1 Harmonic Oscillator in Two Dimensions

The harmonic oscillator in two dimensions is an instructive problem for several reasons. First, it illustrates how certain kinds of problems with more than one degree of freedom may be handled. Second, it can be solved in both Cartesian and cylindrical coordinates under certain circumstances. Third, it introduces the concept of angular momentum in simplified form. This serves a preparation for considering angular momentum in three dimensions, which is a much more complicated topic.

1.1 Solution in Cartesian Coordinates

In Cartesian coordinates x and y, the scaled Hamiltonian has the form

$$H = (p_x^2 + p_y^2 + x^2 + \lambda^2 y^2)/2, \tag{1}$$

where we have retained the option of giving the x and y spring constants different values by setting the constant λ to a value other than unity.

In position representation the time-independent Schrödinger equation is

$$-\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + (x^2 + \lambda^2 y^2)\psi = 2E\psi, \tag{2}$$

where E is the energy and $\psi(x, y)$ is the wavefunction. Solution of this equation by separation of variables is possible. Setting $\psi(x, y) = X(x)Y(y)$ in the above equation and dividing by ψ, we find

$$\left(-\frac{1}{X}\frac{d^2X}{dx^2} + x^2\right) + \left(-\frac{1}{Y}\frac{d^2Y}{dy^2} + \lambda^2 y^2\right) = 2E. \tag{3}$$

Since the first major term is a function only of x and the second is a function only of y, these terms individually must be equal to constants which we define to be $2E_x$ and $2E_y$. This results in two equations,

$$-\frac{d^2X}{dx^2} + x^2 X = 2E_x X \tag{4}$$

and

$$-\frac{d^2Y}{dy^2} + \lambda^2 y^2 Y = 2E_y Y, \tag{5}$$

where it is clear that $E_x + E_y = E$.

We recognize equations (4) and (5) as individual harmonic oscillator equations with respective eigenvalues

$$E_x = 1/2,\ 3/2,\ 5/2, \ldots \qquad E_y = \lambda/2,\ 3\lambda/2,\ 5\lambda/2, \ldots. \tag{6}$$

The total energy is

$$E_{ij} = i + \lambda j + (1+\lambda)/2; \quad i = 0,\ 1,\ 2,\ \ldots;\ j = 0,\ 1,\ 2,\ \ldots, \tag{7}$$

and the associated eigenvectors for the full problem are

$$\psi_{ij}(x, y) = X_i(x) Y_j(y), \tag{8}$$

where $X_i(x)$ is the ith eigenvector of equation (4) and $Y_j(y)$ is the jth eigenvector of equation (5).

Table 1 shows the first 15 energies for the two-dimensional harmonic oscillator as computed from equation (7) and sorted into order of increasing energy. Notice that *degeneracy*, i. e., repeated energy values, occurs for $\lambda = 1$, but not for $\lambda = 0.7$. The existence of degeneracy in the energy means that some other operator must commute with the Hamiltonian. The nature of this operator is no mystery — the angular momentum of the system is conserved when $\lambda = 1$, but not for other values of λ. Thus, it seems plausible that the repeated energy values in table 1 represent states (or superpositions of states) with definite values of the angular momentum.

1.2 Solution in Polar Coordinates

Angular momentum is best understood in polar coordinates for two dimensional problems. We let $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$, and transform equation (2) with $\lambda = 1$ to polar coordinates:

$$-\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + r^2 \psi = 2E\psi. \tag{9}$$

In polar coordinates the kinetic energy may be written (for unit mass)

$$K = (p_r^2 + L^2/r^2)/2, \tag{10}$$

$\lambda = 0.7$			$\lambda = 1$		
i	j	energy	i	j	energy
0	0	0.85	0	0	1.00
0	1	1.55	1	0	2.00
1	0	1.85	0	1	2.00
0	2	2.25	1	1	3.00
1	1	2.55	2	0	3.00
2	0	2.85	0	2	3.00
0	3	2.95	1	2	4.00
1	2	3.25	2	1	4.00
2	1	3.55	0	3	4.00
0	4	3.65	3	0	4.00
3	0	3.85	2	2	5.00
1	3	3.95	0	4	5.00
2	2	4.25	3	1	5.00
0	5	4.35	4	0	5.00
3	1	4.55	1	3	5.00

Table 1: Energies for the two-dimensional harmonic oscillator in terms of the quantum numbers i and j. The first three columns show the results for $\lambda = 0.7$ while the last three columns show $\lambda = 1$.

where p_r is the radial component of the momentum and L is the angular momentum about the axis normal to the x-y plane. Comparison with equation (9) suggests that

$$p_r^2 = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} \tag{11}$$

and

$$L^2 = -\frac{\partial^2}{\partial \theta^2}. \tag{12}$$

One might naively expect the radial momentum squared operator to take the form $p_r^2 = -\partial^2 (\ \ )/\partial r^2$, but the complexities of polar coordinates introduce the additional term. Note that angular momentum has the same units as Planck's constant, so angular momentum is dimensionless in a system of units in which $\hbar = 1$.

Equation (9) may be solved by separation of variables. Setting $\psi(r,\theta) = R(r)\Theta(\theta)$, substituting into the above equation, and dividing by ψ, we find

$$-\frac{1}{R}\frac{d^2R}{dr^2} - \frac{1}{rR}\frac{dR}{dr} - \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + r^2 = 2E. \tag{13}$$

The combination $\Theta^{-1}\partial^2\Theta/\partial\theta^2$ must be equal to a constant, which we call $-m^2$ for reasons that will soon be evident. The variables R and Θ therefore obey the separate equations

$$-\frac{1}{R}\frac{d^2R}{dr^2} - \frac{1}{rR}\frac{dR}{dr} + \frac{m^2}{r^2} + r^2 = 2E \tag{14}$$

and

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0. \tag{15}$$

The Θ equation has obvious solutions $\Theta = \exp(\pm im\theta)$. Since we need to have these solutions single valued in θ, the parameter m must take on integer values only. The reason for choosing a negative separation constant $-m^2$ is now clear; it results in periodic solutions to the Θ equation that make this single valuedness possible.

Solution of the R equation is more difficult. It turns out that equation (14) is actually quite similar to the one-dimensional harmonic oscillator equation, in that it has solutions of the form $R(r) = P(r)\exp(-r^2/2)$ where $P(r)$ is a polynomial. The function $P(r)$ obeys

$$\frac{d^2P}{dr^2} + (1/r - 2r)\frac{dP}{dr} - (m^2/r^2 + 2 - 2E)P = 0. \tag{16}$$

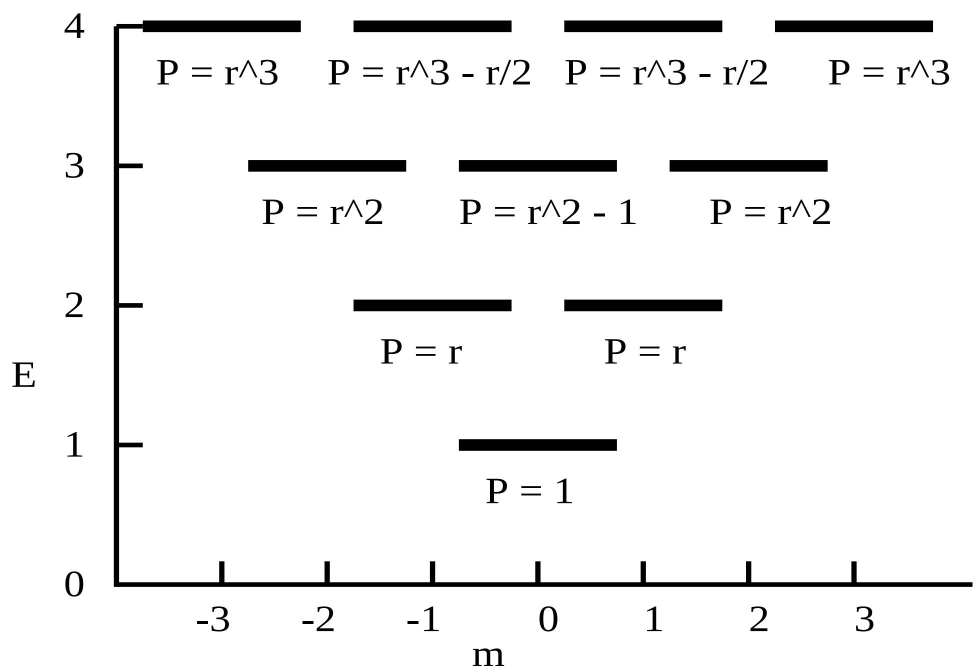

Figure 1: Eigenstates of the two-dimensional harmonic oscillator.

The first four energy levels may be obtained by assuming

$$P(r) = r^3 + ar^2 + br + c. \tag{17}$$

Solutions of this form only exist when E and m take on various integer values. Figure 1 show the actual values of E and m that solve equation (16), along with the corresponding polynomials.

From these few low order solutions the pattern is clear. One solution exists for $E = 1$, two solutions for $E = 2$, etc. This is in agreement with the results shown in table 1. The allowed values of angular momentum for each energy are $-E+1,\ -E+3,\ \ldots,\ E-1$, which means that there are E different allowed angular momentum values for each energy.

1.3 Comparison of Solutions

Comparison of the solutions to the two-dimensional harmonic oscillator in Cartesian and polar coordinates is enlightening. Let us first determine whether these solutions are indeed the same, or at least consistent.

For the lowest energy state it was shown in the section on the ono-dimensional harmonic oscillator that the unnormalized eigenfunction is simply $\exp(-x^2/2)$. Thus, the two-dimensional Cartesian solution is $\exp(-r^2/2)$, which is in agreement with the lowest energy solution in polar coordinates.

For the first excited state, table 1 shows that we either have $i = 1$ and $j = 0$ or $i = 0$ and $j = 1$. Since the first excited eigenfunction of the one-dimensional harmonic oscillator is $x\exp(-x^2/2)$, the full eigenfunctions first excited state of the two-dimensional oscillator from the Cartesian coordinate system analysis are $\psi_{1,0}(x,y) = r\cos(\theta)\exp(-r^2/2)$ and $\psi_{0,1}(x,y) = r\sin(\theta)\exp(-r^2/2)$. However, from the analysis in polar coordinates, the first excited state eigenfunctions are $\psi_{1,-1}(r,\theta) = r\exp(-r^2/2 - i\theta)$ and $\psi_{1,1}(r,\theta) = r\exp(-r^2/2 + i\theta)$, where the subscripts of $\psi(r,\theta)$ indicate respectively the energy and angular momentum quantum numbers. Comparison of these two sets of solutions shows that

$$\psi_{1,-1}(r,\theta) = \psi_{1,0}(x,y) - i\psi_{0,1}(x,y) \tag{18}$$

and

$$\psi_{1,1}(r,\theta) = \psi_{1,0}(x,y) + i\psi_{0,1}(x,y). \tag{19}$$

Thus, the Cartesian coordinate eigenfunctions represent *standing wave* solutions in θ, while the polar coordinate eigenfunctions represent *travelling wave* solutions. Similar arguments can be made about solutions of higher order. The above equations show that travelling waves can be represented in terms of standing waves in quantum mechanics. Since these equations are easily inverted, the reverse is true as well.

The angular momentum equation (15) by itself allows the angular momentum to take on any integer value. However, constraints imposed by the radial equation (14) further limit the possible values of angular momentum. Why is this? Classically, motion in an attractive axially symmetric potential has a maximum possible angular momentum for a given total energy. The limitation $|m| \leq E$ in the quantum mechanical two-dimensional oscillator is thus no surprise.

A more interesting restriction is that when $E = 2$ there is no $m = 0$ state, when $E = 3$ there are no states with $m = \pm 1$, etc. The answer to this lies in the parity operator P, which in two dimensions has the following properties:

$$P\psi(x, y) = \psi(-x, -y). \tag{20}$$

Thus, for instance, from the Cartesian coordinate system analysis, the first excited state has negative parity since $P\psi_{1,0}(x, y) = -\psi_{1,0}(x, y)$. However, all $m = 0$ states must have positive parity from the polar coordinate analysis, since they are axially symmetric. Thus, a first excited state with $m = 0$ cannot exist. Similar arguments can be made about missing m values in higher order states. In particular, states with even values of the energy quantum number n must have even values of m and vice versa.

1.4 General Comments

The difference between the $\lambda = 0.7$ and the $\lambda = 1$ cases is a fundamental one that is not limited to quantum mechanics. Figure 2 shows classical particle trajectories for these two cases in x-y space. Note that for $\lambda = 1$ the particle makes a repeated trajectory which is just a circle. (Depending on the initial conditions, other trajectories are possible, but they are still repeated.) For $\lambda = 0.7$ the trajectory is much more complex, and though it will eventually repeat, the period for repetition being much longer than the harmonic oscillator frequencies in either the x or y direction. In general,

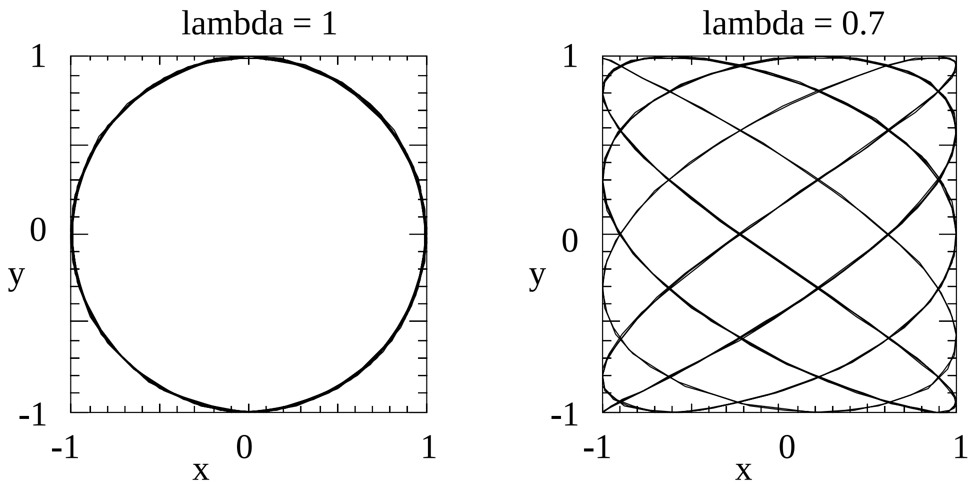

Figure 2: Trajectory of classical two-dimensional harmonic oscillator in x-y plane for two different values of λ.

the trajectory will repeat if λ is a rational number, i. e., a ratio of integers, with the repetition period being longer as the integers become bigger. The trajectory does not repeat if λ is an irrational number. Analogously, there is occasional energy degeneracy in the quantum mechanical problem if λ is rational, but no degeneracy at all if λ is irrational. This follows directly from equation (7).

The two-dimensional harmonic oscillator is a prototype for problems with more than one degree of freedom in quantum mechanics. By chance we can solve this particular problem exactly in Cartesian coordinates for all values of λ. However, we can easily solve it in polar coordinates only for $\lambda = 1$. This we can do because the Hamiltonian has many degenerate eigenvalues when $\lambda = 1$, which means that there exists another operator, namely the angular momentum operator, which commutes with the Hamiltonian. In this case the computation of the eigenvalues and the eigenvectors in the two-dimensional problem reduces to the computation of eigenvalues and eigenvectors in two separate one-dimensional problems. The computational effort is *much* less than is needed to solve the two-dimensional problem directly.

Generally speaking, problems in quantum mechanics with more than one degree of freedom break down into two types, those that factor into a set of smaller problems, as in the $\lambda = 1$ case above, and those that don't. Let us call the former type of problem an *easy* problem and the latter type a *difficult* problem. Difficult problems that differ only slightly from an easy problem are often solved approximately by perturbation theory. It is not an exaggeration to characterize progress in quantum mechanics as consisting of exact solutions to easy problems complemented by approximate perturbation solutions to difficult problems.