1 The Contributions of Louis de Broglie

Prince Louis Victor de Broglie, a French nobleman, won the Nobel Prize in physics for his discovery of the wave nature of electrons. Actually, his contribution to physics was much broader than this — he developed the basic physical ideas that led to the quantum mechanical treatment of all types of matter. He is best known for his relation between particle momentum, p, and the corresponding quantum mechanical wavelength, λ:

$$p = h/\lambda, \tag{1}$$

where h is Planck's constant, i. e., the same constant used by Planck and Einstein to relate the energy and frequency of a photon. Interestingly, de Broglie did all of this for his doctoral dissertation and thereafter sank into relative obscurity!

1.1 Momentum-Wavenumber Relation

Equation (1) is generally presented without justification in elementary physics courses. However, de Broglie showed that this relationship is an inevitable consequence of special relativity plus assuming that the Planck-Einstein result,

$$E = h\nu, \tag{2}$$

is applicable to material particles as well as photons.

The argument goes as follows: Let us first rephrase equations (1) and (2) in terms of angular frequency, $\omega = 2\pi\nu$, and wavenumber, $k = 2\pi/\lambda$:

$$E = \hbar\omega \qquad p = \hbar k, \tag{3}$$

where $\hbar = h/2\pi$. Now, a plane wave can be represented by the formula

$$\psi = \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)], \tag{4}$$

where $\mathbf{k}$ is the wave vector. (Note that $k = |\mathbf{k}|$.) In order for the phase of the wave, $\mathbf{k}\cdot\mathbf{x} - \omega t$, to be a relativistically invariant quantity, it must be a relativistic dot product of two four-vectors, $(\mathbf{k}, \omega/c)$ and $(\mathbf{x}, ct)$. The second of these is known to be a four-vector, since it is just a displacement in spacetime. If the first is also a four-vector, and if $\omega = E/\hbar$, then it follows that this wavenumber-frequency four-vector is just a constant multiple, $1/\hbar c$, of the energy-momentum vector. This leads immediately to the vector form of the de Broglie relation, $\mathbf{k} = \mathbf{p}/\hbar$. Thus, de Broglie's momentum-wavelength equation is not a bolt from the blue, but is a consequence of assuming that the energy of an ordinary particle is related to the associated wave frequency by the Planck-Einstein relation.

1.2 Variable Potential Energy

In his Nobel Prize lecture, de Broglie also alluded to a method for computing the wavefunction of a particle subject to a variable potential energy, $U(\mathbf{x})$, by making an analogy with the behavior of light in a medium of variable refractive index. For light with frequency ω (and hence photon energy $\hbar\omega$), the equation governing the amplitude, ψ, of the light wave is

$$\nabla^2\psi_\omega + k^2\psi_\omega = 0. \tag{5}$$

(Note that this is an approximation to electromagnetic wave theory, in that the polarization of the light is ignored.) The wavenumber, k, can vary with position, and is related to the wavenumber of the light in a vacuum, k_0, by $k = nk_0$, where n is the refractive index of the medium. Note also that $k_0 = \omega/c$.

The trick is in finding the relationship between index of refraction and potential energy. This is easily done for the case in which the potential energy is constant. In this situation, the non-relativistic approximation to the relationship between momentum, p, and kinetic energy, K, is

$$p^2/2m = K = E - mc^2 - U \equiv E' - U, \tag{6}$$

where $E' = E - mc^2$. Using the relationships between momentum and wavenumber and energy and frequency, de Broglie found that

$$\begin{aligned} k &= p/\hbar = [2m(E'-U)/\hbar^2]^{1/2} = (\omega/c)[2mc^2(E'-U)/E^2]^{1/2} \\ &\approx (\omega/c)[2(E'-U)/mc^2]^{1/2}, \end{aligned} \tag{7}$$

where $\hbar\omega = E$ is approximated by the rest energy mc^2 except in terms containing $E - mc^2$. Thus equation (7) tells us that in the non-relativistic limit, massive particles behave like light moving through a medium with index of refraction

$$n = k/k_0 = kc/\omega \approx [(E' - U)/mc^2]^{1/2}. \tag{8}$$

Note that in general the index of refraction in this case is very close to zero, which means that the phase speed of the "light" is much larger than c. However, the group velocity is much less than c, in accord with the non-relativistic assumption.

Assuming that this formula is also valid for variable U leads to a tentative governing equation for non-relativistic matter waves when equations (5) and (7) are combined. Combining equations (5) and (7), we get

$$\nabla^2 \psi_\omega + [2m(E' - U)/\hbar^2]\psi_\omega = 0. \tag{9}$$

This formula is actually a form of the equation generally credited to Erwin Schrödinger. The complete wavefunction is obtained by multiplying by the factor $\exp(-i\omega t)$:

$$\psi(\mathbf{x}, t) = \psi_\omega(\mathbf{x}) \exp(-i\omega t). \tag{10}$$

1.3 A Geometrical Optics Example

De Broglie's insight shows that quantum mechanical problems involving the motion of massive particles subject to conservative forces are analogous to the motion of of photons through a region of variable refractive index. The approximation made when the wavelength of light is much less than the dimensions of the optical system is called geometrical optics. In this approximation, wavefronts are replaced by rays representing the trajectories of photons through the system. The analogous approximation to the quantum mechanics of massive particles is simply classical mechanics. Thus, if de Broglie's theory is correct, the trajectory of a photon through a region of varying refractive index should be the same as the classically computed trajectory of a particle with the corresponding potential function.

We can test this idea with a simple example. Figure 1 shows a situation in which a ray of light passes from a region of large refractive index to one with smaller refractive index. The ray is refracted away from the normal to the interface between the media according to Snell's law:

$$\frac{\sin\theta_0}{\sin\theta} = \frac{n}{n_0}. \tag{11}$$

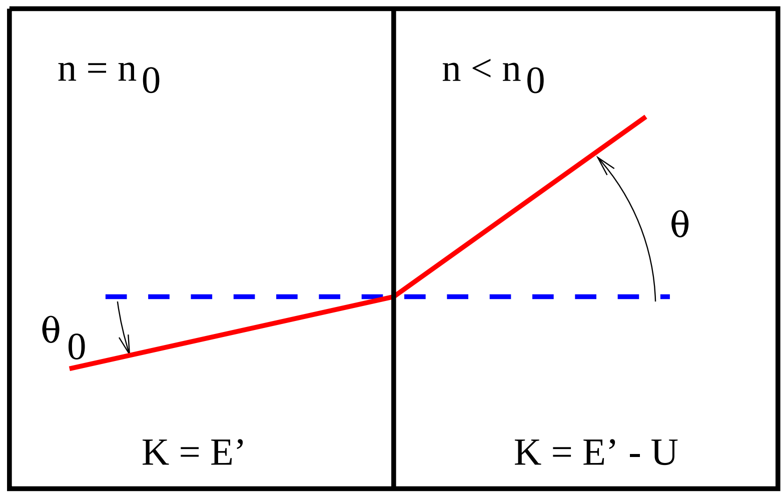

Figure 1: Example of use of de Broglie's index of refraction. The optics problem is analogous to the passage of a ray of light from a medium with high index of refraction into a medium with a lower value.

In general, equation (7) shows that $n = [2(E' - U]/mc^2]^{1/2}$, so that for the left-hand region in figure 1 we have $n = n_0 = (2E'/mc^2)^{1/2}$. The ratio therefore takes on the value

$$\frac{n}{n_0} = \left(\frac{E' - U}{E'}\right)^{1/2}. \tag{12}$$

However, according to non-relativistic classical mechanics, $K = E' = (p_x^2 + p_z^2)/2m$ in the region to the left of the interface, while $K' = E' - U = ({p'}_x^2 + p_z^2)/2m$ to the right, where the ps represent momentum components with x to the right and z upward. The z-component of momentum is conserved because the force $-\nabla U$ has only an x component. Thus only the x-component of momentum changes when the particle crosses the interface.

The angles can be related to the components of the momentum vector: $\sin\theta_0 = p_z/(p_x^2 + p_z^2)^{1/2}$ and $\sin\theta = p_z/({p'}_x^2 + p_z^2)^{1/2}$. Thus,

$$\frac{\sin\theta_0}{\sin\theta} = \left(\frac{{p'}_x^2 + p_z^2}{p_x^2 + p_z^2}\right)^{1/2} = \left(\frac{E' - U}{E'}\right)^{1/2}, \tag{13}$$

which is in accord with the result derived from geometrical optics, equation (12). Thus, the equivalence of geometrical optics and classical mechanics is established at least for this example.