1 Hilbert Space and Fourier Transforms

We first study vectors in ordinary three-dimensional space. We then extend the ideas developed here to higher dimensions and vectors with complex components. A vector space of this type is called a *Hilbert space*. We then study the Fourier transform and show that it is nothing more than a *unitary transformation* in Hilbert space.

1.1 Three-Space

A vector, $\mathbf{A}$, in ordinary space is a quantity with magnitude and direction. Thus vectors are defined without reference to a particular coordinate system.

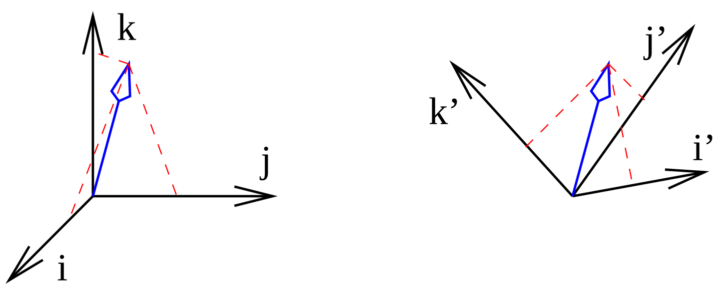

Figure 1: Illustration of the resolution of a vector into components in two different Cartesian coordinate systems rotated with respect to each other.

Vectors can be resolved into components relative to a particular coordinate system, as shown in the lefthand part of figure 1. If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors associated with the coordinate system, then $\mathbf{A} = A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}$. The components can be written in terms of dot products of the vector with the coordinate axis vectors: $A_x = \mathbf{A}\cdot\mathbf{i}$, etc.

As figure 1 shows, one can also resolve $\mathbf{A}$ onto coordinate axes of an alternative coordinate system rotated with respect to the original: $\mathbf{A} = A'_x\mathbf{i}' + A'_y\mathbf{j}' + A'_z\mathbf{k}$, with $A'_x = \mathbf{A}\cdot\mathbf{i}'$, etc. The two sets of components may be related by dotting $\mathbf{A}$ expressed in terms of the unprimed components with $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$:

$$\begin{aligned}
A'_x &= A_x\mathbf{i}\cdot\mathbf{i}' + A_y\mathbf{j}\cdot\mathbf{i}' + A_z\mathbf{k}\cdot\mathbf{i}' \\
A'_y &= A_x\mathbf{i}\cdot\mathbf{j}' + A_y\mathbf{j}\cdot\mathbf{j}' + A_z\mathbf{k}\cdot\mathbf{j}' \\
A'_y &= A_x\mathbf{i}\cdot\mathbf{k}' + A_y\mathbf{j}\cdot\mathbf{k}' + A_z\mathbf{k}\cdot\mathbf{k}'
\end{aligned} \tag{1}$$

This can be written in two different forms. In matrix notation we have

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{2}$$

while in index form it becomes

$$A'_i = \sum_{j=1}^{3} (\mathbf{e}'_i \cdot \mathbf{e}_j) A_j \tag{3}$$

where the changes $(A_x, A_y, A_z) \rightarrow (A_1, A_2, A_3)$ and $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \rightarrow (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ have been made.

The matrix $q_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$ may be expressed

$$q = \begin{pmatrix} \mathbf{e}'_1 \textit{ in unprimed coordinates} \\ \mathbf{e}'_2 \textit{ in unprimed coordinates} \\ \mathbf{e}'_3 \textit{ in unprimed coordinates} \end{pmatrix}, \tag{4}$$

from which it is clear that the rows of q are simply the primed unit vectors resolved in the unprimed frame. This implies that a row dotted with itself is unity, while one row dotted with another is zero. Thus, q is an *orthogonal transformation*. Orthogonal transformations leave the dot products of vectors unchanged.

From inspection of equation (2), it is clear that the inverse of the q transformation, i. e., the operation that sends the primed components to the unprimed components, is obtained simply by transposing q; in other words, $q^{-1} = q'$, where the prime indicates the transpose operation.

1.2 Extension to Hilbert Space

We now extend the above results to a vector space with an arbitrary number of dimensions and complex rather than real vector components. If A_i and B_i are the components of two vectors in an n-dimensional space, then the dot product of A and B becomes

$$(A, B) = \sum_{i=1}^{n} A_i^* B_i, \tag{5}$$

where the superscripted asterisk indicates the complex conjugate. Thus, dot products in Hilbert space don't commute; rather, $(B, A) = (A^*, B^*)$.

In Hilbert space, orthogonal transformations become *unitary transformations*. The only differences are that the extended dot product defined in equation (5) is used and the inverse transformation becomes the transposed complex conjugate or *adjoint* matrix rather than just the transposed matrix. We use the same notation for the adjoint operation as we did for the simple transpose in Hilbert space, namely the "prime" symbol.

1.3 Fourier Transform as a Unitary Transformation

The complex Fourier series representation of a function $f(x)$ over the interval $-L/2 \leq x \leq L/2$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp(ik_n x). \tag{6}$$

The inversion of this representation to obtain C_n is

$$C_m = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \exp(-ik_m x) dx. \tag{7}$$

Suppose we now want to represent $f(x)$ by its values at M equally spaced points, $x_j = j\Delta x$, where $j = -M/2+1, -M/2+2, \ldots, M/2$. The above integral can then be approximated by

$$C_m = \frac{1}{M} \sum_j f(x_j) \exp(-ik_m x_j) \tag{8}$$

where we have used the fact that $M\Delta x = L$. At the same time it makes sense to truncate the sum in equation (6) to M points as well, where $m = -M/2+1, -M/2+2, \ldots, M/2$:

$$f(x_j) \equiv f_j = \sum_m C_m \exp(ik_m x_j). \tag{9}$$

These two equations can be written more symmetrically if we set $g_m = M^{1/2} C_m$:

$$f_j = \frac{1}{\sqrt{M}} \sum_m g_m \exp(ik_m x_j) \qquad g_m = \frac{1}{\sqrt{M}} \sum_j f_j \exp(-ik_m x_j). \tag{10}$$

The above equations define the *finite Fourier transform.* These may be written in matrix form as

$$f = q'g \qquad\qquad g = qf, \tag{11}$$

where

$$q_{mj} = \frac{1}{\sqrt{M}} \exp(-ik_m x_j). \tag{12}$$

Since the inverse transform is the adjoint of the forward transform, the transformation is unitary.