1 Multiple Particles

1.1 Nonidentical Particles

So far we have dealt exclusively with the quantum mechanics of single particles. When multiple particles are involved, we seek a single state vector that describes all particles in the system. As an example, imagine two particles moving along the x-axis. A wave function describing these two particles would take the form $\psi(x_1, x_2)$, which is the amplitude to find simultaneously particle 1 at x_1 *and* particle 2 at x_2.

Decomposing the state vector in energy space rather than position space, we could also represent the state of the system by $A(E_i, E_j)$, which is the amplitude for particle 1 to have energy E_i and particle 2 to have energy E_j.

Normalization of the state vector in these two cases would require

$$\int dx_1 \int dx_2 |\psi(x_1, x_2)|^2 = 1 \tag{1}$$

for the space representation and

$$\sum_{i,j} |A(E_i, E_j)|^2 = 1 \tag{2}$$

in the energy representation.

If the two particles do not interact, then the state vector can be broken down into a sum of products of individual state vectors. For instance, if $\psi_{1i}(x_1)$ and $\psi_{2j}(x_2)$ are the energy eigenvectors for the first and second particles, then the wave function for the system of two particles can be expressed as

$$\psi(x_1, x_2) = \sum_{i,j} A(E_i, E_j)\psi_{1i}(x_1)\psi_{2j}(x_2). \tag{3}$$

The time dependence of each A takes the form $\exp(-iE_it)\exp(-iE_jt) = \exp[-i(E_i + E_j)t]$. If the two particles *do* interact, say, by some force between them, then the wave function still takes the above form, but the A's have a more complex time dependence.

Time independent perturbation theory can be used to obtain the approximate energies of systems of particles which interact via some weak potential energy, say, $U(x_2 - x_1)$. In the case of no interactions, the unperturbed eigenvectors are simply $\psi_{ij}^{(0)}(x_1, x_2) = \psi_{1i}(x_1)\psi_{2j}(x_2)$ and the corresponding energy eigenvalues are just $E_{ij}^{(0)} = E_i + E_j$. Perturbation theory then says that the perturbed energies are given by

$$E_{ij} = E_{ij}^{(0)} + \int dx_1 \int dx_2 \psi_{1i}(x_1)' \psi_{2j}(x_2)' U(x_2 - x_1) \psi_{1i}(x_1) \psi_{2j}(x_2) \qquad (4)$$

to first order, with corresponding expressions for the corrected wave functions.

The above scheme shows how to construct the eigenstates for a two particle system in the case where the particles either interact weakly or don't interact at all. First, construct the energy eigenvectors, $\psi_i(x)$, and eigenvalues, E_i, for the individual particles. Second, construct the combined eigenvectors and eigenvalues, $\psi_{ij}(x_1, x_2) = \psi_{1i}(x_1)\psi_{2j}(x_2)$ and $E_{ij} = E_i + E_j$ for each value of i and j. If the particles don't interact, we are done. However, if they do, the corrected energy eigenvalues can be obtained using equation (4).

1.2 Identical Particles

The above development must be modified if the two particles being considered are identical, e. g., two electrons, two photons, etc. In quantum mechanics, two identical particles are truly indistinguishable, which means that exchanging the two particles cannot change their joint probability distribution. This turns out to have far-reaching consequences.

Suppose that X is an operator that exchanges the particles in a two particle system. Then,

$$X\psi(x_1, x_2) = \psi(x_2, x_1). \qquad (5)$$

If we apply X twice to a two particle system, then the system is returned to its original state, i. e.,

$$X^2\psi(x_1, x_2) = \psi(x_1, x_2). \qquad (6)$$

Thus, X^2 is the identity operator, which has all eigenvalues equal to unity. The operator X must therefore have eigenvalues ± 1. An eigenstate of X must therefore be either symmetric or antisymmetric under the exchange of particles.

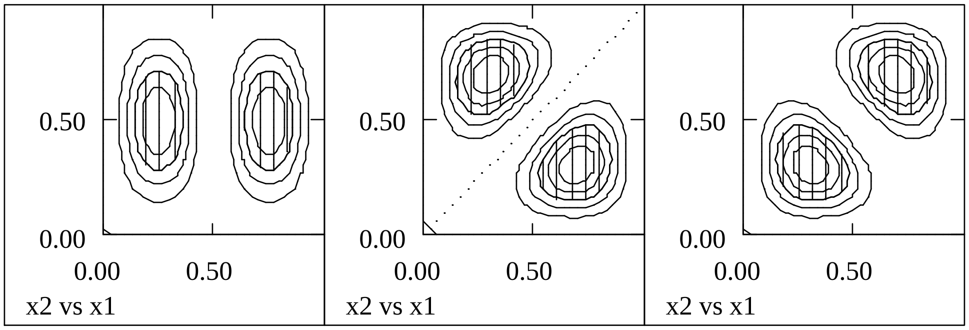

Figure 1: Joint space probability distributions for two particles in a square well, one in the ground state, the other in the first excited state. Left: Non-identical particles. Middle: Identical fermions. Right: Identical bosons.

The Hamiltonian, H, shouldn't change if identical particles are interchanged. Thus, $[X, H] = 0$, and X and H share eigenvalues. Furthermore, once a state is either symmetric or antisymmetric to the interchange of identical particles, it remains that way forever.

It turns out that particles with half-integral spins, i. e., fermions, are antisymmetric with respect to interchange, while particles with integral spins, i. e., bosons, are symmetric. There is a proof of this buried deep in quantum field theory, but at this level we simply accept it as an empirical result.

Joint states of two identical particles may be constructed as was done above if one additional step is taken, namely the enforcement of the symmetry or antisymmetry of the joint state. This is easily done as follows. We write the joint state of two non-interacting bosons or fermions as

$$\psi(x_1, x_2) = [\psi_1(x_1)\psi_2(x_2) \pm \psi_1(x_2)\psi_2(x_1)]/N, \tag{7}$$

where the plus sign applies to bosons and the minus sign applies to fermions. The number N is a normalization factor which enforces equation (1). An equivalent construction for energy space is

$$A_J(E_i, E_j) = [A_1(E_i)A_2(E_j) \pm A_1(E_j)A_2(E_i)]/N, \tag{8}$$

where $A_1(E)$ and $A_2(E)$ are the amplitudes for particle 1 and particle 2 to have energy E and A_J is the joint amplitude.

1.3 Pauli Exclusion Principle

A consequence of equation (8) is that non-interacting identical fermions cannot occupy the same single particle state. If the fermions interact via some potential, then the actual states are modified from the single particle states, but the same result holds — no more than a single fermion may occupy a perturbed single particle state. This result has enormous consequences, for instance, in the structure of matter and the nature of the chemical elements.

Analogous behavior occurs in position space. Figure 1 shows $|\psi(x_1, x_2)|^2$ for two particles in an infinite square well, with one particle in the ground state and the other in the first excited state. Three different cases are shown, (1) two non-identical particles, (2) two identical fermions, and (3) two identical bosons. Notice that the fermions are "antisocial", i. e., they try to remain as far from each other as possible. In particular, the probability of finding the two particles in the same location (i. e., where $x_1 = x_2$) is zero. If one of the fermions inhabits the left end of the box, the other is most likely to be found at the right end.

1.4 Bose-Einstein Condensation

Figure 1 shows that, unlike fermions, bosons behave like "sheep", in that they like to stick together. In particular, if one particle is found at a particular location, the other is most likely to be found there as well. This is behavior which is accentuated as more particles are added to the system. If many bosons are involved, a macroscopically observable quantum state may be created in which multiple bosons move in lockstep. Superconductivity and superfluidity in liquid Helium are examples of such behavior. In the former case, the bosons are electron pairs in which the total spin adds to zero. In the later case the bosons are He^4 atoms.

1.5 Entwinement of States

Figure 1 graphically indicates that identical particles cannot be considered individually, but must be treated collectively. This can even be true for identical particles separated by large distances.

For free photons the spin is either aligned with the direction of motion, resulting in right circular polarization, or antialigned, resulting in left circular

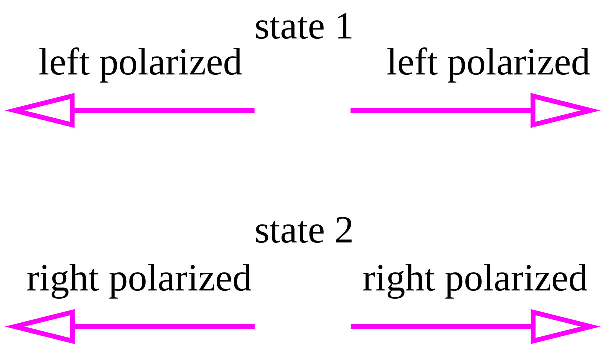

Figure 2: Sketch of two possible joint circular polarization states of two photons emitted in opposite directions with total spin zero.

polarization. (Though the photon has spin one and thus should have a third spin orientation in which the component of angular momentum along the direction of motion is zero, this orientation does not occur for free photons.)

Suppose two photons are emitted in opposite directions by a piece of apparatus as shown in figure 2 such that the spins of the photons are not individually known, but are known to be oppositely aligned, so that the total spin angular momentum is zero. Since the photons are moving in opposite directions, they both have either left circular polarization or right circular polarization. In this situation the polarization of each individual photon is not known a priori, as the state of each photon is not an eigenstate of that photon's angular momentum. It is only known that the polarizations are the same. However, if the polarization of *one* of the photons is *measured*, then the polarization of the *other* photon immediately becomes known. This is true even if the two photons are, say, 10 light years apart!

The states of the two photons are said to be *entwined*. This really means that there is a joint state in which particular polarization or spin amplitudes for different particles are correlated with each other. This aspect of quantum mechanics bothered Einstein so much that he put a great deal of effort into trying to disprove the theory. However, experiments have shown that Einstein's "spooky action at a distance", which results from the above-described entwinement of states, actually occurs.