Chapter 5

Shallow Water on a Sphere

In this section we derive the shallow water equations on a sphere as an approximation to flow on the earth's surface. Contours of constant geopotential (and hence constant elevation) aren't really spherical, but are somewhat ellipsoidal in shape. However, the difference is small enough to be ignored for most purposes.

We first develop the continuity and momentum equations and then derive the tools for potential vorticity inversion on a sphere. We then introduce common approximations in which a small patch of a spherical surface can be treated in a simple way. Figure 5.1 shows the spherical coordinate system we use. Note that the longitude λ is the azimuthal coordinate and the latitude ϕ is the elevation coordinate. This system differs from the usual spherical coordinates in which the elevation angle is the co-latitude or $\pi/2 - \phi$.

5.1 Mass continuity

Figure 5.2 shows a "rectangular" region of fluid in latitude ϕ and longitude λ on a sphere. The depth of the fluid h itself is a function of λ and ϕ. If the earth's radius is a, then the linear dimensions of this region are given by

$$\Delta x_\lambda = a\cos\phi\Delta\lambda \qquad \Delta x_\phi = a\Delta\phi. \tag{5.1}$$

The volume of the region of fluid is therefore $a^2 h\cos\phi\Delta\lambda\Delta\phi$.

If v_λ is the fluid velocity component in the direction of increasing longitude λ and v_ϕ is the component in the direction of increasing latitude ϕ, then the rate at which fluid volume enters the region from the sides is

$$\begin{aligned} a^2\cos\phi\Delta\lambda\Delta\phi\frac{\partial h}{\partial t} &= [v_\lambda(\lambda)h(\lambda) - v_\lambda(\lambda')h(\lambda')]a\Delta\phi \\ &+ [v_\phi(\phi)h(\phi)\cos(\phi) - v_\phi(\phi')h(\phi')\cos(\phi')]a\Delta\lambda \end{aligned} \tag{5.2}$$

where $\lambda' = \lambda + \Delta\lambda$ and $\phi' = \phi + \Delta\phi$. Dividing by $a^2\cos\phi\Delta\lambda\Delta\phi$, bringing all terms to the left side, and taking the limit of small $\Delta\lambda$ and $\Delta\phi$, we get the mass continuity equation on a sphere:

$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\phi}\left[\frac{\partial}{\partial\lambda}(hv_\lambda) + \frac{\partial}{\partial\phi}(hv_\phi\cos\phi)\right] = 0. \tag{5.3}$$

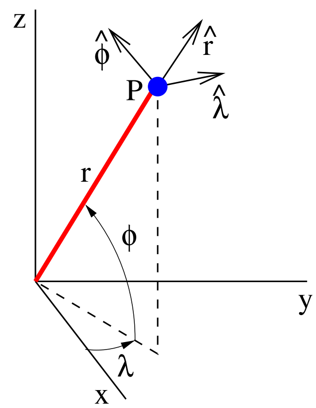

Figure 5.1: Spherical coordinate system used in this chapter. The longitude is λ and the latitude is ϕ. A local Cartesian coordinate system is defined at point P with eastward, northward, and upward unit vectors defined as $\hat{\lambda}$, $\hat{\phi}$, and $\hat{\mathbf{r}}$.

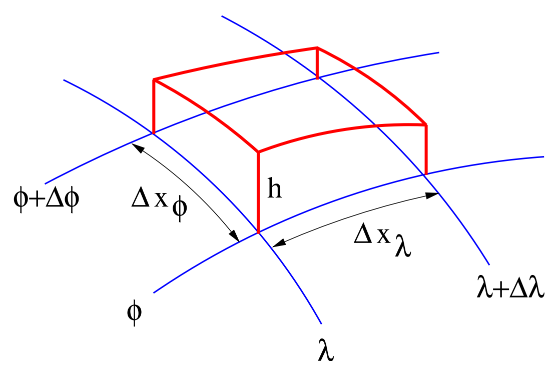

Figure 5.2: Sketch used to obtain shallow water mass continuity equation on a sphere.

5.2 Momentum equations

The difficult part of deriving the momentum equations is calculating the total time derivative of the horizontal velocity. The difficulty is that the orientation of the unit vectors defining the local east-north-up coordinate system changes for different locations on the sphere. Thus, the variability in these unit vectors needs to be taken into account when taking spatial derivatives.

If $\mathbf{v}_h(\lambda, \phi, t) = v_\lambda \hat{\lambda} + v_\phi \hat{\phi}$ is the horizontal flow velocity, then

$$\frac{d\mathbf{v}_h}{dt} = \frac{\partial \mathbf{v}_h}{\partial t} + \frac{d\lambda}{dt}\frac{\partial \mathbf{v}_h}{\partial \lambda} + \frac{d\phi}{dt}\frac{\partial \mathbf{v}_h}{\partial \phi}. \tag{5.4}$$

It is possible to show that

- $\partial\hat{\lambda}/\partial\lambda = \hat{\phi}\sin\phi - \hat{\mathbf{r}}\cos\phi$;
- $\partial\hat{\phi}/\partial\lambda = -\hat{\lambda}\sin\phi$;
- $\partial\hat{\lambda}/\phi = 0$;
- $\partial\hat{\phi}/\partial\phi = -\hat{\mathbf{r}}$.

Furthermore

$$\frac{d\lambda}{dt} = \frac{v_\lambda}{a\cos\phi} \qquad \frac{d\phi}{dt} = \frac{v_\phi}{a} \tag{5.5}$$

where a is the radius of the earth as before. Finally

$$\nabla h = \frac{\partial h}{\partial x_\lambda}\hat{\lambda} + \frac{\partial h}{\partial x_\phi}\hat{\phi} = \frac{1}{a\cos\phi}\frac{\partial h}{\partial \lambda}\hat{\lambda} + \frac{1}{a}\frac{\partial h}{\partial \phi}\hat{\phi}. \tag{5.6}$$

Putting all of this together and splitting into longitudinal and latitudinal components, we get

$$\frac{\partial v_\lambda}{\partial t} + \frac{v_\lambda}{a\cos\phi}\frac{\partial v_\lambda}{\partial \lambda} + \frac{v_\phi}{a}\frac{\partial v_\lambda}{\partial \phi} - \frac{v_\lambda v_\phi \tan\phi}{a} + \frac{g}{a\cos\phi}\frac{\partial h}{\partial \lambda} - f v_\phi = 0 \tag{5.7}$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_\lambda}{a\cos\phi}\frac{\partial v_\phi}{\partial \lambda} + \frac{v_\phi}{a}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\lambda^2 \tan\phi}{a} + \frac{g}{a}\frac{\partial h}{\partial \phi} + f v_\lambda = 0 \tag{5.8}$$

where the Coriolis parameter $f = 2\Omega\sin\phi$ now varies with latitude. We have dropped terms pointing in the $\hat{\mathbf{r}}$ direction. These terms actually enter the vertical momentum equation. However, they are small compared to the other terms in this equation for ordinary velocities, and therefore they don't significantly perturb hydrostatic balance. The geostrophic wind is obtained by dropping all terms related to $d\mathbf{v}_h/dt$:

$$v_{g\phi} = \frac{g}{fa\cos\phi}\frac{\partial h}{\partial \lambda} \qquad v_{g\lambda} = -\frac{g}{fa}\frac{\partial h}{\partial \phi}. \tag{5.9}$$

This treatment breaks down near the north and south poles where $\cos\phi \to 0$, and an alternate coordinate system needs to be used. Global numerical models deal with the problem of the poles in a variety of ways which will not be discussed here.

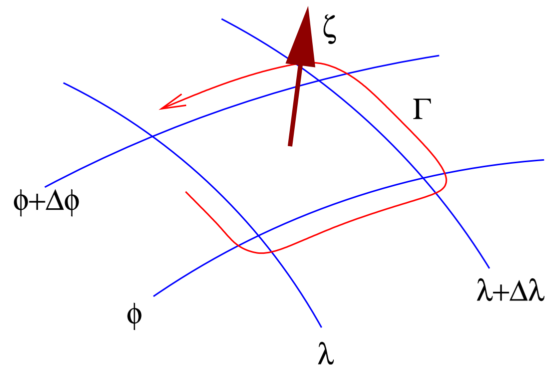

Figure 5.3: Sketch indicating circulation about a region rectangular in latitude-longitude space.

5.3 Vorticity and divergence

We now use the circulation theorem to obtain an expression for the vertical component of the vorticity on a sphere. Employing equation (5.1, the relative circulation is computed to be

$$\begin{aligned} \Gamma_r &= v_\lambda(\phi) a \cos\phi \Delta\lambda \\ &+ v_\phi(\lambda + \Delta\lambda) a \Delta\phi \\ &- v_\lambda(\phi + \Delta\phi) a \cos(\phi + \Delta\phi) \Delta\lambda \\ &- v_\phi(\lambda) a \Delta\phi. \end{aligned} \tag{5.10}$$

For small $\Delta\lambda$ and $\Delta\phi$ this becomes

$$\Gamma_r = \Delta\lambda\Delta\phi \left[\frac{\partial}{\partial\lambda}(a v_\phi) - \frac{\partial}{\partial\phi}(a v_\lambda \cos\phi) \right]. \tag{5.11}$$

Recall that by Stokes' theorem $\Gamma_r = \zeta_r A$, where $A = \Delta x_\lambda \Delta x_\phi = a^2 \cos\phi \Delta\lambda\Delta\phi$ is the area of the enclosed region and ζ_r is the vertical component of the relative vorticity. The relative vorticity is thus

$$\zeta_r = \frac{1}{a\cos\phi} \left(\frac{\partial v_\phi}{\partial\lambda} - \frac{\partial v_\lambda \cos\phi}{\partial\phi} \right). \tag{5.12}$$

Substitution of the geostrophic wind components results in the geostrophic approximation to the relative vorticity

$$\zeta_{gr} = \frac{g}{a^2 \cos\phi} \left[\frac{\partial}{\partial\lambda} \left(\frac{1}{f\cos\phi} \frac{\partial h}{\partial\lambda} \right) + \frac{\partial}{\partial\phi} \left(\frac{\cos\phi}{f} \frac{\partial h}{\partial\phi} \right) \right] \tag{5.13}$$

where the Coriolis parameter $f = 2\Omega\sin\phi$ must be retained inside the ϕ derivative since it depends on ϕ. The potential vorticity is as usual $q = (f + \zeta_r)/h$.

A similar calculation of the horizontal divergence D of the velocity yields

$$D = \frac{1}{a\cos\phi} \left(\frac{\partial v_\lambda}{\partial\lambda} + \frac{\partial v_\phi \cos\phi}{\partial\phi} \right). \tag{5.14}$$

5.4 Beta-plane approximation

Equations (5.3), (5.7), and (5.8) are difficult to solve due to the sines and cosines of latitude which enter. A useful approximation is to treat a region of the earth's surface as being locally flat in all respects except in the latitudinal variation in the Coriolis parameter, for which a Taylor series expansion is made about the central latitude ϕ_0 of the region of interest:

$$f \approx 2\Omega \sin\phi_0 + (2\Omega \cos\phi_0/a)y \equiv f_0 + \beta y. \tag{5.15}$$

where Ω is the angular rotation rate of the earth and a is the earth's radius. A local Cartesian coordinate system centered on the region is employed with x increasing to the east and y increasing to the north. This approximation can be justified if the diameter of the region of interest is much less than the diameter of the earth. This is called the *beta-plane* approximation.

A special case of this approximation obtains when $\phi_0 = 0$. This is called the *equatorial beta-plane* approximation. In this case

$$f \approx 2\Omega\phi = (2\Omega/a)y = \beta y. \tag{5.16}$$

The equatorial beta-plane approximation is valid for a larger domain in the east-west direction, i. e., for the entire equatorial strip around the globe as long as the north-south width of the strip is much less than the earth's diameter.

The *f-plane approximation* is like the beta-plane approximation except that the region is assumed to be small enough that the latitudinal variation in the Coriolis parameter can be ignored as well; f is replaced by a constant representative value.

5.5 Rossby wave on a mid-latitude beta-plane

A fluid of constant depth h_0 at rest on a mid-latitude beta-plane exhibits a north-south gradient in potential vorticity, and should therefore be expected to support Rossby waves. The potential vorticity in this case takes the form

$$q = \frac{f_0 + \beta y}{h_0} = q_0(1 + \beta y/f_0), \tag{5.17}$$

where $q_0 = f_0/h_0$. If we now introduce nearly geostrophic motion in the form of a fractional thickness perturbation η^* and a potential vorticity perturbation q^*, then the linearized potential vorticity inversion equation takes the same form as for Rossby waves due to a tilted bottom:

$$L_R^2 \nabla^2 \eta^* - \eta^* = q^*/q_0, \tag{5.18}$$

where the Rossby radius is defined here by $L_R^2 = gh_0/f_0^2$.

The geostrophic potential vorticity advection equation linearizes to

$$\frac{1}{q_0}\frac{\partial q^*}{\partial t} + \beta L_R^2 \frac{\partial \eta^*}{\partial x} = 0. \tag{5.19}$$

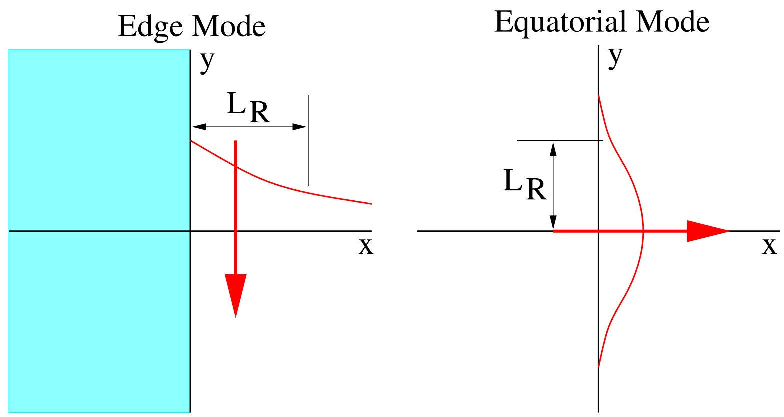

Figure 5.4: Sketch showing environments of two kinds of Kelvin waves. The large arrows indicate the allowed direction of propagation in the two cases. (We assume $f > 0$ in the left panel.) In each case the transverse extent of the wave scales with the appropriately defined Rossby radius and the curve shows how the wave amplitude varies in the transverse direction.

Assuming as before an east-west channel of width w, the solution becomes perfectly analogous to the tilted bottom case, with the dispersion relation

$$\omega = -\frac{kL_R^2\beta}{1 + L_R^2(k^2 + \pi^2/w^2)}. \tag{5.20}$$

Comparison with the solution for a tilted bottom shows that the two dispersion relations are identical if we replace $f\mu$ by β. Though the physical mechanisms for generating a north-south gradient in potential vorticity are different in the two cases, the dynamical consequences are the same.

5.6 Kelvin waves

Kelvin waves are waves in a rotating environment which for various reasons act like gravity waves in the equivalent non-rotating environment. This is made possible in all cases by the imposition of a constraint on parcel motions normal to the direction of the wave propagation.

We study two important types of Kelvin waves here, edge waves and equatorially trapped modes. In the former case a Kelvin wave mode is found which propagates along a lateral boundary to the fluid. The wave amplitude is maximal at the boundary and decays exponentially away from it.

Equatorially trapped Kelvin modes propagate eastward along the equator and exist because of the spatial variability of the Coriolis parameter with latitude. The wave amplitude is maximal on the equator and has a Gaussian structure in latitude.

5.6.1 Edge waves

Let us first examine edge waves on the east side of a north-south boundary, as illustrated in the left panel of figure 5.4. By hypothesis $v_x = 0$ in this case. The linearized continuity and momentum equations are therefore

$$\frac{\partial \eta}{\partial t} + \left\{ \frac{\partial v_x}{\partial x} \right\} + \frac{\partial v_y}{\partial y} = 0 \tag{5.21}$$

$$\left\{ \frac{\partial v_x}{\partial t} \right\} + gh_0 \frac{\partial \eta}{\partial x} - f v_y = 0 \tag{5.22}$$

$$\frac{\partial v_y}{\partial t} + gh_0 \frac{\partial \eta}{\partial y} + \{ f v_x \} = 0, \tag{5.23}$$

where the terms in curly brackets are to be omitted. Equations (5.21) and (5.23) together are equivalent to the equations for gravity waves moving in the $\pm y$ directions in a non-rotating environment. We therefore know that the dispersion relation is $\omega = \pm kc$ where $c = (gh_0)^{1/2}$. We also have from equation (5.21) that $v_y = \pm c\eta$. Equation (5.22) thus becomes

$$\frac{\partial \eta}{\partial x} = \pm \frac{f}{c} \eta = \pm \frac{\eta}{L_R}, \tag{5.24}$$

where $L_R = c/f = (gh_0)^{1/2}/f$ is the Rossby radius.

Equation (5.24) has the solution

$$\eta \propto \exp(\pm \eta / L_R). \tag{5.25}$$

The plus sign causes this solution to blow up for $x \to +\infty$, which is unacceptable. Exponential decay of η with increasing x occurs for the minus sign. This represents the lateral structure of the Kelvin wave, which we therefore infer can only move in the minus y direction in this case. For an arbitrary orientation of the fluid boundary, the Kelvin wave moves so as to keep the boundary on the right in the northern hemisphere and on the left in the southern hemisphere. This wave is called an edge wave because its amplitude decays exponentially away from the fluid boundary on the scale of the Rossby radius.

5.6.2 Equatorial waves

We investigate equatorially trapped Kelvin waves using the linearized shallow water equations on an equatorial beta-plane:

$$\frac{\partial \eta}{\partial t} + \frac{\partial v_x}{\partial x} + \left\{ \frac{\partial v_y}{\partial y} \right\} = 0 \tag{5.26}$$

$$\frac{\partial v_x}{\partial t} + gh_0 \frac{\partial \eta}{\partial x} - \{ \beta y v_y \} = 0 \tag{5.27}$$

$$\left\{ \frac{\partial v_y}{\partial t} \right\} + gh_0 \frac{\partial \eta}{\partial y} + \beta y v_x = 0, \tag{5.28}$$

where consistent with the right panel of figure 5.4, we assume that v_y (confined to the curly brackets) is zero. Equations (5.26) and (5.27) tell a similar story to before, i. e., they represent a wave with free gravity wave characteristics moving in the $\pm x$ direction. Equation (5.28) gives the transverse structure of the wave:

$$\frac{\partial \eta}{\partial y} \pm \frac{\beta y}{c}\eta = 0, \tag{5.29}$$

where as before $c = (gh_0)^{1/2}$ and $v_x = \pm c\eta$, depending on the direction of wave motion. Equation (5.29) has the solution

$$\eta \propto \exp[-\beta y^2/(2c)] \tag{5.30}$$

for the plus sign, i. e., for eastward-moving waves. This solution decays away rapidly from the equator on the space scale $L_R = (2c/\beta)^{1/2}$, known as the *equatorial Rossby radius.* The minus sign, corresponding to waves moving to the west, results in the blowup of η far from the equator, and is thus not acceptable. Therefore, equatorial Kelvin waves can propagate only to the east.

5.7 Laboratory

1. Grab a small selection of FNL files and compute longitudinal averages using the Candis package. Plot these as a function of latitude and pressure. Then determine how accurately the geostrophic balance condition

$$f v_{g\lambda} = -\frac{1}{a}\frac{\partial \Phi}{\partial \phi}$$

holds, where $\Phi = gh$ is the geopotential. The latitudinal or ϕ derivative is taken at constant pressure. Since the vertical coordinate in FNL files is pressure, this derivative is easy to compute. Remember that for this to turn out correctly, the latitude must be represented in radians rather than the native units of the FNL file which is degrees. Recall also that the Coriolis parameter is a function of latitude, $f = 2\Omega \sin\phi$.

5.8 Problems

1. Derive equation (5.14) using the methods described in this chapter.

2. Try a solution for the Rossby wave on a beta-plane of the form $(\eta^*, q^*) = (\eta_0^*, q_0^*)\exp[i(kx + ly - \omega t)]$:

   (a) Find the dispersion relation for this wave, $\omega = \omega(k, l)$.

   (b) Find the x and y components of the group velocity of this wave, $u_x = \partial\omega/\partial k$ and $u_y = \partial\omega/\partial l$. Comment on the direction wave packets move relative to the phase propagation of the wave. In particular, under what conditions is the group velocity toward the east?

3. Show that in order to keep the geostrophic relative vorticity from blowing up at the equator due to the Coriolis parameter in the denominator, one must have a thickness field of the form $h = h_0 + A(\lambda)\phi^n$ where h_0 is a constant, $A(\lambda)$ is an arbitrary (but with bounded second derivatives) function of λ, and $n \geq 3$.