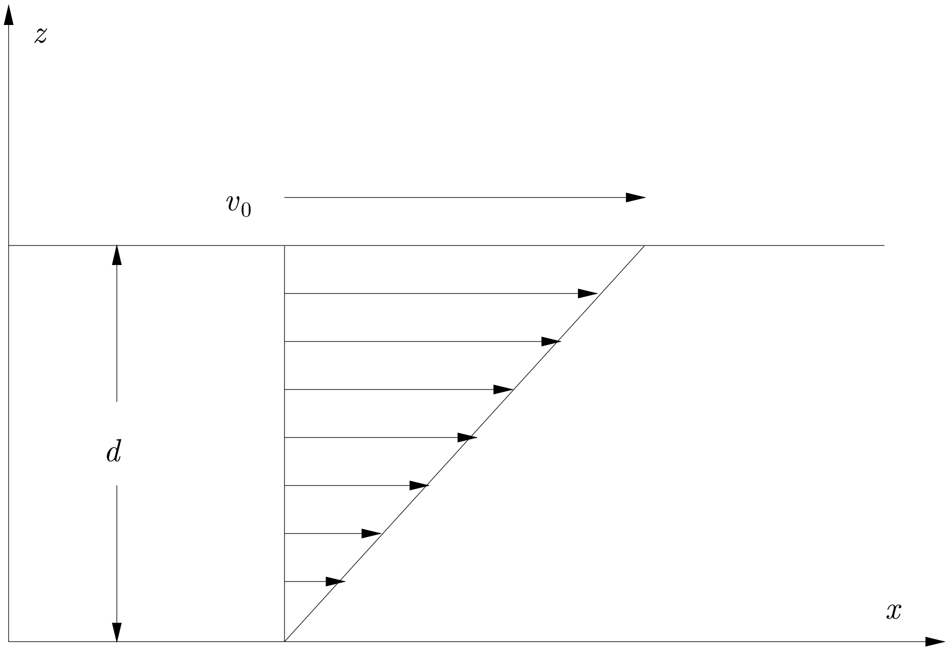

Figure 9.1. Creeping flow between two parallel plates, one of which is moving.

$$T_{ij} = \begin{pmatrix} 0 & 0 & \mu v_0/d \\ 0 & 0 & 0 \\ \mu v_0/d & 0 & 0 \end{pmatrix} . \tag{9.5}$$

The pressure has been ignored, since by hypothesis it is constant. It is clear from the above equation that the upper plate exerts a tangential traction $\mathbf{t} = (\mu v_0/d)\mathbf{i}$, on the fluid, which is consistent with equation (8.3).

Pipe Flow

A somewhat more complex flow occurs when a pressure gradient drives a viscous fluid through a pipe of circular cross-section. As figure 9.2 illustrates, creeping flow under these circumstances is parallel to the axis of the pipe, with zero velocity at the walls of the pipe and maximum velocity on the axis. Defining a cylindrical coordinate system, (r, θ, z), with its z axis aligned with the axis of the pipe, and assuming that the velocity points only in the z direction and is a function of r alone, then

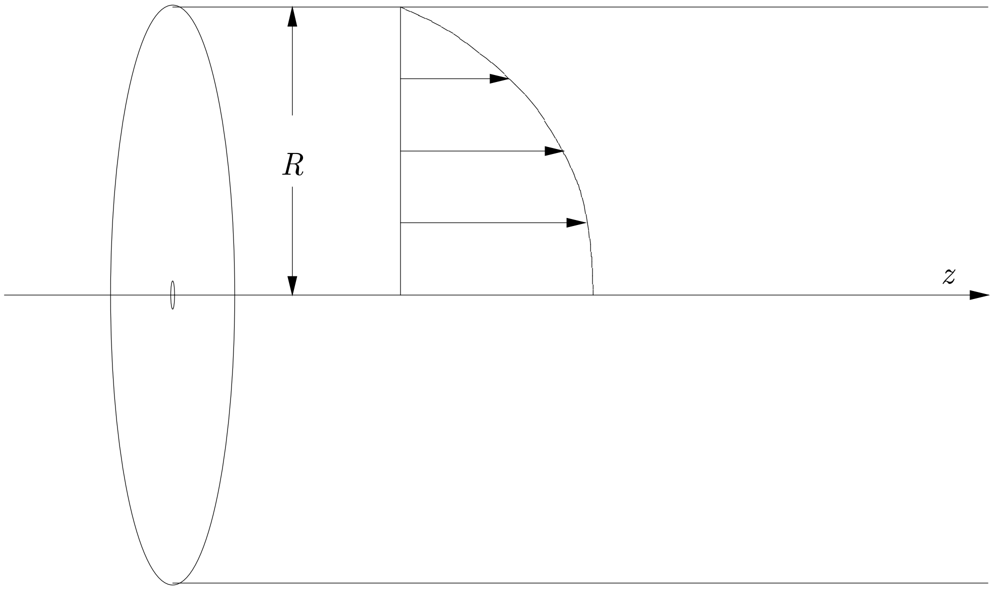

Figure 9.2. Sketch of laminar flow through a pipe of radius R.

$$\nabla^2 \mathbf{v} = \left(\mathbf{r} \frac{\partial}{\partial r} + \frac{\boldsymbol{\theta}}{r} \frac{\partial}{\partial \theta} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{r} \frac{\partial}{\partial r} + \frac{\boldsymbol{\theta}}{r} \frac{\partial}{\partial \theta} + \mathbf{k} \frac{\partial}{\partial z} \right) v_z(r) \mathbf{k} =$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v_z(r) \mathbf{k} = \frac{\mathbf{k}}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) . \tag{9.6}$$

Assuming that the only pressure gradient is a uniformly decreasing one in the z direction, the left side of equation (9.1) becomes

$$\nabla p = -\, \alpha \mathbf{k} = constant \; . \tag{9.7}$$

Equation (9.1) is therefore

$$-\, \alpha = \frac{\mu}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) , \tag{9.8}$$

which integrates to

$$v_z = -\, \frac{\alpha\, r^2}{4\mu} + A \ln r + B \; , \tag{9.9}$$

where A and B are constants of integration. This velocity satisfies mass continuity, since there are no velocity components normal to the axis of the pipe, and since v_z is not a function of z.

We rule out infinite velocity on the pipe axis on physical grounds, and therefore set $A = 0$. Adjusting B so that $v_z(R) = 0$ results in the final solution

$$v_z(r) = \alpha(R^2 - r^2)/(4\mu) \ . \tag{9.10}$$

The volume flow rate, F, through the pipe is simply v_z integrated over the cross-section of the pipe:

$$F = \int_0^{2\pi}\int_0^R v_z \ r \ dr \ d\theta = \frac{2\pi\alpha}{4\mu}\int_0^R (R^2 - r^2) \ r \ dr = \frac{\pi\alpha R^4}{8\mu} \ . \tag{9.11}$$

Note that this flow rate is very sensitive to the diameter of the pipe, with a doubling of pipe diameter yielding a factor of 16 increase in flow rate for a given pressure gradient and viscosity. Note also that the flow rate is a linear function of pressure gradient.

Flow in Porous Media

The underground flow of water or oil is an example of creeping flow. In this case the fluid is moving slowly through small pores in the rock, resulting in a flow with extremely small Reynolds number. Such a flow is quite complex in detail, but may be relatively simple in some averaged sense.

Generally speaking, gravity must be considered in the underground flow of fluids. Thus, equation (9.1) becomes

$$\nabla p = \mu\nabla^2\mathbf{v} - \rho g\mathbf{k} \ , \tag{9.12}$$

where g is the acceleration of gravity, assumed to act in the negative z direction. In the case of a homogeneous fluid in which the density ρ is a constant independent of position, equation (9.12) may be written

$$\nabla(p + g\rho z) = \mu\nabla^2\mathbf{v} \ . \tag{9.13}$$

The quantity $h = p + g\rho z$, or something closely related, is generally called the *total head*. As we have defined it, it is the pressure with the hydrostatic burden of the overlying fluid subtracted. When gravity is present, creeping flow occurs when a gradient of head rather than of pressure occurs. (See problem 5.) The pipe flow example explored above may be solved with gravity present simply by replacing the pressure gradient with the head gradient.

For flow in porous media, the volume flux of fluid, $\mathbf{q}$, is normally considered to be the most interesting observable. $\mathbf{q}\cdot\mathbf{n}$ is the volume of fluid per unit area per unit time passing through a surface with unit normal $\mathbf{n}$. This has the same units as velocity, but is less than the spatially averaged fluid velocity by a factor ϵ, the *porosity*, because the fluid only occupies the pores in the medium. The porosity is defined as the void volume in a medium divided by the total volume.

Darcy's Law is an empirical law, established by observing the flow of water through a bed of sand, that relates $\mathbf{q}$ to the head gradient:

$$\mathbf{q} = -\,\frac{k}{\mu}\,\nabla h\ . \tag{9.14}$$

The constant k is called the *permeability*, and has the units of length squared. It plays the role of a "fluid conductivity", i. e., the larger its value, the greater the fluid flow for a given head gradient and viscosity. For an incompressible fluid, the volume flux obeys

$$\nabla\cdot\mathbf{q} = 0\ . \tag{9.15}$$

A simple flow model will serve to clarify the physical basis of Darcy's Law. Consider a block of solid material, as shown in figure 9.3. The block has numerous holes of radius R drilled through it, amounting to N holes per unit area in the $x-y$ plane. We consider the flow of a fluid with viscosity μ and density ρ through these holes subject to a head gradient in the z direction $\partial h/\partial z = -\,\alpha$. The volume flow for each hole is $\pi\alpha R^4/8\mu$ for each hole. The volume flux in the z direction is just N times this, or

$$q_z = \frac{N\pi\alpha R^4}{8\mu} = -\,\frac{\epsilon R^2}{8\mu}\,\frac{\partial h}{\partial z}\ , \tag{9.16}$$

where $\epsilon = N\pi R^2$ is the fractional area in the $x-y$ plane covered by holes, or just the porosity. The permeability is evidently $k = \epsilon R^2/8$ in this case, as comparison with the z component of equation (9.14) shows. For a given porosity, equation (9.16) shows that bigger holes result in more flow for a given head gradient, since the permeability then scales with the square of hole radius.

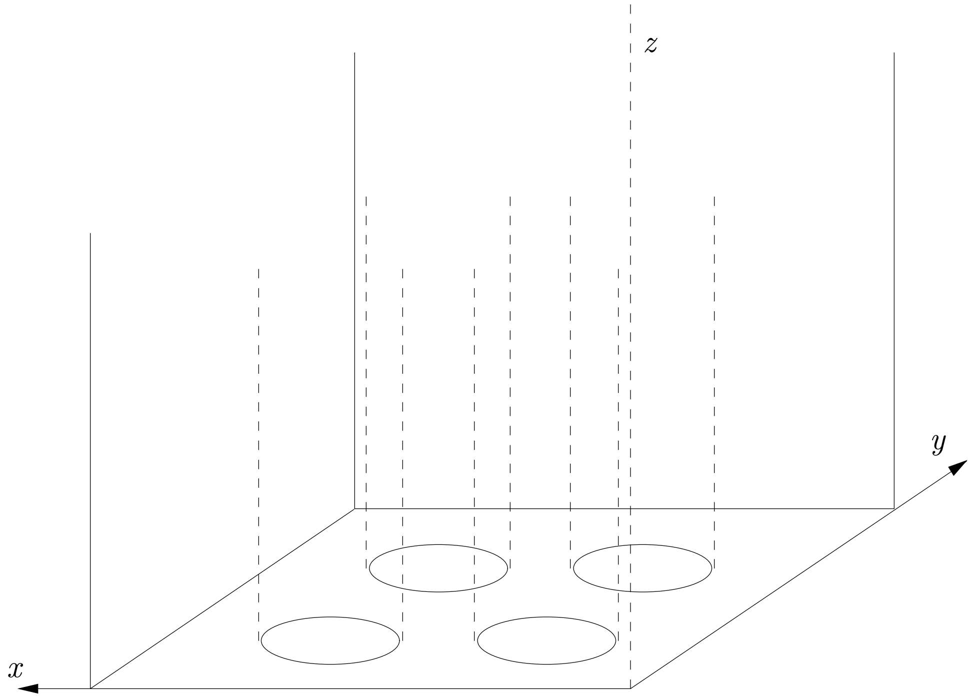

Figure 9.3. Flow through a porous medium idealized as flow through a parallel array of pipes.

Real porous media have much more complex flow channels than the example of figure 9.3. However, a scale analysis of equation (9.13) leads to the same conclusion. The Laplacian of the velocity in this equation will be of order $V/L^2 = Q/(\epsilon L^2)$ where L is a characteristic pore dimension, V is a characteristic fluid velocity, and $Q = \epsilon V$ is an estimate of the volume flux. Equation (9.13) therefore becomes (in an averaged sense)

$$\nabla h = -\frac{C\mu}{\epsilon L^2}\mathbf{q}\,, \tag{9.17}$$

where C is a numerical constant of order unity. The minus sign arises because the cross channel second derivative of the velocity arising from the Laplacian is negative for a positive flow with zero velocity on the boundaries of the channel. Identifying $k = \epsilon L^2/C$ as the permeability makes equations (9.14) and (9.17) identical. Thus, for a pore pattern of the same shape, but scaled down by a factor of two, the permeability would be decreased by a factor of four.

Actually, the porous medium illustrated in figure 9.3 is an example of an anisotropic medium, in the sense that fluid can move through it in only one direction. The usual form of Darcy's law shown in equation (9.14) assumes that the medium is isotropic. A generalization of Darcy's law that takes anisotropy into account is

$$\mathbf{q} = -\frac{1}{\mu}\mathbf{K}\cdot\nabla h \; . \tag{9.18}$$

The permeability becomes a symmetric tensor, $\mathbf{K}$, in this formulation, and the fluid flow is not necessarily in the direction of minus the head gradient. For the unidirectional medium of figure 9.3, we would have

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon R^2/8 \end{pmatrix} . \tag{9.19}$$

It is clear that a head gradient in the x or y direction would result in no flow for the medium of figure 9.3. This is easily verified by substituting equation (9.19) into equation (9.18).

Problems

1. From equation (8.19) we see that the energy dissipated per unit volume per unit time by viscous forces in an incompressible fluid is

$$\mu\left(\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\right) .$$

Using this result, show that the work per unit time done by the moving plate in plane Couette flow just balances the viscous dissipation in the fluid between the plates.

2. Show that equation (8.19) is satisfied in the case of Couette flow by a balance between the last term in square brackets on the left side of the equal sign and the term considered in problem 1.

3. Solve the incompressible Couette flow problem (i. e., the flow between two parallel plates with one plate moving) for the situation in which a pressure gradient $\partial p/\partial x = -\,\alpha = constant$ exists between the plates. When the velocity of both plates is zero, this is called plane Poiseuille flow. It is the slab-symmetric

Figure 9.4. Two horizontal layers of soil with differing thickness and permeability. (See problem 7.)

analog of pipe flow.

4. Find the strain rate and stress tensors for pipe flow. (Hint: You can obtain these in Cartesian coordinates by realizing that $r^2 = x^2 + y^2$, where x and y are the Cartesian axes normal to the axis of the pipe.) Using the stress tensor, show that the force per unit length exerted by the pipe on the fluid just counterbalances the pressure gradient integrated over the cross-section of the pipe.

5. Show that for a fluid of constant density at rest, the total head is constant.

6. Prove equation (9.15) from first principles. Why wouldn't a version of (9.15) with $\mathbf{q}$ replaced by a spatially smoothed velocity work? (Hint: Suppose that the porosity varied from place to place.)

7. Consider flow of water downward through two horizontal layers of saturated soil, the upper layer having thickness d_1 and permeability k_1, the lower layer with thickness and permeability d_2 and k_2. (See figure 9.4) If the total head is h_0 at $z = 0$, find the head at all levels in the soil. Assume that the pressure is the same below the bottom layer as it is above the top layer.

8. Given the permeability tensor in equation (9.19), find the flow for a head gradient $\nabla h = -\ A[\cos(\alpha)\mathbf{i} + \sin(\alpha)\mathbf{k}]$, where A and α are constants. What is the component of the flow in the direction of the head gradient?

Chapter 10 -- High Reynolds Number Flow

When the Reynolds number is very large it is reasonable to assume that the viscous terms in the Navier-Stokes equation can be ignored. We now explore the consequences of this assumption. When a fluid is in contact with a solid surface, we will find that the assumption is not strictly defensible. However, it remains a useful approximation under many circumstances.

When the viscous terms are dropped, the Navier-Stokes equation reduces to the Euler equation,

$$\rho \frac{d\mathbf{v}}{dt} = - \boldsymbol{\nabla} p + \rho \mathbf{B} \ . \tag{10.1}$$

Since terms involving second order spatial derivatives have been ignored, the boundary conditions needed to insure a unique physical solution are changed. The no slip boundary conditions required for the Navier-Stokes equation overconstrain the Euler equation, resulting in no solution in most circumstances. It turns out that free slip boundary conditions are appropriate for the case in which viscosity is ignored. Physically, this means that the only constraint on a fluid adjacent to a solid boundary is that the fluid not flow through the boundary. Flow tangent to the boundary is unconstrained. Mathematically, the free slip condition may be written

$$(\mathbf{v} - \mathbf{v}_s) \cdot \mathbf{n} = 0 \ , \tag{10.2}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{v}_s$ is the velocity of the bounding surface (often zero), and $\mathbf{n}$ is the unit normal to the bounding surface.

Figure 10.1 illustrates the flow next to a plate for large, small, and no viscosity. The sheared region near the plate in the first two cases is called a boundary layer, as discussed in chapter 8. Other things being equal, as viscosity decreases, the boundary layer becomes thinner. However, the shear in the boundary layer becomes stronger, leaving the product of the shear and the boundary layer thickness the same. Thus, as long as there is any viscosity, no matter how small, there will be a thin sheared boundary layer. The third case in figure 10.1, in which the viscosity is identically zero, is quite different. Since the boundary condition here is free slip, no shear layer exists.

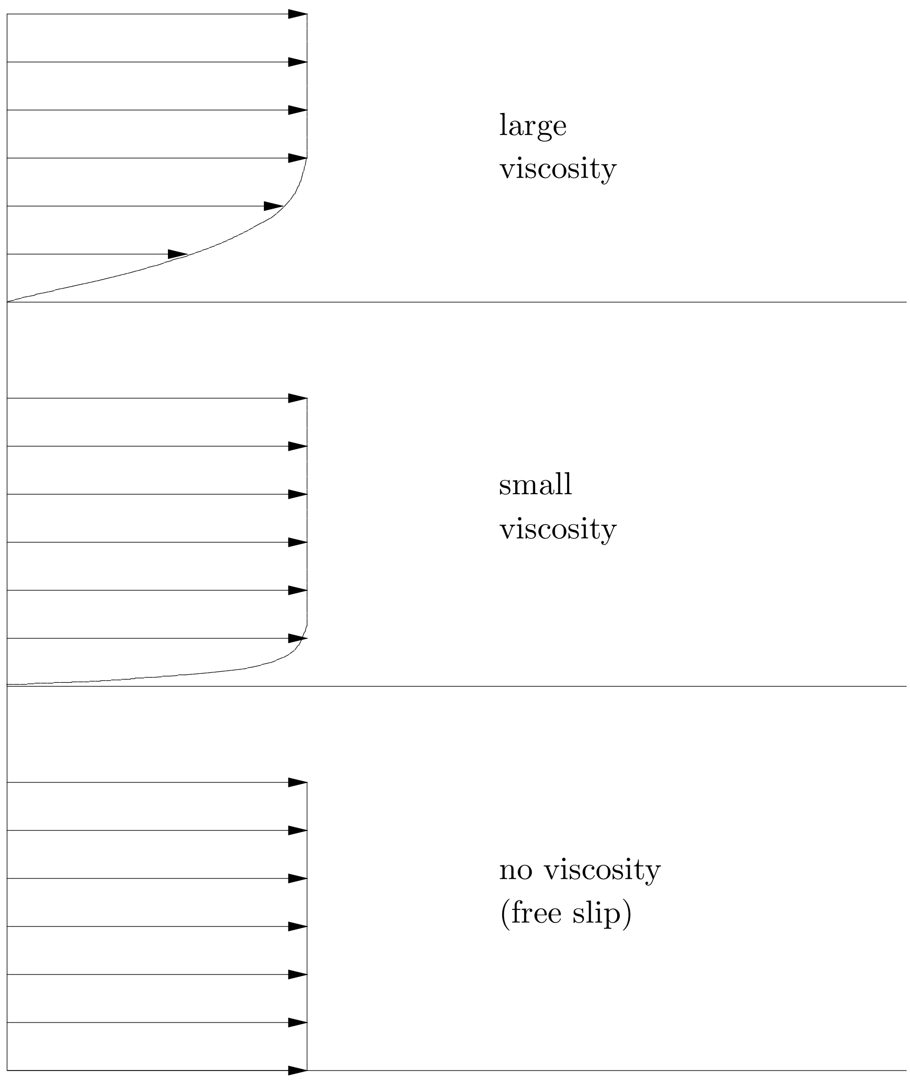

Figure 10.1. Illustration of flow parallel to a flat plate in the case of large viscosity, small viscosity, and no viscosity. The last case differs fundamentally from the other two in that no layer of vorticity exists at or just above the surface.

The distinction between a very thin shear layer near the surface and no shear layer may seem trivial, and so it is as long as fluid adjacent to the surface never departs from it. However, as we shall see, the distinction is crucial when the phenomenon of *flow separation* occurs. In the case of small viscosity, the shear layer is sometimes stripped away from the surface and transported into the interior of the fluid, where it has significant dynamical consequences. An inviscid fluid has no shear layer available for transport into the interior.

In order to obtain a better understanding of these matters, we delve into the concepts of vorticity and circulation. We then study Bernoulli's equation and irrotational flows. Finally, we use these tools to investigate flow in the presence of solid surfaces.

Kelvin's Theorem

In chapter 4 the vorticity was introduced as twice the dual vector of the rotation rate tensor, and it was shown to be equal to the curl of the velocity:

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} \; . \tag{10.3}$$

The vorticity plays a crucial role in high Reynolds number fluid dynamics.

A closely related quantity, the *circulation*, is defined

$$\Gamma = \int \mathbf{v} \cdot d\mathbf{l} \; , \tag{10.4}$$

where the integral is a line integral around a closed loop. The circulation around a loop is therefore the average value of the component of the velocity tangent to the loop times the circumference of the loop. Figure 10.2 illustrates streamlines of a flow pattern, a circulation loop, and the velocity $\mathbf{v}$ and line element $d\mathbf{l}$ vectors at one point on the circulation loop.

The relationship with vorticity comes from applying Stokes' theorem to equation (10.4):

$$\Gamma = \int \boldsymbol{\nabla} \times \mathbf{v} \cdot \mathbf{n} \, dA = \int \boldsymbol{\omega} \cdot \mathbf{n} \, dA = \overline{\omega}_n A \; . \tag{10.5}$$

The integral is now an area integral over the surface (actually, any of the many surfaces) bounded by the circulation loop. Since $\mathbf{n}$ is the unit normal to the surface (see figure 10.2), the integral picks out the component of vorticity normal to the surface. If $\overline{\omega}_n$ is the average value of the component of vorticity normal to the surface, then the integral can also be written as the area A of the surface times $\overline{\omega}_n$.

In the special case of two-dimensional flow the only nonzero component of the vorticity is that component normal to the plane of the flow. A point vortex

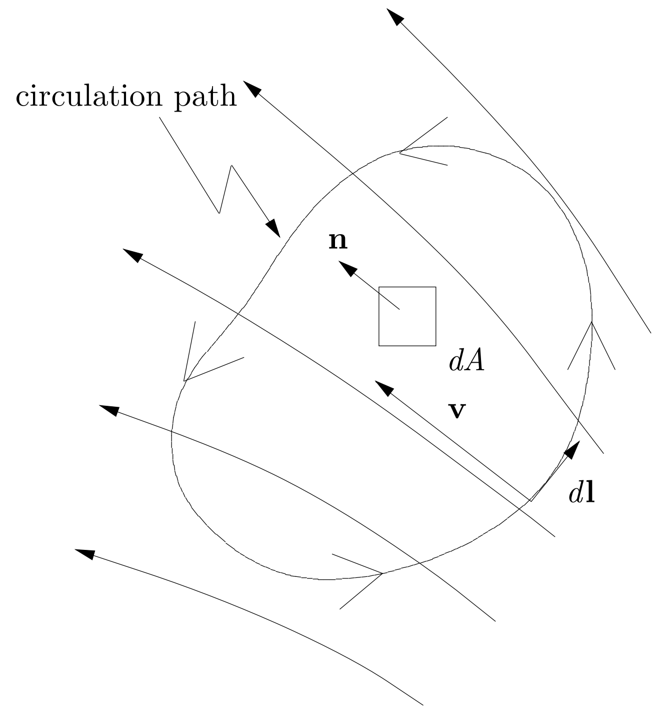

Figure 10.2. Definition sketch for circulation theorem.

results when all vorticity is concentrated in a very small region. Any circulation loop that encompasses the vorticity has the same circulation in this case, by virtue of equation (10.5). Thus, in particular, two circular loops concentric with the concentration of vorticity have the same circulation, as illustrated in figure 10.3. By symmetry, the circulation around a concentric, circular loop of radius r is simply $\Gamma = 2\pi r v_t$, where v_t is the tangential velocity around the loop. As a consequence, $v_t = \Gamma/(2\pi r)$ for a point vortex. Thus, the circulation is a convenient measure of the strength of a point vortex. The equivalent of a point vortex occurs in three dimensions when all vorticity is concentrated into a pencil-shaped region aligned with the vorticity vector.

Let us see if we can compute the time rate of change of the circulation around a loop that moves with the fluid. To do this it is easiest to approximate the circulation integral as a finite sum:

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int \mathbf{v} \cdot d\mathbf{l} \simeq \frac{d}{dt}\sum_i \mathbf{v}_i \cdot (\mathbf{x}_{i+1} - \mathbf{x}_i) \; . \qquad (10.6)$$

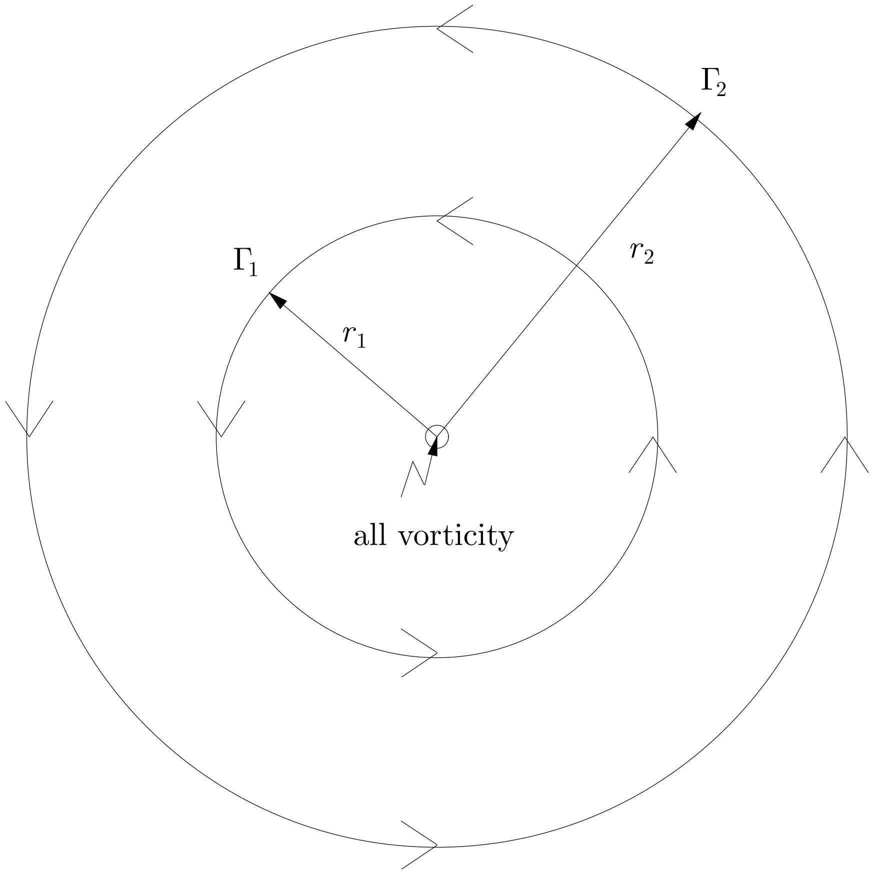

Figure 10.3. Two circulation loops around a point vortex. The circulations are equal, i. e., $\Gamma_1 = \Gamma_2$.

The finite difference analog of $d\mathbf{l}$ is taken as the difference between two position vectors representing successive points along the circulation path. By hypothesis these points move with the fluid, so $d\mathbf{x}_i/dt = \mathbf{v}_i$. Applying the product rule for differentiation to the right side of equation (10.6), we see that

$$\frac{d\Gamma}{dt} = \int \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} + \int \mathbf{v} \cdot d\mathbf{v} \qquad (10.7)$$

upon passing back to the exact integral form.

The second term on the right side of equation (10.7) is easily disposed of by noting that $\mathbf{v} \cdot d\mathbf{v} = d(v^2/2)$. This is an exact differential, and integration around a closed loop yields zero. The first integral may be evaluated by eliminating $d\mathbf{v}/dt$ with the Euler equation, (10.1). If the body force per unit mass is conservative, i. e., $\mathbf{B} = -\nabla U$, where U is the potential energy per unit mass of fluid, then the integral of $\nabla U \cdot d\mathbf{l} = dU$ around a closed loop is zero, since dU is also a perfect differential. As $\nabla p \cdot d\mathbf{l} = dp$, equation (10.7) becomes

$$\frac{d\Gamma}{dt} = -\int \frac{dp}{\rho}\,. \tag{10.8}$$

This is called the Kelvin circulation theorem, and it states that the only way the circulation around a closed loop moving with an inviscid fluid can change is when the gradient of pressure and the gradient of density don't point in the same direction.

Imagine a tank filled with water containing a variable concentration of dissolved salt, and hence a variable density. If left to itself, the saltier, more dense water will end up at the bottom of the tank, making the density gradient vector point vertically downward. The pressure gradient also points downward due to the hydrostatic law. If the tank is somehow perturbed so that the denser fluid is moved to the left side of the tank and the lighter fluid to the right, the density gradient will then have a horizontal component, while the pressure gradient will remain approximately vertical, as shown in figure 10.4. According to equation (10.8), a circulation will then develop which will tend to restore the fluid to its equilibrium configuration. However, the flow will overshoot, which will result in a tilt of the constant density surfaces the other way. This will slow down the circulation that has developed, eventually stopping and reversing it. Oscillations will continue until damped by viscosity.

A fluid for which the density can vary independent of the pressure is called a *baroclinic* fluid, and the process described above is called the baroclinic generation of vorticity. A *barotropic* fluid has a unique relationship between pressure and density, $\rho = \rho(p)$. A simple example is a homogeneous, incompressible fluid for which the density is constant. Another example is an ideal gas with constant specific entropy. It is easy to show from equation (8.14) that $p = C\rho^{\gamma}$ in this case, where C is a constant and $\gamma = C_p/C_v$. Thus, $\nabla p = \gamma C\rho^{(\gamma-1)}\nabla\rho$, and $\nabla\rho \times \nabla p = 0$ since the gradients of pressure and density are automatically parallel. Equation (10.8) states that the circulation around any loop that moves with the fluid is constant in time for a barotropic fluid.

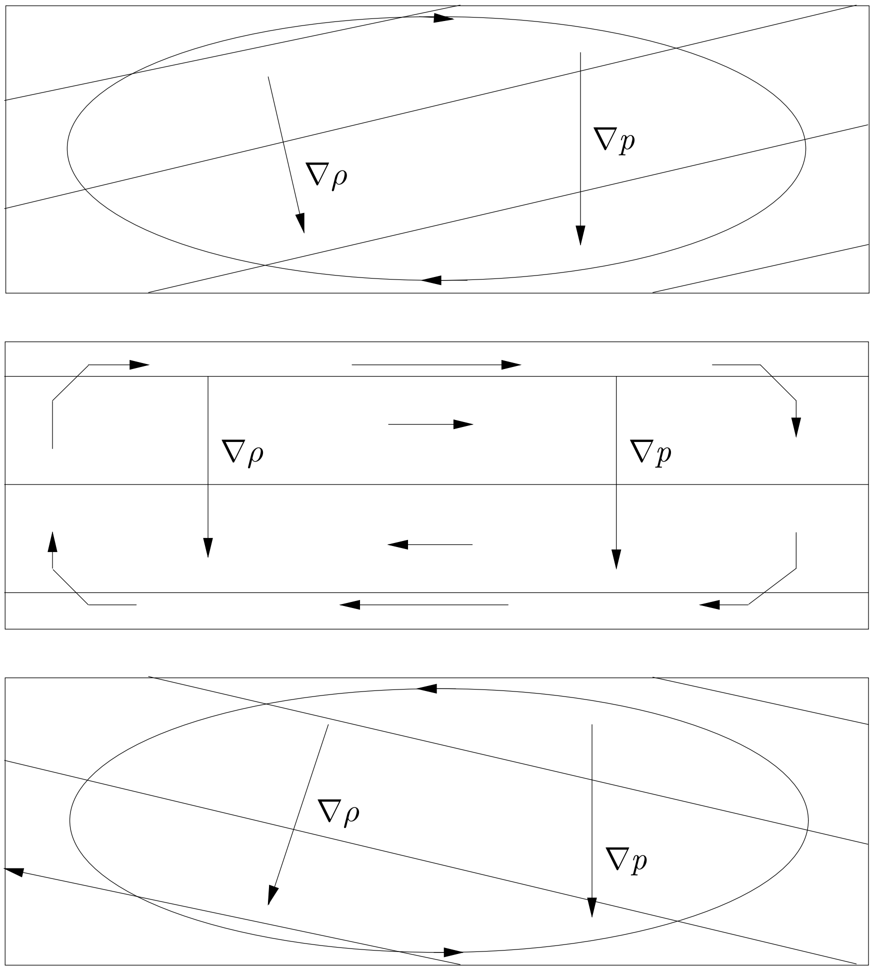

Figure 10.4. Evolution of the orientation of the constant density surfaces in a tank with salt water of varying concentration. In the upper panel the density gradient has a component to the left, which induces a clockwise circulation. The middle panel shows the velocity field when the circulation has made the constant density surfaces horizontal. The fluid configuration in the last panel is opposite of that in the first panel. In all panels, $\boldsymbol{\nabla}\rho$ is the the density gradient and $\boldsymbol{\nabla}p$ is the pressure gradient.

Lines everywhere parallel to the vorticity vector are called *vortex lines*. The more closely spaced the lines, the stronger the vorticity. Since the vorticity is the curl of the velocity, its divergence is zero. This means that vortex lines cannot begin or end in the fluid. For a barotropic fluid, vortex lines can be thought to move with the fluid. This follows from the Kelvin circulation theorem. (Why?)

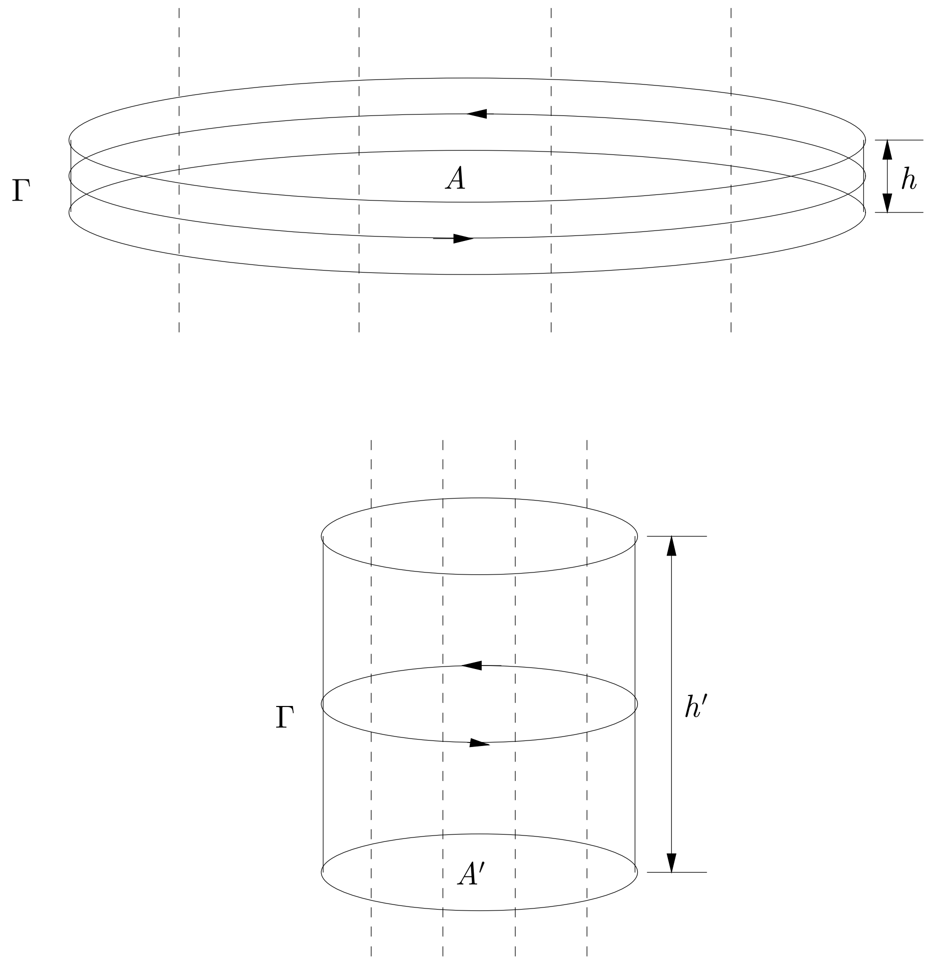

Figure 10.5. Illustration of “vortex stretching”. The dashed lines are vortex lines.

The development of strong vortices in a barotropic fluid can be understood using the circulation theorem and equation (10.5). If such a fluid deforms so that vortex lines are brought closer together, as in figure 10.5, then the vorticity is intensified. If the fluid is incompressible, the shrinking of the circulation loop around the vortex lines must be accompanied by stretching in the perpendicular direction, as illustrated by the change in shape of the cylinder shown in figure 10.5. This constant volume deformation is often characterized as a process of stretching the vortex lines. Any increase in the length of vortex lines must be accompanied by a decrease in their spacing such that the volume, $Ah = A'h'$, remains constant for an incompressible fluid. For this reason, the spinup of a vortex from weak ambient vorticity by such a deformation process is often called *vortex stretching.* Examples of vortex stretching are the development of the "bathtub vortex" when water drains out of a bathtub, and the formation of tornados and hurricanes. Note, however, that the spinup is not a consequence of the "stretching", but of the reduction in the cross-sectional area normal to the vorticity vector.

Inviscid Irrotational Flows and Bernoulli's Equation

An irrotational flow is one in which the vorticity is everywhere zero. If the fluid is also inviscid and barotropic, as we shall generally assume in this section, it will remain irrotational for all time if it starts out that way -- a consequence of the Kelvin circulation theorem. It is therefore useful to study this special case of fluid flow.

If $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} = 0$, then the velocity field can be represented as the gradient of a scalar, called the *velocity potential*:

$$\mathbf{v} = \boldsymbol{\nabla}\phi \qquad (\textit{irrotational flow}) \ . \tag{10.9}$$

If the flow is also incompressible, then

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \nabla^2\phi = 0 \ , \tag{10.10}$$

i. e., the velocity potential, ϕ, satisfies Laplace's equation. The free slip boundary condition evaluated in terms of ϕ is derived from equation (10.2):

$$\boldsymbol{\nabla}\phi \cdot \mathbf{n} = \mathbf{v}_s \cdot \mathbf{n} \ . \tag{10.11}$$

Equation (10.10) and these boundary conditions contain all of the physics of incompressible, inviscid, irrotational flows, and represent a considerable simplification over the more general problem of fluid motion. Equation (10.10) also contains no time derivative, so the fluid flow at any time depends only on the conditions at that time, and not on previous times. In other words, such a fluid does not remember its history, and any time dependence can only enter through the boundary conditions.

We now derive the Bernoulli equation for an inviscid, irrotational, barotropic fluid. Using a vector identity, $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla}(v^2/2) - \mathbf{v}\times\boldsymbol{\omega}$, and assuming that the body force is derivable from a potential, $\mathbf{B} = -\ \boldsymbol{\nabla}U$, the Euler equation, (10.1), can be written

$$\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t} + v^2/2 + \int \frac{dp}{\rho} + U\right) = 0\ , \tag{10.12}$$

which immediately integrates to Bernoulli's equation,

$$\frac{\partial\phi}{\partial t} + v^2/2 + \int \frac{dp}{\rho} + U = H\ , \tag{10.13}$$

where H is a constant called the Bernoulli constant. We are able to take the density inside the gradient operation in equation (10.12) because the fluid is barotropic by hypothesis. The density is a unique function of the pressure, and the integral is therefore unique. In the case of a time-independent flow of a homogeneous, incompressible fluid, Bernoulli's equation reduces to the more familiar form

$$v^2/2 + p/\rho + U = H\ . \tag{10.14}$$

Since the velocity is determined by $\nabla^2\phi = 0$ and the free slip boundary conditions, Bernoulli's equation may be thought of as a diagnostic equation for the pressure in the case of inviscid, incompressible, irrotational flow. Thus, between Bernoulli's equation and Laplace's equation for the velocity potential, the entire problem of computing this type of flow is in principle solved.

In the special case of two-dimensional flow in the $x - y$ plane, further simplification occurs. In this case the continuity equation reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\ . \tag{10.15}$$

This can be satisfied by assuming that

$$v_x = -\ \frac{\partial\psi}{\partial y} \qquad\qquad v_y = \frac{\partial\psi}{\partial x}\ . \tag{10.16}$$

The variable ψ is called the *streamfunction*, since lines of constant ψ are streamlines. This is easily verified by noting that the above equations imply $\mathbf{v}\cdot\boldsymbol{\nabla}\psi = 0$. Thus, $\mathbf{v}$ is parallel to lines of constant ψ. A corollary is that $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\phi = 0$, i. e., lines of constant ψ and ϕ are perpendicular. Finally, since the vorticity is zero in irrotational flow,

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \; . \tag{10.17}$$

Thus both the velocity potential and the streamfunction satisfy Laplace's equation in two-dimensional, incompressible flow, and solving for either one leads to a solution for the velocity field. In a steady flow situation (i. e., the boundaries don't move), ψ is simply constant on each boundary.

Boundary Layers and Forces on Immersed Objects

With new tools in hand, we now return to the question of flow near solid boundaries. In particular, we try to understand the forces exerted on an object as it moves through a homogeneous, incompressible fluid at very high Reynolds number.

An alternative way to interpret figure 10.1 is that a layer of vorticity exists where a moving viscous fluid comes in contact with a solid body. As the viscosity is decreased, the layer becomes thinner, but the vorticity becomes stronger, thus keeping constant the circulation around a segment of this layer. Vorticity generated at the surface moves outward by molecular diffusion, so the smaller the viscosity, which can be thought of as the diffusion coefficient for momentum and vorticity, the smaller the spread.

The main effect of the flow is to transport this vorticity downstream. The result for a streamlined body, as shown in the upper part of figure 10.6, is the creation of a narrow wake behind the body. The wake is a region in which the downstream flow speed is less than the upstream flow speed. It is created by the two boundary layers from the opposite sides of the body coming together.

In the inviscid approximation, these layers of vorticity, and hence the wake, don't exist. For high Reynolds number the boundary layers and the wake become very thin for a streamlined body. It seems reasonable under these circumstances to suppose that in the limit of large Reynolds number the inviscid approximation to the flow, and hence to the force on the body, is valid.

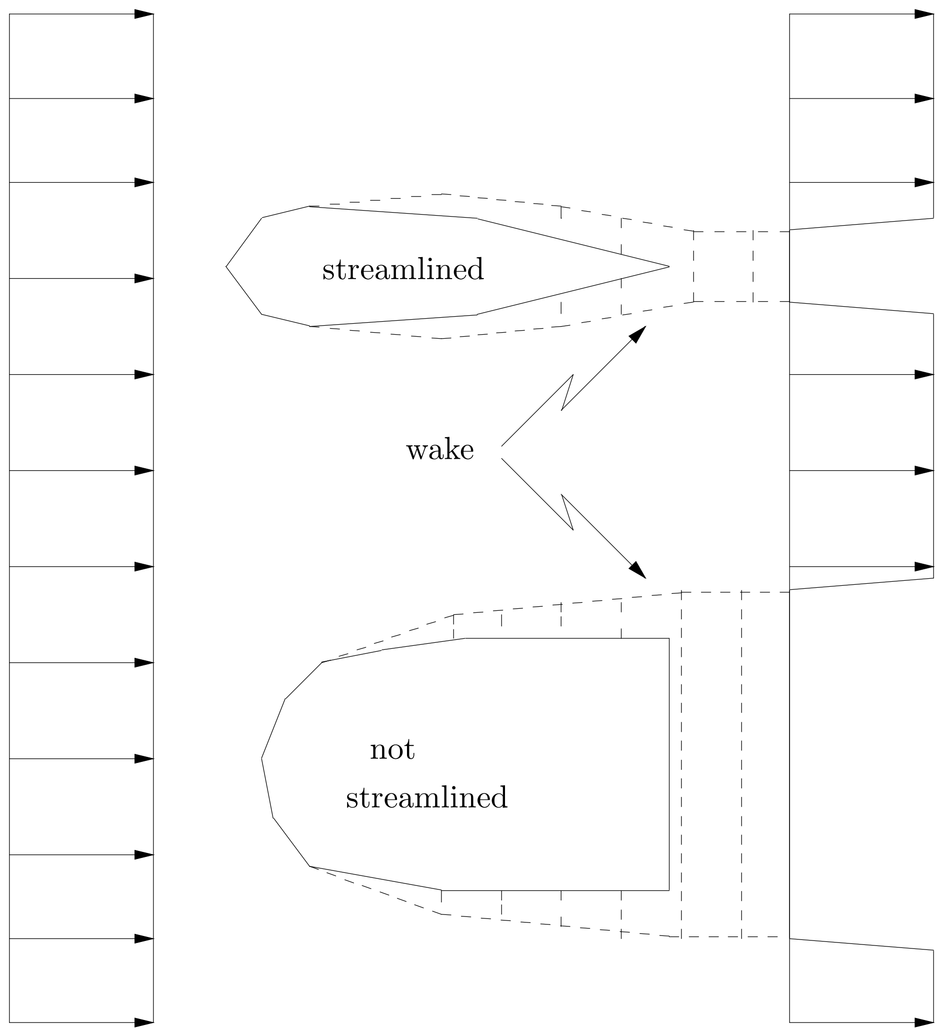

Figure 10.6. Development of a wake behind streamlined and unstreamlined bodies.

How do we evaluate the force on a body due to its motion through an inviscid, irrotational fluid? The most straightforward way is to integrate the traction of the fluid on the body over the surface of the body, i. e.,

$$\mathbf{F} = \int \mathbf{T} \cdot \mathbf{n} \, dA = - \int p\mathbf{I} \cdot \mathbf{n} \, dA = - \int p\mathbf{n} \, dA \ , \tag{10.18}$$

where $\mathbf{n}$ is the outward unit normal to the surface of the body, and the stress for an inviscid fluid, $-\,p\mathbf{I}$, has been substituted. If we place ourselves in the reference frame of the body so the flow is steady, the pressure can be obtained from the flow field with Bernoulli's equation, assuming no body forces:

$$p = -\ \rho v^2/2 + constant\ . \tag{10.19}$$

For flow around a sphere or a cylinder, or any other object that exhibits side-to-side and upstream-downstream symmetry, substitution of equation (10.19) into equation (10.18) yields zero force.

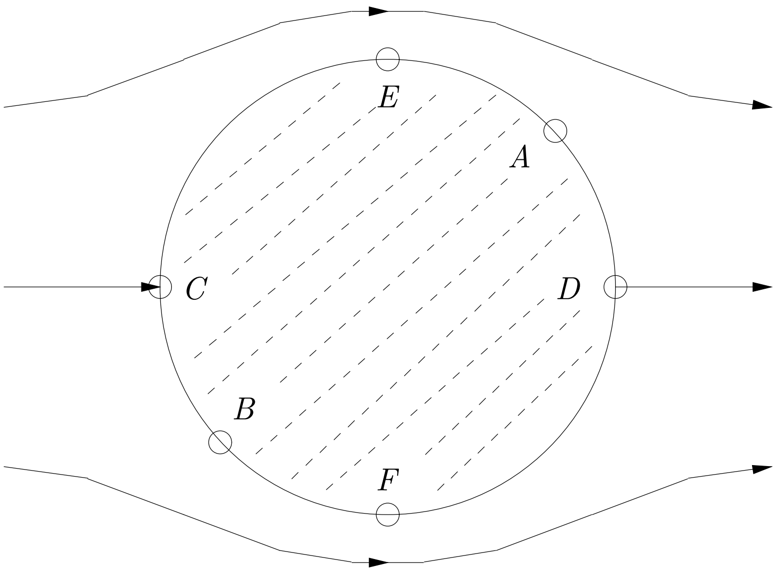

Figure 10.7. Schematic of flow around a sphere or cylinder.

This may be seen by examining figure 10.7, which illustrates such a flow. The symmetry of the object causes the flow to have similar symmetry. For instance, the points C and D both have zero velocity, and are called *stagnation points*. By Bernoulli's equation, C and D have pressures exceeding the pressure in the distant fluid by $\rho v_0^2/2$, where v_0 is the flow speed far from the object. On the other hand, the points E and F both experience pressures less than the distant fluid value, since the flow speeds at these points exceeds v_0. In the more general case of points A and B, the pressures are equal by symmetry, and therefore equal but opposite tractions are exerted on the object at these two points. Since the total force on the object can be expressed as the sum of pairs of tractions on opposite sides of the object, the total force on the object is zero.

It turns out that the above argument can be generalized to a finite-sized object of *any* shape. Thus, any such object moving through a homogeneous, incompressible, inviscid fluid experiences no drag force. This result, called

D'Alembert's paradox, is not in accord with our intuition (derived, say, from riding a bicycle into a strong wind!), so there must be a flaw in the argument.

The flaw turns out to be the assumption that the wake always tends to zero thickness as the Reynolds number increases. The lower part of figure 10.6 shows what more commonly happens when an object moves through a fluid. In the illustrated case the vorticity generated in the boundary layer of the object separates from the object before it meets its companion vorticity from the other side of the object. In the illustrated case the separation is induced by the sharp corner at the back of the unstreamlined object, but even in less extreme cases such as actual (as opposed to ideal) flow around a sphere, separation occurs at high Reynolds numbers. The result is a very broad wake, comparable in diameter to the cross-stream diameter of the object. In such a wake region the flow is typically time-dependent and turbulent, though for simplicity it is shown as being zero in the illustration.

An estimate of the drag on such a blunt object can be made using the following reasoning. The pressure at the front stagnation point on the bottom object in figure 10.6 is $\rho v_0^2/2$ by Bernoulli's equation. If the object has a projected cross-sectional area normal to the flow of A, the drag from pressure perturbation forces on the forward face of the object is of order $\rho A v_0^2/2$. For pure irrotational flow, this force would be countered by an equal and opposite force on the rear face of the object, and the net force would be zero. However, the pressure in the wake region must be the same as the pressure in the ambient flow far from the object, since there can be no pressure jump across the shear line that defines the boundary of the wake (why?). Thus, there is no rear pressure perturbation, and no countervailing thrust force. The net drag can therefore be written

$$F_{drag} = C_D \rho A v_0^2/2 \,, \tag{10.20}$$

where C_D is a dimensionless coefficient of order one called the drag coefficient. The drag coefficient accounts for the fact that we only estimated the drag on the front faces of the object and didn't precisely calculate it. Drag coefficients are hard to calculate from first principles in any but the most idealized cases, and are generally obtained by measuring the force on an object placed in a moving stream of air or water.

We now need to ask why flow separation takes place. Let us examine the behavior of parcels moving around the sphere in figure 10.7. For truly inviscid flow, a parcel starting out near point C will have near-zero velocity and higher than normal pressure. The pressure gradient is such that it will accelerate in its trajectory around the sphere until it reaches point E, the location of lowest pressure. From point E to point D it will then decelerate until it reaches nearly zero velocity at the rear stagnation point. However, when viscous drag is added, the acceleration from C to E is less than in the idealized case, and the deceleration afterwards is greater. Thus, the fluid near the surface of the sphere comes nearly to a halt before it reaches point D. By mass continuity, fluid cannot pile up at this new stagnation point, so it must move away from the surface, carrying its boundary layer vorticity with it. Thus, separation of the flow from the surface of the object is an inevitable consequence of viscous drag in the boundary layer.

As might be expected, the separation process is one that is very sensitive to many things, and is very hard to predict. In addition, it turns out that the boundary layer vortex sheet that peels off of the object is itself unstable. This generally causes the wake to evolve in a complex and unpredictable fashion. The subject of high Reynolds number flow past objects thus becomes one of the most difficult areas of study in fluid mechanics. Nevertheless, the simple arguments made here suffice to give a qualitative picture of what happens in these circumstances.

We now turn to the subject of lift. Lift is the component of the fluid force on an object normal to the direction of the flow. Imagine an infinitely long cylinder of radius R aligned normal to a flow that is uniform far from the cylinder. As is shown in problem 3, a solution for the streamfunction that satisfies free slip boundary conditions at $x^2 + y^2 = R^2$ and has uniform flow in the minus-x direction at large distances is

$$\psi = v_0 y \left(1 - \frac{R^2}{x^2 + y^2}\right) . \tag{10.21}$$

However, this isn't the most general solution possible, since

$$\psi = C \log(x^2 + y^2) , \tag{10.22}$$

where C is a constant, also satisfies Laplace's equation. This solution has a singularity at $x = y = 0$, but is irrotational away from this point. Application of equation (10.16) shows that this represents the flow around a point vortex at the center of the cylinder with circulation $\Gamma = 4\pi C$. Thus, the vorticity associated with this flow is confined to the region inside the cylinder, and is imaginary in the sense that no vorticity is actually found in the fluid. Nevertheless, since the circulation around the cylinder in this case is nonzero, a *bound vortex* is said to exist inside the cylinder.

The bound vortex flow satisfies free slip boundary conditions on the surface of the cylinder, as does the zero circulation flow represented by equation (10.21). Since the governing equation for streamfunction in the irrotational flow case, $\nabla^2\psi = 0$ is linear, a linear combination of equations (10.21) and (10.22) is a valid flow solution that also satisfies boundary conditions on the cylinder. The characteristics of this combined solution, written here in cylindrical coordinates,

$$\psi = v_0 r \sin\theta\,(1 - R^2/r^2) + (\Gamma/2\pi)\log r\ , \tag{10.23}$$

are quite interesting. Since the tangential velocity is simply $\partial\psi/\partial r$, the tangential velocity at the surface of the cylinder is

$$v_t(R,\theta) = 2v_0 \sin\theta + \frac{\Gamma}{2\pi R}\ , \tag{10.24}$$

and the pressure at the surface of the cylinder is

$$p = -\,\rho\left(2v_0^2 \sin^2\theta + \frac{v_0\Gamma\sin\theta}{\pi R} + \frac{\Gamma^2}{8\pi^2 R^2}\right) + \mathit{constant} \tag{10.25}$$

by Bernoulli's equation. Inserting this into equation (10.18) results in the following force per unit length of cylinder:

$$\mathbf{F} = \rho v_0 \Gamma \mathbf{j} \tag{10.26}$$

This is the Kutta-Zhukovskii theorem, which states that the force on an infinitely long cylinder oriented normal to a uniform flow is proportional to the product of the flow speed and the circulation around the cylinder, and is directed perpendicular to the direction of the flow. This theorem can actually be proved for a cylinder of arbitrary cross-section. A condition of the Kutta-Zhukovskii theorem is that there be no significant wake. Though we have proved the theorem for the special case of a circular cylinder, this condition can be realized in practice only with streamlined shapes so that boundary layer separation doesn't take place.

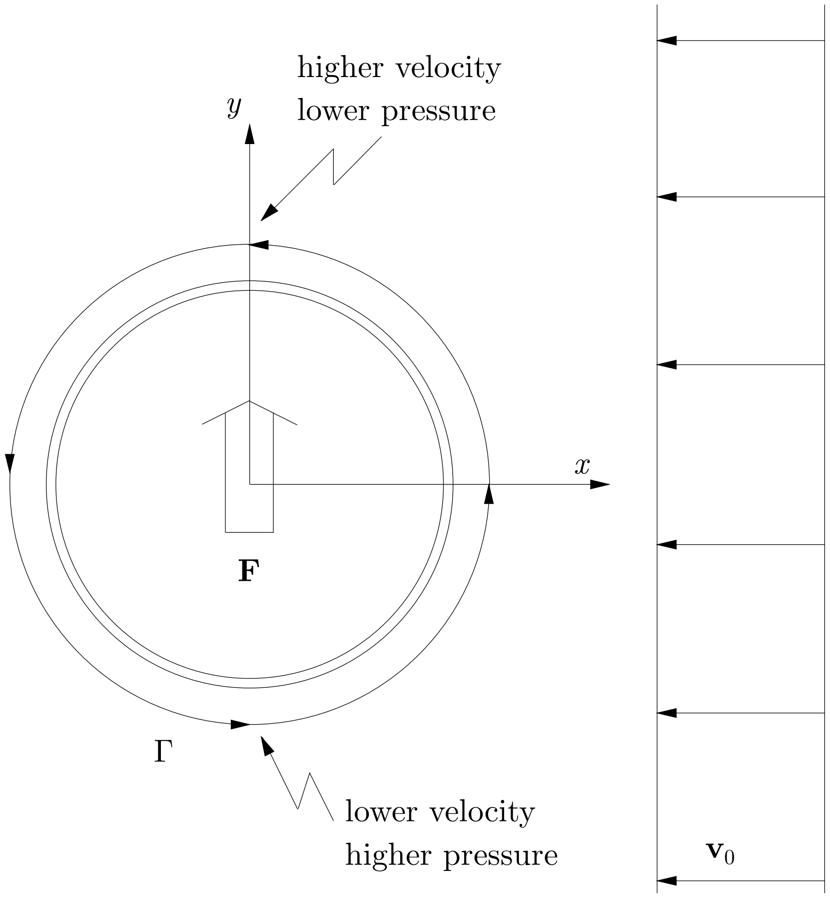

Figure 10.8. Lift acting on a rotating cylinder.

It is possible to understand the Kutta-Zhukovskii theorem from figure 10.8, in which a circular cylinder is subjected to the translational flow plus bound vortex described by equation (10.23). On top of the cylinder the circulation associated with the bound vortex and the translational flow reinforce resulting in a larger velocity than at the bottom of the cylinder, where they act against each other. By Bernoulli's equation, the pressure is therefore lower on the top than on the bottom of the cylinder, resulting in a net upward force. Note that this lift force requires both translation of the cylinder through the fluid and the existence of a circulation. Neither effect by itself will generate lift.

The Kutta-Zhukovskii theorem strictly applies only to infinitely long cylinders. However, it is approximately correct for finite cylinders that are much

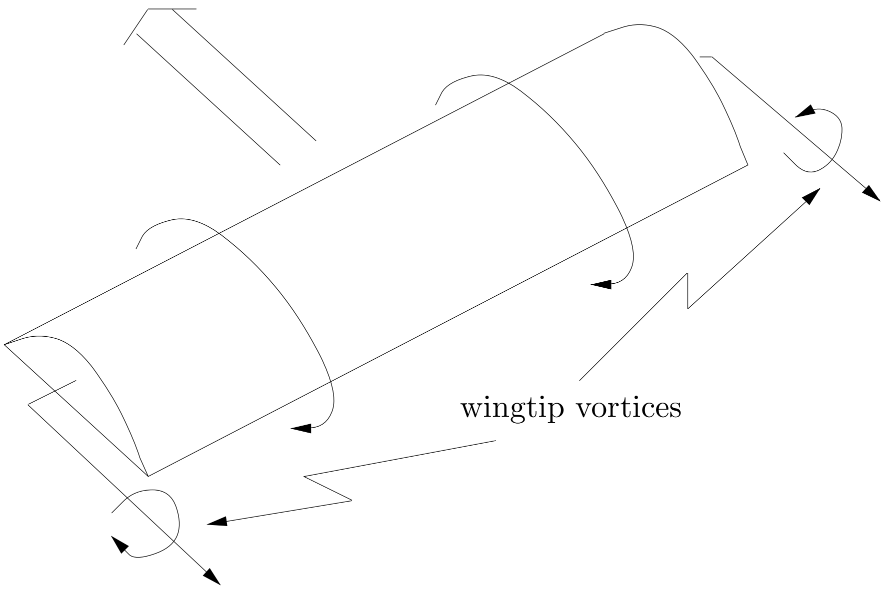

Figure 10.9. Lift from a wing of finite length.

longer than they are broad. For a finite cylinder, the bound vortex inside the cylinder must emerge into the fluid at some point, as vortex lines cannot end. This occurs typically at the end of the cylinder. As the cylinder is moving through the fluid, the vortex line trails downstream from the end of the cylinder. This is called a wingtip vortex, and is illustrated in figure 10.9, which shows the circulations around the airfoil and the wingtip vortices for an idealized "flying wing".

Problems

1. Water flows irrotationally in a channel as shown in figure 10.10. When it is going around the curve, is the water on the inside or outside of the curve flowing faster? Hint: Apply the circulation theorem around the path shown.

2. Compute the motions of two point vortices with known circulations separated by a distance d in two cases, a) when the circulations around the vortices are the same, and b) when they are equal but opposite in sign. Hint: Recall that vorticity moves with the fluid flow, including that flow induced by other vortices.

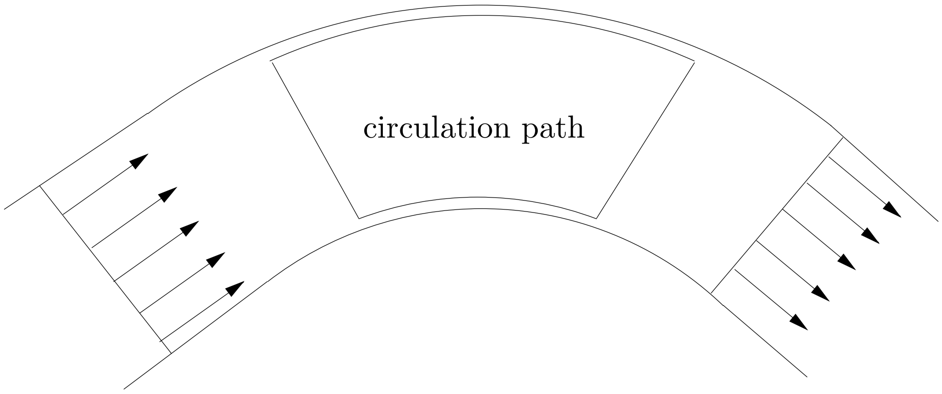

Figure 10.10. Water flowing irrotationally in a curved channel.

3. Show that the streamfunction

$$\psi = v_0 y \left(1 - \frac{R^2}{x^2 + y^2} \right)$$

represents two-dimensional, inviscid, irrotational flow about a cylinder of radius R with axis perpendicular to the flow.

4. Find the pressure field surrounding a point vortex using Bernoulli's equation in a fluid of constant density. Also find the streamfunction. Ignore body forces and assume two-dimensionality.

5. Show that if a flow is steady, inviscid, and has conservative body forces, but is rotational, Bernoulli's equation is valid along streamlines, but that different streamlines may have different Bernoulli "constants".

6. For an ideal gas with constant specific entropy, show that

$$\int dp/\rho = C_p T/m + constant \ ,$$

where C_p is the molar specific heat of the gas at constant pressure, m is its molecular weight, and T is the temperature.

7. Apply Bernoulli's equation for an ideal gas to a rocket engine to find the exhaust velocity of the engine as a function of the gas temperature inside the engine, the molecular weight of the gas, and its molar specific heat at constant pressure. Assume that no body forces are acting, and that the rocket is in

vacuum.

8. For a viscous fluid show that vortex lines cannot end on a solid surface. How then are "bound vortices" explained in the viscous case?