Electromagnetism and the Extension of Potential Energy to Relativity

David J. Raymond
Physics Department and Geophysical Research Center
New Mexico Tech
Socorro, NM 87801
USA

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Summary

1 Introduction

The idea of potential energy as it is commonly expressed is not relativistically invariant, even with Galilean relativity. To see this, it is sufficient to postulate a one-dimensional potential energy $U(x)$ in the rest frame. In a reference frame moving with velocity V relative to the rest frame, the potential energy function becomes $U(x' + Vt)$ where x' is the space coordinate in the moving frame and t is the time. The potential energy in the moving frame is thus an explicit function of time, and total energy is not conserved.

A simple example of this occurs for a space probe moving through the gravitational field of a planet. In the frame in which the planet is stationary, the probe moves about the planet in a hyperbolic orbit and potential plus kinetic energy is everywhere conserved. Thus, in particular, at large distances where the potential energy is negligible, the incoming kinetic energy of the probe equals its outgoing kinetic energy.

The picture is very different in a reference frame in which the planet is moving. Depending on the directions of approach and recession, the probe may either gain or loose energy. This fact is of course central to the use of planets as "slingshots" to boost the energy of space probes to the outer planets.

With Einsteinian relativity the problems are even more severe. The four-force must take on a covariant form. This is complicated by the fact that the potential energy U must be the timelike component of some four-momentum. Thus, $\mathbf{F} = -\nabla U$ is clearly not the spacelike part of a four-force.

2 Notation and some basics

In this paper a four-vector is denoted by an underscore. Thus, for instance, the position vector in spacetime is $\underline{x} = (\mathbf{x}, ct)$ where c is the speed of light, while the momentum four-vector is $\underline{p} = (\mathbf{p}, E/c)$ where $\mathbf{p}$ is the ordinary momentum and E is the energy. The length of a four-vector is taken as the square of the spacelike part minus the square of the timelike part in agreement with the usual definition of invariant interval. Thus, for a particle with four-momentum $\underline{p}$, we have $\underline{p} \cdot \underline{p} = |\mathbf{p}|^2 - E^2/c^2 = -m^2c^2$, where m is the mass of the particle. The four-gradient is defined

$$\underline{\Box} = \left(\nabla, -\frac{1}{c}\frac{\partial}{\partial t}\right). \tag{1}$$

The minus sign in the timelike part is needed so that the four-divergence of a four-current $\underline{j} = (\mathbf{J}, \rho c)$ comes out right:

$$\underline{\Box} \cdot \underline{j} = \nabla \cdot \mathbf{J} - \left(-\frac{1}{c}\frac{\partial \rho c}{\partial t}\right) = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}. \tag{2}$$

The relativistically covariant form of Newton's second law is

$$\underline{f} = \frac{d\underline{p}}{d\tau}, \tag{3}$$

where $\underline{f} = (\gamma \mathbf{F}, \gamma P/c)$ is the four-force and $d\tau = (dt^2 - |d\mathbf{x}|^2/c^2)^{1/2} = dt/\gamma$ is the proper time increment. The quantity $\mathbf{F}$ is the ordinary force and $P = \mathbf{F} \cdot \mathbf{v}$ is the power exerted by the force on the particle of interest, $\mathbf{v}$ being the particle's ordinary velocity. The four-velocity is defined $\underline{u} = \gamma(\mathbf{v}, c)$ where $\gamma = (1 - |\mathbf{v}|^2/c^2)^{-1/2}$ as usual. The four-velocity and four-momentum are related by $\underline{p} = m\underline{u}$.

A requirement on any physically valid four-force is that it not change the mass of the particle on which the force is being exerted. This is equivalent to not changing the length of the four-momentum vector, which means that $\underline{f}$ and $\underline{p}$ (and hence $\underline{u}$) must be perpendicular in a spacetime sense:

$$\underline{f} \cdot \underline{p} = 0 \quad \text{and} \quad \underline{f} \cdot \underline{u} = 0. \tag{4}$$

This is guaranteed by the definition of power:

$$\underline{f} \cdot \underline{u} = \gamma^2(\mathbf{F} \cdot \mathbf{v} - (P/c)c) = \gamma^2(\mathbf{F} \cdot \mathbf{v} - \mathbf{F} \cdot \mathbf{V}) = 0. \tag{5}$$

3 Potential energy and relativity

The four-force ought to reduce to the form

$$\underline{f} = (-\nabla U, P/c) \qquad \text{(non-relativistic form)} \tag{6}$$

in the non-relativistic limit. Let us leave the power P undefined for now and concentrate on obtaining a relativistically valid form for the spacelike part of the four-force. As noted in the introduction, U, being an energy, ought to be the timelike component of a four-momentum. This implies the existence of a *potential momentum*, which we shall call $\mathbf{Q}$, to complement the potential energy. Combining the two into a four-vector results in the potential four-momentum,

$$\underline{Q} = (\mathbf{Q}, U/c). \tag{7}$$

Notice that the dot product of the particle velocity and the potential four-momentum $\underline{u} \cdot \underline{Q} = \gamma(\mathbf{v} \cdot \mathbf{Q} - U)$ reduces to just $-U$ for sufficiently small velocities. Thus, it makes sense to replace U by $-\underline{u} \cdot \underline{Q}$ in equation (6), since this is a scalar which reduces to U in the non-relativistic case. Thus, the four-force tentatively becomes

$$\underline{f} = (\nabla(\underline{u} \cdot \underline{Q}), P/c) \qquad \text{(first tentative form).} \tag{8}$$

Given this form it is tempting to replace P/c by a time derivative of $\underline{u} \cdot \underline{Q}$ since this would make the force formula manifestly covariant:

$$\underline{f} = \left(\nabla(\underline{u} \cdot \underline{Q}), -\frac{1}{c}\frac{\partial(\underline{u} \cdot \underline{Q})}{\partial t}\right) = \Box(\underline{u} \cdot \underline{Q}) \qquad \text{(second tentative form).} \tag{9}$$

The problem with this form is that the timelike part is not a true power, because $\underline{f} \cdot \underline{u} \neq 0$. However, an additional term solves this problem:

$$\underline{f} = \underline{\Box}(\underline{u} \cdot \underline{Q}) - (\underline{u} \cdot \underline{\Box})\underline{Q} \quad \text{(third tentative form).} \tag{10}$$

Notice that with this form we have $\underline{u} \cdot \underline{f} = (\underline{u} \cdot \underline{\Box})(\underline{u} \cdot \underline{Q}) - (\underline{u} \cdot \underline{\Box})(\underline{u} \cdot \underline{Q}) = 0$! Thus, equation (10) enforces the constant mass condition. The price of this modification is the addition of a new term to the spacelike part of the four-force. The implications of this addition are profound as we shall see below.

Let us first see if there are any alternative possible forms for a covariant four-force. If we try to construct a four-vector from the three four-vectors $\underline{Q}$, $\underline{u}$, and $\underline{\Box}$, and recall that $\underline{\Box}$ operates on $\underline{Q}$ but not $\underline{u}$, the most arbitrary vector $\underline{f}'$ we can construct is

$$\underline{f}' = A(\underline{u} \cdot \underline{\Box})\underline{Q} + B\underline{\Box}(\underline{u} \cdot \underline{Q}) + C(\underline{\Box} \cdot \underline{Q})\underline{u}, \tag{11}$$

where A, B, and C are arbitrary constants. Dotting this with $\underline{u}$ results in $\underline{u} \cdot \underline{f}' = (A + B)(\underline{u} \cdot \underline{\Box})(\underline{u} \cdot \underline{Q}) - C(\underline{\Box} \cdot \underline{Q})c^2$ since $\underline{u} \cdot \underline{u} = -c^2$. The only way this can be zero is if $A + B = 0$ and $C(\underline{\Box} \cdot \underline{Q}) = 0$. The former relation is satisfied by setting $B = -A$. However, aside from a trivial scaling factor, this is precisely what is done in equation (10). The latter relation suggests that we must enforce either $C = 0$ or a "gauge condition"

$$\underline{\Box} \cdot \underline{Q} = 0. \tag{12}$$

The upshot of this analysis is that the tentative form given in equation (10) is the *only* possible form for a relativistically covariant four-force, given that the force must be derived from a potential four-momentum.

4 Relationship to electromagnetism

The relationship of equation (10) to electromagnetism becomes clear from an examination of this equation's spacelike component:

$$\mathbf{F} = \nabla(\mathbf{v} \cdot \mathbf{Q} - U) - \left(\mathbf{v} \cdot \nabla + \frac{\partial}{\partial t}\right)\mathbf{Q}. \tag{13}$$

Rearranging and applying the vector identity $\mathbf{v} \times (\nabla \times \mathbf{Q}) = \nabla(\mathbf{v} \cdot \mathbf{Q}) - (\mathbf{v} \cdot \nabla)\mathbf{Q}$, and substituting into equation (3) results in

$$\frac{d\mathbf{p}}{dt} = -\nabla U - \frac{\partial \mathbf{Q}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{Q}), \tag{14}$$

which is precisely the electromagnetic Lorentz force if we let $U = q\phi$ and $\mathbf{Q} = q\mathbf{A}$, where q is the electric charge on the particle of interest, ϕ is the electrostatic potential, and $\mathbf{A}$ is the vector potential. Thus, the form of the Lorentz force is no accident — *it is the only possible form for a covariant four-force derivable from a potential four-momentum.*

The timelike part of Newton's second law is also interesting when this formula for the force is plugged in. After minor manipulation we find

$$\frac{dE}{dt} = -\mathbf{v} \cdot \left(\nabla U + \frac{\partial \mathbf{Q}}{\partial t} \right) . \qquad (15)$$

Note that as expected, the right side is in the form of the time rate of change of momentum (i. e., the force as conventionally defined), as given by equation (14), dotted with the velocity, and is therefore the power exerted on the particle by the force. The time rate of change of energy is thus equal to the applied power.