1 Energy and Momentum in Special Relativity

In these notes we explore the treatment of force, energy, and momentum in special relativity. It turns out that Newton's second law works in special relativity just as it does in non-relativistic mechanics *except* for modifications imposed by the geometry of spacetime. The approach we will take is as follows: We will assume that $F = ma$ retains its validity as long as speeds are low. We will then see how $F = ma$ changes when we change our reference frame to one in which the mass in question is moving at high speed. This will result in the modifications needed to adapt Newton's second law to special relativity. The key step in this procedure is developing the relativistic formula for the addition of velocities. From this we can determine how acceleration appears to change between reference frames. This leads to the relativistic form of $F = ma$, from which the concepts of relativistic energy and momentum emerge.

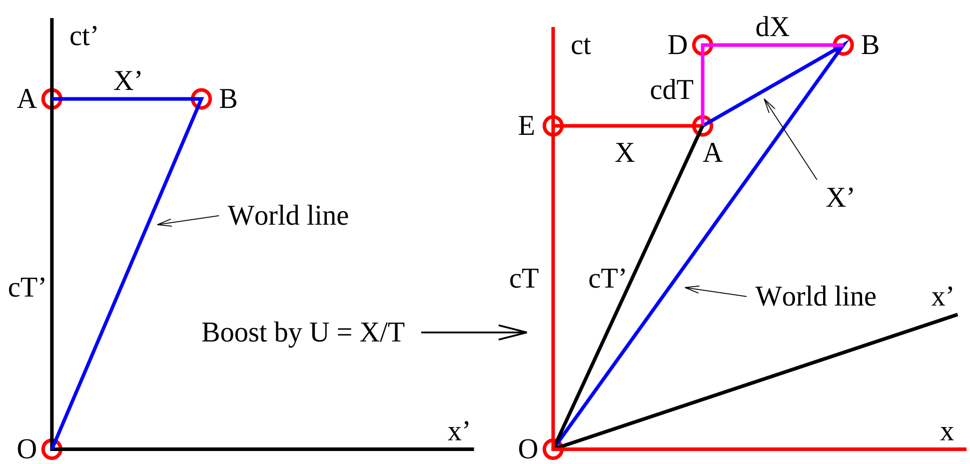

Figure 1: Definition sketch for relativistic velocity addition. The two panels show the world line of a moving object relative to two different reference frames.

1.1 Addition of Velocities

Figure 1 shows the world line of a moving object from the point of view of two different reference frames, with the primed frame (left panel) moving to the right at speed U relative to the unprimed frame (right panel). The object is to calculate the velocity of the object relative to the unprimed frame, v, assuming its velocity in the primed frame, v' is known. The classical result simply has

$$v = U + v' \qquad (classical\ result), \tag{1}$$

but this is inconsistent with the speed of light being constant in all reference frames.

From what we have learned about world lines, we know that

$$\frac{v'}{c} = \frac{X'}{cT'}, \tag{2}$$

$$\frac{U}{c} = \frac{X}{cT}, \tag{3}$$

and

$$\frac{v}{c} = \frac{X + dX}{cT + cdT}. \tag{4}$$

(Verify this for yourself!) Furthermore, the slope of the line of simultaneity A-B in the right panel of figure 1 is

$$\frac{cdT}{dX} = \frac{U}{c}. \tag{5}$$

The left side of equation (5) just results from the geometry of figure 1, while the right side comes from the fact that the slope of a line of simultaneity associated with some object is equal to the velocity of that object divided by the speed of light. The final pieces of information we need come from applying the spacetime Pythagorean theorem to the triangle A-B-D in the right panel of figure 1,

$$X' = (dX^2 - c^2dT^2)^{1/2}, \tag{6}$$

and to the triangle A-E-O:

$$cT' = (c^2T'^2 - X^2)^{1/2}. \tag{7}$$

Equations (2) — (7) can be combined to yield the relativistic equivalent to equation (1), but the analysis is a bit tricky. From equations (2), (3), (6), and (7), we find that

$$v' = \frac{X'}{T'} = \frac{dX^2 - c^2 dT^2}{T^2 - X^2/c^2} = \frac{dX^2 - (U/c)^2 dX^2}{T^2 - (U/c)^2 T^2} = \frac{dX}{T}. \tag{8}$$

This result may now be combined with equations (4) and (5) to get

$$v = \frac{X + dX}{T + dT} = \frac{X + v'T}{T + Uv'T/c^2}. \tag{9}$$

Finally, combining this with equation (3) results in the relativistic formula for the addition of velocities:

$$v = \frac{U + v'}{1 + Uv'/c^2} \qquad (\textit{special relativity}). \tag{10}$$

Notice how this equation behaves in various limits. If $|Uv'| \ll c^2$, the denominator of equation (10) is nearly unity, and the special relativistic formula reduces to the classical case. On the other hand, if $v' = c$, then equation (10) reduces to $v = c$. In other words, if the object in question is moving at the speed of light in one reference frame, it is moving at the speed of light in all reference frames, i. e., for all possible values of U. Thus, we have found a velocity addition formula that 1) reduces to the classical formula for low velocities and 2) gives the observed results for very high velocities as well.

1.2 Acceleration

We use the above results to determine how acceleration transforms from one reference frame to another. Figure 2 shows the world line of an *accelerated reference frame*, with a time-varying velocity $U(t)$ relative to the unprimed inertial frame. Defining $\delta U = U(T) - U(0)$ as the change in the velocity of the accelerated frame (relative to the unprimed frame) between events A and C, we can relate this to the change of velocity, $\delta U'$, of the accelerated frame *relative to an inertial frame moving with the initial velocity*, $U(0)$, of the accelerated frame. Applying equation (10), we find

$$U(T) = U(0) + \delta U = \frac{U(0) + \delta U'}{1 + U(0)\delta U'/c^2}. \tag{11}$$

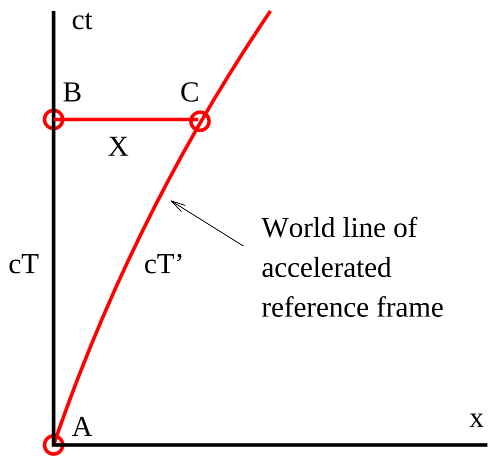

Figure 2: World line of the origin of an accelerated reference frame.

We now note that the mean acceleration of the reference frame between events A and C in the unprimed reference frame is just $a = \delta U/T$, whereas the mean acceleration in the primed frame between the same two events is $a' = \delta U'/T'$. From equation (11) we find that

$$\delta U = \frac{\delta U'(1 - U(0)^2/c^2)}{1 + U(0)\delta U'/c^2}, \tag{12}$$

and the acceleration of the primed reference frame as it appears in the unprimed frame is the velocity change over time, or

$$a = \frac{\delta U}{T} = \frac{\delta U'(1 - U(0)^2/c^2)}{T(1 + U(0)\delta U'/c^2)}. \tag{13}$$

Since we are interested in the instantaneous rather than the average acceleration, we let T become small. This has three consequences. First, δU and $\delta U'$ become small, which means that the term $U(0)\delta U'/c^2$ in the denominator of equation (13) can be ignored compared to 1. This means that

$$a \approx \delta U'(1 - U(0)^2/c^2)/T, \tag{14}$$

with the approximation becoming perfect as $T \to 0$. Second, the "triangle" with the curved side in figureaccel becomes a true triangle, with the result that $T' = T(1 - U(0)^2/c^2)^{1/2}$. The acceleration of the primed frame with respect to an inertial frame moving at speed $U(0)$ can therefore be written

$$a' = \frac{\delta U'}{T'} = \frac{\delta U'}{T((1 - U(0)^2/c^2)^{1/2}}. \tag{15}$$

Third, we can replace $U(0)$ with U, since the velocity of the accelerated frame doesn't change very much over a short time interval.

Dividing equation (14) by equation (15) results in a relationship between the two accelerations:

$$a = a'(1 - U^2/c^2)^{3/2}, \tag{16}$$

which shows that the apparent acceleration of a rapidly moving object, a, is less than its intrinsic acceleration, a', by the factor $(1 - U^2/c^2)^{3/2}$. This is purely the result of the geometry of spacetime, but it has interesting dynamical consequences. If we equate a' to F/m using Newton's second law, and further write the acceleration in the unprimed frame as the time derivative of the velocity of the accelerated frame, $a = dU/dt$, then we find

$$F = \frac{m}{(1 - U^2/c^2)^{3/2}} \frac{dU}{dt} = \frac{d}{dt}\left[\frac{mU}{(1 - U^2/c^2)^{1/2}}\right], \tag{17}$$

which is the relativistic version of Newton's second law. (The step from the middle expression to the right expression in this equation can be verified by carrying out the time differentiation with respect to U in the right expression.) Notice that this just reduces to $F = d(mU)/dt = ma$ when $U^2 \ll c^2$. In other words, the relativistic version of Newton's second law reduces to the classical version for low velocities.

Some of you may have read that the mass increases with velocity in special relativity. The mass being referred to in this case is the *relativitic mass*,

$$m_R = \frac{m}{(1 - U^2/c^2)^{1/2}}. \tag{18}$$

Notice that the relativistic version of Newton's second law can be written $F = d(m_R U)/dt$ — in other words, it looks like the non-relativistic version except that the rest mass is replaced by the relativistic mass. In this context, m is referred to as the *rest mass*. In this course, when we refer to mass with no qualifier, we mean rest mass. The rest mass doesn't change with velocity, but remains constant.

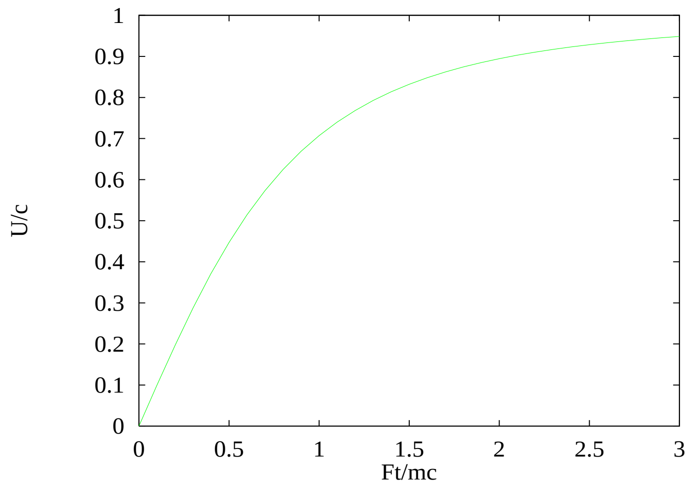

Figure 3: Velocity over the speed of light as a function of the product of the time and the (constant) force divided by the speed of light time the rest mass.

What does equation (17) tell us about the motion of an object at high speeds? Integrating equation (17) in time and assuming that $U = 0$ at $t = 0$ results in

$$Ft = \frac{mU}{(1 - U^2/c^2)^{1/2}}, \tag{19}$$

which may be solved for v/c,

$$\frac{U}{c} = \frac{Ft/mc}{[1 + (Ft/mc)^2]^{1/2}}, \tag{20}$$

which is plotted in figure 3. Classically, the velocity would reach the speed of light when $Ft/mc = 1$. However, as figure 3 shows, the rate at which the velocity increases with time slows as the object moves faster, such that U approaches c asymptotically, but never reaches it.

1.3 Momentum and Energy

The quantity

$$p = \frac{mU}{(1 - v^2/c^2)^{1/2}} \tag{21}$$

is identified as the momentum in special relativity. (Notice that we are now referring to the velocity as v rather than U). Thus, Newton's second law can be written compactly as $F = dp/dt$, which is formally identical to the nonrelativistic form. This form for the momentum is the only one that leads to the conservation of momentum in relativity, which is a powerful reason for defining it this way. Notice that the momentum of a particle with $v = c$ would be infinite unless the mass were zero. Furthermore, particles with zero mass (such as the photon and possibly the neutrino) *must always* travel with the speed of light — otherwise they would carry no momentum! Of course, for the special case of zero mass particles, equation (21) is formally invalid, since one is dividing zero by zero. In this case the momentum must be obtained from other considerations which will be discussed later.

The work done by a force F acting over a distance dx is just

$$dW = Fdx = Fvdt, \tag{22}$$

where we have used the fact that $v = dx/dt$. Substituting equation (17) in this results in

$$dW = \frac{d}{dt}\left[\frac{mv}{(1 - v^2/c^2)^{1/2}}\right] vdt = \frac{d}{dt}\left[\frac{mc^2}{(1 - v^2/c^2)^{1/2}}\right] dt, \tag{23}$$

where the equality of the middle and right-hand parts are best verified by differentiating each term out and comparing the results. The change in the kinetic energy may be obtained by integrating equation (23) in time. Since we know that the kinetic energy is zero for $v = 0$, we find a formula for the relativistic kinetic energy, K:

$$K = mc^2\left[\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right]. \tag{24}$$

What about the *total* energy? Einstein guessed from the form of this equation that the total energy, E, of an object of mass m is

$$E = mc^2\left[\frac{1}{(1 - v^2/c^2)^{1/2}}\right]. \tag{25}$$

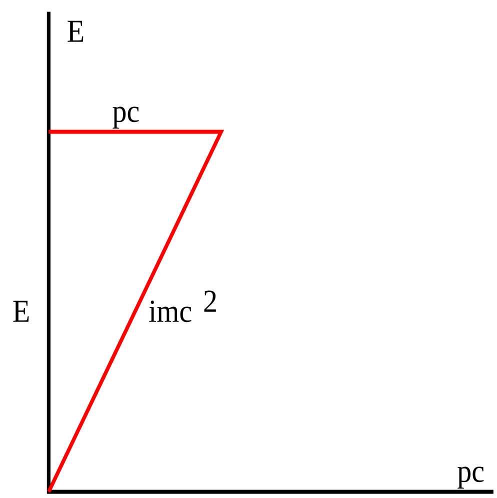

Figure 4: Energy-momentum diagram.

Subtracting equations (25) and (24) leaves what we call the *rest energy* of the object, which is a kind of potential energy:

$$E = mc^2. \tag{26}$$

This is a guess because there was no a priori reason for assuming that a potential energy of this form existed.

Not many people realize that Einstein's famous equation is actually based on a *guess*, but it is true! Everything up through equation (24) is based on solid extensions of classical physics, but the subsequent steps are not. However, Einstein's guess was an informed guess, because only by assuming that the total energy of a mass m is given by equation (25) do the total energy and the momentum obey the following equation:

$$p^2c^2 - E^2 = -m^2c^4. \tag{27}$$

(Verify this!!!) Why is this equation so important? Notice that it is in the form of the spacetime Pythagorean theorem that we discussed earlier, except that the sides of the triangle have units of energy rather than length.

One can think of an analog to the spacetime diagram in which the vertical axis is the energy of a mass and the horizontal axis is the speed of light times its momentum. This we call an energy-momentum diagram, and an example is shown in figure 4. There are two nice things about an energy-momentum diagram. First, the length of the hypotenuse of the triangle with sides pc and E is imc^2, or i times the rest energy of the mass m. This is an invariant quantity, so energy-momentum diagrams have something in common with spacetime diagrams. Second, using equations (21) and (25), the slope of the hypotenuse of the triangle in the energy-momentum diagram is

$$\frac{E}{pc} = \frac{c}{v}, \tag{28}$$

which is the same as the slope of the world line of the particle in the corresponding spacetime diagram! Thus, we think of $(E,\ pc)$ as being a "vector" in spacetime pointing in the direction of the world line and with length imc^2. The energy is the time-like component of the vector, while pc is the space-like component. This vector is called the *energy-momentum* vector.

In physics we often find that the simplest and most beautiful theory out of a range of possible theories is the one that is true. Einstein's guess leads to the simplest possible theory for energy and momentum in special relativity. It also turns out to be true, since it's predictions, among them the momentous $E = mc^2$, have been well verified.

1.4 Four-Vectors

Just as we represent quantities with direction and magnitude in ordinary space by a vector, we call quantities with direction and magnitude in spacetime *four-vectors*. The prototypical four-vector is the displacement vector $(\mathbf{x}, ct)$, where the first part is an ordinary three-vector in space, called the *space component*, and the second part is called the *time component*. The squared length of a vector in space time is the square of the space component *minus* the square of the time component. Thus our definition of spacetime interval generalized to three space dimensions is

$$I^2 = \mathbf{x} \cdot \mathbf{x} - c^2 t^2. \tag{29}$$

The squared length of a four-vector is a scalar, i. e., it's value doesn't depend on coordinate system.

The other spacetime vector we have seen so far is the energy-momentum vector of a particle, $(\mathbf{p}, E/c)$, where $\mathbf{p}$ is the ordinary momentum of the particle and E is its energy. We have actually shown graphically in figure 4 what the energy-momentum four-vector looks like. The length squared of the energy-momentum vector is

$$|(\mathbf{p}, E/c)|^2 = |\mathbf{p}|^2 - E^2/c^2, \tag{30}$$

which by comparison with equation (27) is seen to be $-m^2c^2$. Notice that with this vector the time component is divided by c to give it the same dimensions as the space component, namely the dimensions of momentum.

Another four-vector is the relativistic four-velocity. This is simply defined as the energy-momentum vector divided by m. The space component is thus the ordinary velocity divided by $(1 - v^2/c^2)^{1/2}$. The squared length of the four-velocity is simply $-c$. (Show this!!!)

The length squared of a four-vector is just the relativistic dot product of the vector with itself. In general, the dot product of two different four-vectors is a scalar, and is written as the ordinary dot product of the space parts of the two vectors minus the product of the time parts. Thus, for example, the dot product of the displacement and energy-momentum vectors is

$$\mathbf{p} \cdot \mathbf{x} - (E/c)ct = \mathbf{p} \cdot \mathbf{x} - Et. \tag{31}$$