1 Time Evolution and the Energy

If the possible values of frequency for some particle are ω_j, $j = 1, 2, 3, \ldots$, and the amplitude to have frequency ω_j is A_j, then the expectation value of frequency is

$$< \omega >= \sum_j A_j^* \omega_j A_j = (A, \omega A). \tag{1}$$

The position space representation of the particle state represented by the values A_i must simply be given by

$$\psi(x_m) = \psi_m = \sum_j {q'}_{mj} A_j, \tag{2}$$

or

$$\psi = q' A \tag{3}$$

where ${q'}_{mj}$ is the matrix representing the unitary transformation from the energy representation of the state vector to the position representation. The inverse transformation is just

$$A = q\psi. \tag{4}$$

Proceeding as with the momentum operator, we can therefore write

$$< \omega >= (q\psi, \omega q\psi) = (\psi, q'\omega q\psi), \tag{5}$$

where $q'\omega q$ is the frequency operator in position space.

So far we have not asked how state vectors in *any* representation vary in time. However, we know from the work of Planck, Einstein, and de Broglie that the amplitude A_j for a particle with definite frequency ω_j has a time dependence $\exp(-i\omega_j t)$. We can therefore construct the time-dependent state vector resolved in position space by assuming that $A_j \propto \exp(-i\omega_j t)$:

$$\psi(t) = q' A(t). \tag{6}$$

Notice that

$$i\frac{\partial \psi(t)}{\partial t} = q' i \frac{\partial A}{\partial t} = q'\omega A = (q'\omega q) q' A = (q'\omega q)\psi. \tag{7}$$

Thus, the frequency operator in position space is simply i times the time derivative.[1]

[1] More properly, to keep our analogy with finite matrices, we should write this representation of the frequency operator as $i\delta(t - t')(\partial/\partial t')$ and the left side of equation 7 as $i \int \delta(t - t')[\partial\psi(t')/\partial t']dt'$.

Equation (7) is generally multiplied by $\hbar$, turning the frequency into energy. The operator $\hbar q' \omega q'$ is called the *Hamiltonian*, H, and it is just the operator whose eigenvalues are the possible energies of the system. The resulting system is just a generalization of Schrödinger's equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{8}$$

This may be seen to be true for a single non-relativistic particle if the Hamiltonian, or energy operator, is related to the position and momentum operators as in classical physics. In particular, for a non-relativistic particle of mass m subject to the potential $U(x)$,

$$H = \frac{p^2}{2m} + U(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x), \tag{9}$$

and this equation becomes

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x} + U(x)\psi. \tag{10}$$

The mathematical structure underlying quantum mechanics is now almost complete. The goal in quantum mechanics is to obtain the Hamiltonian operator for the physical system of interest in some representation. Then the Schrödinger equation (8) is either solved directly or the eigenvectors, $\psi_j(x)$, and eigenvalues, $E_j = \hbar\omega_j$, of the Hamiltonian are found by some other means. In the latter case the evolution of the state vector in the original representation is obtained from equation (6), with A at time $t = 0$ being obtained from equation (4). The advantage of this approach over de Broglie's and Schrödinger's is that it applies to *any* system for which the Hamiltonian can be found — not just the case of a single non-relativistic particle.