1 Angular Momentum by Operator Methods

The section on the hydrogen atom showed how orbital angular momentum can be analyzed by separating variables in configuration space. We found that the eigenvalues of angular momentum squared are $\hbar^2 l(l+1)$ where l takes on integer values, $l = 0,\ 1,\ 2, \ldots$. The z-component of angular momentum takes on values $-\hbar l,\ -\hbar(l-1), \ldots,\ \hbar l$.

The Stern-Gerlach experiment was found to yield results in conflict with this picture of angular momentum in atoms. In particular, half-integal values of l were inferred from the number of beams emerging from the apparatus for certain atoms. In this section we analyze angular momentum using operator methods similar to those used in calculating the eigenvalues and eigenvectors of the harmonic oscillator.

1.1 The Angular Momentum Operator

The angular momentum operator is defined as in classical physics:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1}$$

where $\mathbf{r}$ is the position vector and $\mathbf{p}$ is the momentum. In Cartesian coordinates,

$$L_x = yp_z - zp_y, \tag{2}$$

etc. Using $[x,\ p_x] = i$ (henceforth in this section we set $\hbar = 1$ for simplicity) and the corresponding commutation relations for the y and z components of position and momentum, it is easy to show that

$$[L_x,\ L_y] = iL_z, \quad [L_y,\ L_z] = iL_x, \quad [L_z,\ L_x] = iL_y. \tag{3}$$

The square of the angular momentum vector is

$$L^2 \equiv |\mathbf{L}|^2 = L_x^2 + L_y^2 + L_z^2. \tag{4}$$

Using the identity $[A,\ B^2] = [A,\ B]B + B[A,\ B]$, it is also easy to show that

$$[L_x,\ L^2] = [L_y,\ L^2] = [L_z,\ L^2] = 0. \tag{5}$$

From equations (3) and (5) we see that the angular momentum squared and one of its components form a compatible set of operators. However, different components of the angular momentum are incompatible, so only one component may be known at a time. It is conventional to select the z component, but since the z-axis can be aligned arbitrarily, this constitutes no loss of generality.

1.2 Angular Momentum and the Hamiltonian

We now show that the square of the angular momentum and its z component commute with Hamiltonians that have spherically symmetric potentials. Defining $r^2 = x^2 + y^2 + z^2$, such Hamiltonians take the form

$$H = p^2/(2M) + U(r). \tag{6}$$

It is sufficient to prove that L^2 and L_z each commute with p^2 and r^2. Several preliminary commutation relations are need to carry out this proof.

It is easy to show that

$$[x,\ L_x] = 0, \quad [y,\ L_x] = -iz, \quad [z,\ L_x] = iy, \tag{7}$$

with similar commutation relations for L_y and L_z obtained by cyclic permutation. In addition,

$$[p_x,\ L_x] = 0, \quad [p_y,\ L_x] = -ip_z, \quad [p_z,\ L_x] = ip_y, \tag{8}$$

etc. From these one can show that

$$[r^2,\ L_x] = [r^2,\ L_y] = [r^2,\ L_z] = 0 \tag{9}$$

and

$$[r^2,\ L^2] = 0, \tag{10}$$

which means that the spherically symmetric potential energy commutes with the components and squared magnitude of angular momentum.

It is similarly possible to show that

$$[p^2,\ L_x] = [p^2,\ L_y] = [p^2,\ L_z] = [p^2,\ L^2] = 0, \tag{11}$$

which concludes the proof:

$$[H,\ L_z] = [H,\ L^2] = 0 \quad \text{(spherical symmetry)}. \tag{12}$$

By substituting equation (2) etc. into equation (4) and performing a lengthy but routine series of manipulations, we can show that

$$L^2 = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2 + i \mathbf{r} \cdot \mathbf{p}. \tag{13}$$

Note that the order of terms containing $\mathbf{r}$ and $\mathbf{p}$ and their magnitudes is important, since these operators don't commute. Solving for p^2, the Hamiltonian can be written

$$H = \frac{1}{2Mr^2} \left[(\mathbf{r} \cdot \mathbf{p})^2 - i \mathbf{r} \cdot \mathbf{p} + L^2 \right]. \tag{14}$$

In configuration space

$$\mathbf{r} \cdot \mathbf{p} = -i \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) = -ir \frac{\partial}{\partial r}. \tag{15}$$

Simplifying the derivatives shows that

$$p^2 = \frac{L^2}{r^2} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \tag{16}$$

Comparison with the analysis of the hydrogen atom in configuration space agrees with the form of the L^2 operator deduced there:

$$L^2 = -\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2}. \tag{17}$$

1.3 Determination of Eigenvalues

The determination of the eigenvalues of L^2 and L_z follows from a rather extended argument. The operators

$$L^+ = L_x + iL_y, \qquad L^- = L_x - iL_y. \tag{18}$$

will play an important role in this argument, and will ultimately be identified as raising and lowering operators for the z component of angular momentum. The left and right products of L^+ and L^- can be written in terms of L^2 and L_z:

$$L^+ L^- = L^2 - L_z^2 + L_z, \qquad L^- L^+ = L^2 - L_z^2 - L_z. \tag{19}$$

Notice further that the raising and lowering operators commute with L^2,

$$[L^{\pm},\ L^2] = 0, \tag{20}$$

while

$$[L^{\pm},\ L_z] = \mp L^{\pm}. \tag{21}$$

Let us assume that $\psi_{\lambda,m}$ is an eigenstate of L^2 (eigenvalue λ) and L_z (eigenvalue m). At this time we don't presume to know the values of λ and m. From equation (20) we see that

$$L^2 L^{\pm}\psi_{\lambda,m} = L^{\pm} L^2 \psi_{\lambda,m} = \lambda L^{\pm}\psi_{\lambda,m}, \tag{22}$$

which shows that $L^{\pm}\psi_{\lambda,m}$ is an eigenstate of L^2 with eigenvalue λ. Thus, $L^{\pm}$ doesn't alter the squared magnitude of the angular momentum.

On the other hand, equation (21) shows that

$$L_z L^{\pm}\psi_{\lambda,m} = L^{\pm} L_z \psi_{\lambda,m} \pm L^{\pm}\psi_{\lambda,m} = (m \pm 1) L^{\pm}\psi_{\lambda,m}, \tag{23}$$

which shows that $L^{\pm}\psi_{\lambda,m}$ is an eigenstate of L_z with eigenvalue $m \pm 1$. Thus, L^+ raises the z component of the angular momentum by 1, while L^- lowers it by 1. This justifies the identification of L^+ and L^- as raising and lowering operators. We may thus write

$$L^{\pm}\psi_{\lambda,m} = C^{\pm}_{\lambda,m}\psi_{\lambda,m\pm 1}, \tag{24}$$

where the $C^{\pm}_{\lambda,m}$ are constants.

The combination $L^- L^+$ first raises the value of m by 1 and then lowers it by 1, thus returning the original eigenstate multiplied by a constant. Using equation (19), we find

$$\begin{aligned} L^- L^+ \psi_{\lambda,m} &= C^-_{\lambda,m+1} C^+_{\lambda,m}\psi_{\lambda,m} = \\ &\quad (L^2 - L_z^2 - L_z)\psi_{\lambda,m} = (\lambda - m^2 - m)\psi_{\lambda,m}, \end{aligned} \tag{25}$$

from which we infer that $C^-_{\lambda,m+1} C^+_{\lambda,m} = \lambda - m^2 - m$. A similar exercise with $L^+ L^-$ shows that $C^+_{\lambda,m-1} C^-_{\lambda,m} = \lambda - m^2 + m$. These equations have the solutions

$$C^+_{\lambda,m} = (\lambda - m^2 - m)^{1/2}, \qquad C^-_{\lambda,m} = (\lambda - m^2 + m)^{1/2}, \tag{26}$$

which are unique up to a phase factor.

Further insight into possible values of m may be obtained by noting that the operator $L_x^2 + L_y^2$ must have non-negative eigenvalues since it is composed of the sum of the squares of Hermitian operators. However, $L_x^2 + L_y^2 = L^2 - L_z^2$, has the eigenvalues $\lambda - m^2$, which implies that $m^2 \leq \lambda$. Thus, m is bounded above and below. If t and b are the top and bottom values of m, then $L^+\psi_{\lambda,t} = 0$ and $L^-\psi_{\lambda,b} = 0$ — otherwise states violating the boundedness condition on m^2 would exist. Equation (26) thus implies $\lambda = t^2 + t = b^2 - b$, which means that either $b = t + 1$ or $b = -t$. The first solution violates the identification of t as the top value of m, so the second solution is the valid one.

The final piece of the puzzle comes from the realization that $t - b = n$, where n is some non-negative integer. Combining this with $b = -t$ results in $t = n/2$. Recalling that $\lambda = t^2 + t$, we rename t to l to be consistent with previous terminology, so that $\lambda = l^2 + l = l(l + 1)$.

To summarize, the eigenvalues of L^2 are

$$\lambda = l(l + 1), \qquad l = 0,\ 1/2,\ 1,\ 3/2,\ 2,\ \ldots, \tag{27}$$

and the eigenvalues of L_z are

$$m = -l,\ -l + 1,\ \ldots,\ l. \tag{28}$$

There are $2l + 1$ values of m.

Notice that this analysis has found solutions with half-integral values of l. These solutions have no configuration space representation, and must therefore represent a type of angular momentum that is distinct from that associated with the orbital motion of a particle. Note in particular that the solution with $l = 1/2$ has two possible values of m, and thus constitutes a possible explanation for the (hypothesized) behavior of ground state hydrogen passing through the Stern-Gerlach apparatus. The evidence indicates that the electron possesses an *intrinsic* angular momentum. We now change our terminology to distinguish this case from the orbital angular momentum case by replacing l with s. This is called the *spin* quantum number, and the electron is said to have a spin $s = 1/2$. Possible values of L_z are $\pm 1/2$.