1 Electromagnetism and Quantum Mechanics

We first discuss how the idea of potential energy extends to the relativistic case. In this case it is necessary to define a corresponding potential momentum. The implications of potential momentum for quantum mechanics are explored. We then see how electromagnetic theory provides a potential energy and momentum in the form of the electric charge times the electrostatic potential and the vector potential.

1.1 Potential Energy and Relativity

The idea of potential energy, as conventionally developed, violates the principle of relativity. To understand this, recall that the kinetic energy K is the total energy E minus the potential energy U in the non-relativistic case. The relationship between kinetic energy and momentum $\mathbf{p}$ of a particle of mass M is

$$E - U = K = \frac{p^2}{2M}. \tag{1}$$

In relativity the energy E and momentum $\mathbf{p}$ togther form a four-vector $\underline{p} = (\mathbf{p}, E/c)$, where c is the speed of light. The square of the length of this vector is

$$\underline{p} \cdot \underline{p} = \mathbf{p} \cdot \mathbf{p} - E^2/c^2 = -M^2c^2. \tag{2}$$

Extending the idea of potential energy to relativity in a manner which obeys the principle of relativity and reduces to the non-relativistic case in the proper circumstances requires that the potential energy have an accompanying *potential momentum* $\mathbf{Q}$, which together with the potential energy forms a four-vector: $\underline{Q} = (\mathbf{Q}, U/c)$. The above equation then becomes

$$(\underline{p} - \underline{Q}) \cdot (\underline{p} - \underline{Q}) = (\mathbf{p} - \mathbf{Q}) \cdot (\mathbf{p} - \mathbf{Q}) - (E - U)^2/c^2 = -M^2c^2. \tag{3}$$

Figure 1 shows an example of a four-vector which has no spacelike component in a particular reference frame. In this reference frame only, equation (3) simplifies to

$$|\mathbf{p}|^2 - (E - U)^2/c^2 = -M^2c^2. \tag{4}$$

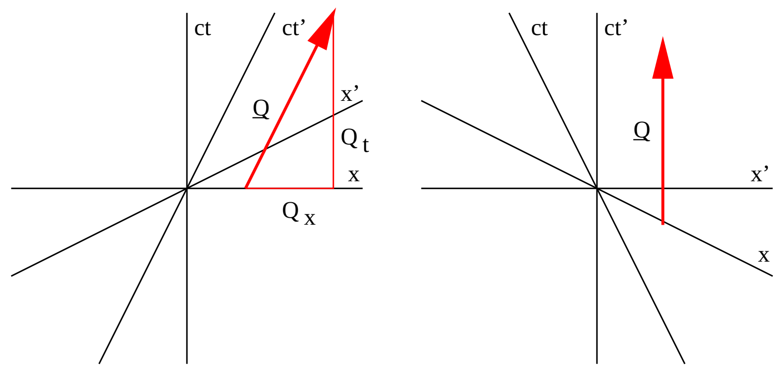

Figure 1: The left panel shows an example of a four-vector $\underline{Q}$ in what we denote the unprimed reference frame. Note how the vector has space and time components Q_x and Q_t in this frame. However, in a reference frame in which the time axis is parallel to the vector (right panel), there is no space component.

If in addition we have $|\mathbf{p}|^2 \equiv p^2 \ll M^2c^2$ in this reference frame, we can approximately solve for $E - U$, with the result

$$E - U = Mc^2 + \frac{p^2}{2M}. \tag{5}$$

This is just like equation (1) except that the rest energy Mc^2 is included in the definition of energy.

More generally, one may not wish to work in the coordinate system in which $\underline{Q}$ is purely timelike. If $|\mathbf{p} - \mathbf{Q}|^2 \ll M^2c^2$ in some reference frame in which $\mathbf{Q}$ is non-zero, then equation (3) becomes

$$E - U = Mc^2 + \frac{|\mathbf{p} - \mathbf{Q}|^2}{2M}. \tag{6}$$

1.2 Potential Momentum in Quantum Mechanics

Equation (1) leads to the Schrödinger equation as it is normally defined:

$$\frac{1}{2M}p^2\psi + U\psi = E\psi. \tag{7}$$

The Schrödinger equation may be extended to the case in which there is a potential momentum by using equation (6):

$$\frac{1}{2M}(\mathbf{p}-\mathbf{Q})\cdot(\mathbf{p}-\mathbf{Q})\psi + U\psi = E\psi \tag{8}$$

The Planck relationship $E = \hbar\omega$ between the energy and the frequency ω applies to the *total energy* E. Similarly, the de Broglie relation $\mathbf{p} = \hbar\mathbf{k}$ between the momentum and the wave vector $\mathbf{k}$ applies to the *total momentum.* Thus, the dispersion relation for the associated quantum mechanical waves becomes

$$\omega = \frac{U}{\hbar} + \frac{\hbar(\mathbf{k}-\mathbf{Q}/\hbar)\cdot(\mathbf{k}-\mathbf{Q}/\hbar)}{2M}. \tag{9}$$

The group velocity of waves for constant values of U and $\mathbf{Q}$ is therefore

$$\mathbf{u}_g = \nabla_k\omega = \frac{\hbar\mathbf{k}-\mathbf{Q}}{M} = \frac{\mathbf{p}-\mathbf{Q}}{M}. \tag{10}$$

In other words, the group velocity equals the *kinetic momentum* $\mathbf{p}-\mathbf{Q}$ divided by the mass.

It is straightforward to show that

$$\frac{d}{dt}\int \psi^*\psi dV = 0 \tag{11}$$

for the extended Schrödinger equation by identifying E with $i\hbar\partial/\partial t$ in equation (8). Thus, probability is still conserved. The probability flux turns out to be

$$\mathbf{F} = \frac{1}{M}\mathrm{Re}[\psi^*(\mathbf{p}-\mathbf{Q})\psi] = \frac{1}{M}\mathrm{Re}[\psi^*(-i\hbar\nabla-\mathbf{Q})\psi] \tag{12}$$

for the extended Schrödinger equation.

1.3 Potential Four-Momentum for Electromagnetism

In electromagnetism the scalar and vector potential together form a four-vector, $\underline{a} = (\mathbf{A}, \phi/c)$. Since the potential energy U of a particle with charge e in the electrostatic potential field ϕ is $U = e\phi$, it follows that the electromagnetic potential four-momentum for a particle with charge e is just $\underline{Q} = (e\mathbf{A}, e\phi/c)$. The condition $\nabla\cdot\mathbf{Q} = 0$ translates to $\nabla\cdot\mathbf{A} = 0$, which is just the Coulomb gauge condition in electromagnetism.

As an example of the effects of a potential momentum, let us consider the problem of a uniform magnetic field $\mathbf{B}$ normal to the plane of a two-dimensional harmonic oscillator. A vector potential which yields such a field is

$$\mathbf{A} = \frac{B}{2}(-y, x, 0). \tag{13}$$

It is easily verified that $\mathbf{B} = \nabla \times \mathbf{A} = B\mathbf{k}$ in this case. Furthermore, $\nabla \cdot \mathbf{A} = 0$, so the Coulomb gauge condition is satisfied.

Let us further assume that the scalar potential is of the form

$$\phi = \frac{C}{2}(x^2 + y^2) = \frac{C}{2}r^2 \tag{14}$$

where C is a constant. The scalar potential thus provides the harmonic oscillator potential energy.

The resulting Schrödinger equation takes the form

$$\frac{1}{2M}\left[-\hbar^2\nabla^2 + (eB)i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi + \left(\frac{\kappa r^2}{2} - E\right)\psi = 0 \tag{15}$$

where the effective spring constant is $\kappa = eC + e^2B^2/(4M)$. Notice that the term involving first spatial derivatives is just the two-dimensional angular momentum operator:

$$L \equiv -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{16}$$

The Hamiltonian thus takes the form

$$H = H_0 + H' = \left[\frac{p^2}{2M} + \frac{kr^2}{2}\right] + \left[-\frac{eB}{2M}L\right], \tag{17}$$

where we have divided the Hamiltonian into a part which we know how to solve explicitly, i. e., the two-dimensional harmonic oscillator, and a part which is due to the magnetic field. Note that since $eL/(2M)$ is just the magnetic dipole moment of the oscillating particle, we could have written down its contribution to the Hamiltonian directly from the formula for the potential energy of a magnetic dipole $\boldsymbol{\mu}$ in a magnetic field: $U = -\boldsymbol{\mu} \cdot \mathbf{B}$. If the magnetic field isn't too strong, we can apply perturbation theory to obtain the resulting energy shifts.

We have already solved the above problem with $H' = 0$. The energies depend on the principal and angular momentum quantum numbers, n, and m as follows:

$$E_{nm} = n\hbar\omega_0 \quad n = 1, 2, 3, \ldots, \tag{18}$$

where $\omega_0 = (\kappa/M)^{1/2}$ is the classical harmonic oscillator frequency. The condition on the angular momentum quantum number m is that it must take on integer values such that $|m| < n$. In addition, only even values of m are allowed when n is odd and vice versa. The actual value of the angular momentum is $\hbar m$. Notice that the energies are degenerate with respect to the angular momentum quantum number.

If ψ_{nm} are the associated eigenstates, first order perturbation theory says that the perturbation energies associated with H' are given by $(\psi_{nm}, H'\psi_{nm})$. The energies with these corrections included are therefore

$$E_{nm} = n\hbar\omega_0 - \frac{eB}{2M}(\psi_{nm}, L\psi_{nm}) = n\hbar\omega_0 - \frac{eB\hbar m}{2M} \tag{19}$$

since ψ_{nm} is also an eigenstate of L with eigenvalue $\hbar m$. The magnetic field thus breaks the degeneracy of the original Hamiltonian in angular momentum.

1.4 Gauge Transformations

In electromagnetism the scalar and vector potentials are not unique in that the transformation

$$\begin{aligned} e\mathbf{A} &\rightarrow e\mathbf{A} + \nabla\chi \\ e\phi &\rightarrow e\phi - \frac{\partial\chi}{\partial t} \end{aligned} \tag{20}$$

doesn't change the electric and magnetic fields. This is called a *gauge transformation.* In quantum mechanics one additional transformation needs to be made to insure that the extended Schrödinger equation doesn't change form:

$$\psi \rightarrow \exp(i\chi)\psi. \tag{21}$$

Making the above transformations in the extended Schrödinger equation (8) leaves this equation unchanged. The phase factor $\exp(i\chi)$ doesn't change any observable results because it cancels out when the probability is computed: $[\exp(i\chi)\psi]^* \exp(i\chi)\psi = \exp(-i\chi)\psi^* \exp(i\chi)\psi = \psi^*\psi$.