1 Position and Momentum Operators

We found that the expectation value of the position could be represented as follows in the finite approximation:

$$< x >= \sum_{i,j} \psi_i^* x_i \delta_{ij} \psi_j, \qquad \sum_i \psi_i^* \psi_i = 1. \tag{1}$$

we can now take the limit as the domain over which x is defined is allowed to grow to infinite size and at the same time let the interval, Δx, at which the wavefunction is sampled go to zero. However, for this to work, we must impose a different normalization on the wave function by making the replacement $\psi_i \to \sqrt{\Delta x}\psi(x)$. In this way, the normalization condition in the case of continuous x becomes

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1 \tag{2}$$

and the formula for the expectation value becomes

$$< x >= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x)\delta(x - x')x'\psi(x')\ dx\ dx'. \tag{3}$$

The Dirac delta function has the property that

$$\int_{-\infty}^{\infty} f(x')\delta(x - x')dx' = f(x) \tag{4}$$

for any non-pathological function $f(x)$. Thus, it is common to reduce equation (3) to

$$< x >= \int_{-\infty}^{\infty} \psi^*(x)x\psi(x)dx. \tag{5}$$

However, the x operator must technically be written $\delta(x-x')x'$ if the analogy with the finite matrix case is to be maintained.

Let us now turn to the expectation value of the wavenumber:

$$< k >= \int_{-\infty}^{\infty} \phi^*(k)k\phi(k)dk. \tag{6}$$

Realizing that $\phi(k)$ is the infinite Fourier transform of $\psi(x)$,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\exp(-ikx)dx, \tag{7}$$

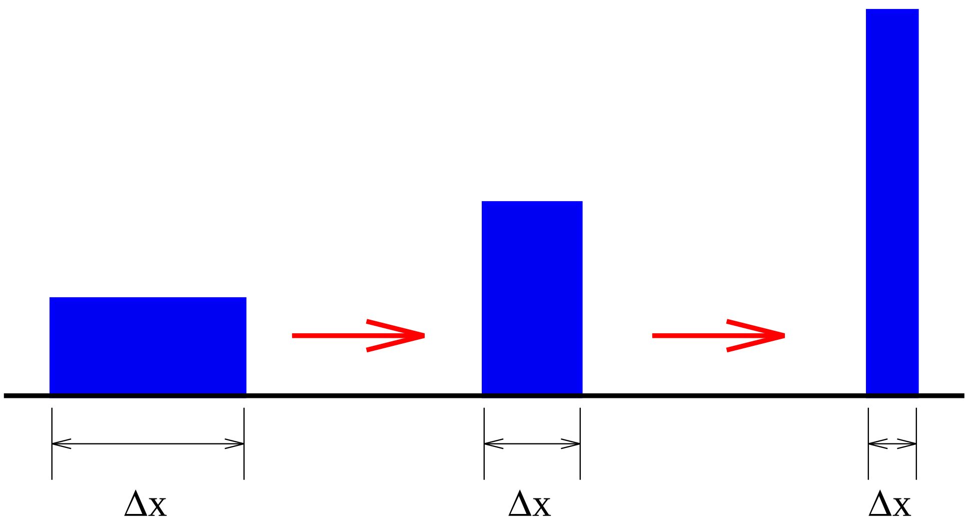

Figure 1: Limiting process in which $\delta_{ij}/\Delta x$ is turned into the Dirac delta function, $\delta(x)$. As Δx becomes smaller, $\delta_{ij}/\Delta x$ becomes taller in the region where $i = j$, i. e., in an interval of width Δx on the x-axis, such that the area of each rectangle is unity.

we can rewrite equation (6) as

$$< k >= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} k \exp[ik(x - x')] dk \right\} \psi(x') dx dx'. \quad (8)$$

The integral in curly brackets is actually divergent, and as a consequence the interchange of the order of integration we used to derive this equation is not strictly justified. However, we can extract the physical meaning by the following process. We note that

$$\int_{-\infty}^{\infty} k \exp[ik(x - x')] dk = i \frac{\partial}{\partial x'} \int_{-\infty}^{\infty} \exp[ik(x - x')] dk = i \frac{\partial}{\partial x'} \delta(x - x'). \quad (9)$$

Thus,

$$< k >= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) i \frac{\partial \delta(x - x')}{\partial x'} \psi(x') dx dx'. \quad (10)$$

Finally, integrating by parts leads us to

$$< k >= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \delta(x - x') i \frac{\partial \psi(x')}{\partial x'} dx dx' \quad (11)$$

leads us to conclude that the wavenumber operator in x space is just

$$k = -i\delta(x - x')\frac{\partial}{\partial x'}. \tag{12}$$

Equation (11) is generally simplified to

$$< k >= -i\int_{-\infty}^{\infty} \psi^*(x)\frac{\partial\psi(x)}{\partial x}dx, \tag{13}$$

and the wavenumber operator is correspondingly written

$$k = -i\frac{\partial}{\partial x}. \tag{14}$$

However, as with the position operator, the analogy with the finite matrix form is maintained more closely with the form containing the Dirac delta function, (12).

Since the momentum is just the wavenumber times $\hbar$, the momentum operator is just $p = \hbar k$.