# 1 Waves in Special Relativity

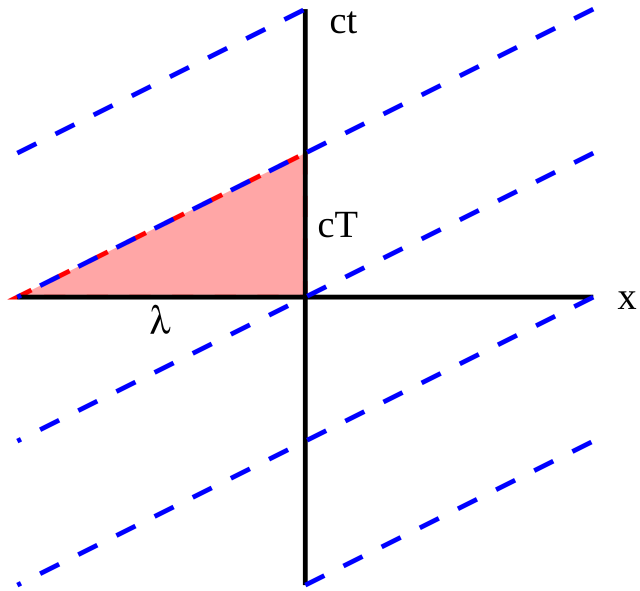


Figure 1: Wave fronts in spacetime are represented by the dashed lines. The wavelength $\lambda$ and the period $T$ are shown for the illustrated wave.

A spacetime picture of a wave is shown in figure 1. The wave fronts are represented by the dashed lines in this figure. The phase speed of the wave divided by the speed of light is $\lambda/cT$, which is also the inverse of the slope of the wave fronts. The wavelength, $\lambda$, is the interval between successive wave fronts in the space direction, while $c$ times the period, $T$, is the interval in the time direction.

Since "space direction" and "time direction" are not relativistically invariant concepts, it is clear that the period and wavelength of a wave will be different in different coordinate systems. Let us in particular transform to the primed coordinate system in which the wave fronts are horizontal. This view of the wave is shown in figure 2. The velocity of the primed coordinate system divided by the speed of light is $U/c = cT/\lambda$. This follows from the earlier conclusion that the slope of the wave fronts in the unprimed frame is $cT/\lambda$. In the primed frame of figure 2, these wave fronts are horizontal, or parallel to the lines of simultaneity of the primed frame. Thus, they are

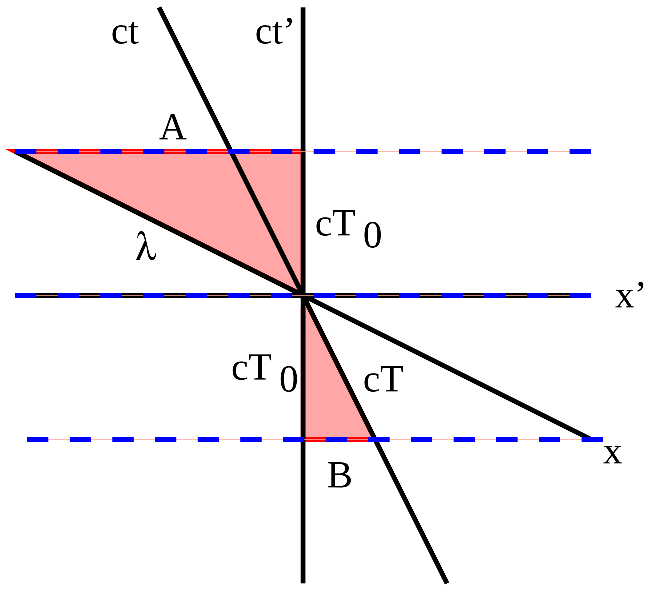


Figure 2: View of the wave of figure 1 in a reference frame in which the wave fronts are horizontal.

parallel to the wave fronts in the unprimed frame as well, and therefore also have the slope $cT/\lambda$ in this frame. Finally, we note that the slope of a line of simultaneity is equal to the velocity of the associated reference frame divided by $c$, from which the above assertion follows.

If the period of the wave in the primed frame is $T_0$, we can relate the wavelength and period in the unprimed frame to $T_0$ and $U$ using the Pythagorean theorem of spacetime. In particular, $A = cT_0/(U/c) = c^2T_0/U$ and $B = cT_0(U/c) = UT_0$. Then, since $\lambda^2 = A^2 - c^2T_0^2$ and $c^2T^2 = c^2T_0^2 - B^2$, we find that $\lambda = cT_0/\gamma$ and $T = T_0/\gamma$, where $\gamma = (1 - U^2/c^2)^{-1/2}$.

If we express wave parameters in terms of angular velocity, $\omega = 2\pi/T$, and wavenumber, $k = 2\pi/\lambda$, and further define $\omega_0 = 2\pi/T_0$, then we easily show that

$$\omega = \omega_0\gamma \qquad k = \omega_0 U\gamma/c^2. \tag{1}$$

we further note that

$$\omega^2 - k^2c^2 = \omega_0^2\gamma^2 - \omega_0^2\gamma^2U^2/c^2 = \omega_0^2. \tag{2}$$

This is the dispersion relation for relativistic waves. The phase and group velocities of a wave can be obtained from the dispersion relation. Since $\omega_0$ is a constant, we immediately see that $\omega d\omega = c^2 k dk$, which means that the group velocity, $u_g \equiv d\omega/dk$ is proportional to the inverse of the phase velocity: $u_\phi \equiv \omega/k = \lambda/T = c^2/U$. In particular, we find that

$$u_g = \frac{d\omega}{dk} = \frac{c^2 k}{\omega} = U. \tag{3}$$

What does all this mean? First of all, if the idea of a wave front is relativistically invariant, then the dispersion relation of the wave must take the form given by equation (2). Second, the group velocity resulting from this dispersion relation behaves sensibly when a Lorentz transformation is performed, in that it transforms like other velocities. Note in particular that the group velocity is zero when the wave fronts are horizontal in spacetime. This follows from equation (3) because $k = 0$ in this case. Furthermore, in a coordinate system moving with velocity $-U$ with respect to this special coordinate system, the group velocity is $+U$. This may seem like a trivial result, but in fact the *phase velocity* does *not* behave this way at all!

Planck and Einstein established that light consisted of photons with energy $E = \hbar\omega$. Louis de Broglie extended this idea to ordinary particles and further showed by arguments related to those given above that the particle momentum had to be related to the wavenumber by $p = \hbar k$. The key point in this argument is to note the relationship between the relativistic wave dispersion relation, $\omega^2 - k^2c^2 = \omega_0^2$ and the relativistic relationship between energy and momentum, $E^2 - p^2c^2 = m^2c^4$. In addition to the relationship between momentum and wavenumber, we see that $\omega_0 = mc^2$.