Chapter 4

Vorticity and Potential Vorticity

In this chapter we explore a way of solving the shallow water equations for motions with characteristic time scales longer than the rotation period of the earth. Under this constraint the motions of parcels are very nearly in geostrophic balance, i. e., the Coriolis and pressure gradient forces nearly counterbalance each other.

4.1 Circulation and vorticity

The *vorticity* of a fluid flow is defined as the curl of the velocity field:

$$\boldsymbol{\zeta} \equiv \nabla \times \mathbf{v}. \tag{4.1}$$

The vorticity plays a key role in the dynamics of an incompressible fluid, as well as in the sub-sonic flow of a compressible fluid. For shallow water flow only the vertical component of the vorticity is of interest:

$$\boldsymbol{\zeta} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} \tag{4.2}$$

The vorticity is related to a quantity called the *circulation*, defined as the closed line

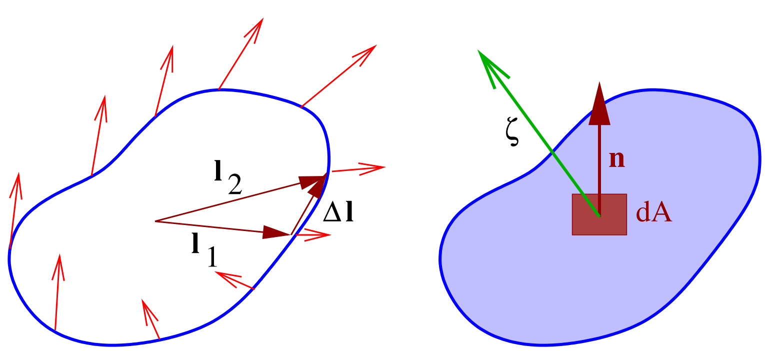

Figure 4.1: Sketch of a circulation loop which advects with the fluid flow, symbolized by the arrows on the left. Stokes' theorem relates the circulation to an area integral of the vorticity, as shown on the right.

integral of the fluid velocity component parallel to the path:

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \int \nabla \times \mathbf{v} \cdot \mathbf{n} dA = \int \boldsymbol{\zeta} \cdot \mathbf{n} dA. \tag{4.3}$$

The second form of the circulation involving the vorticity is obtained using Stokes' theorem. Hence the area integral is over the region bounded by the circulation path. Figure 4.1 illustrates the circulation loop.

Of particular interest is the circulation loop which moves and deforms with the fluid flow. The area, shape, and orientation of this loop evolve with time. However, the time rate of change of the circulation around such a loop obeys a surprisingly simple law, as we now show.

We wish to take the time derivative of Γ. However, the fact that the circulation loop evolves with time complicates this calculation. It is simplest to write the circulation integral in finite sum form while taking the derivative in which $d\mathbf{l} \to \Delta\mathbf{l}_i = \mathbf{l}_{i+1} - \mathbf{l}_i$ as illustrated in the left panel of figure 4.1:

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \sum \mathbf{v}_i \cdot (\mathbf{l}_{i+1} - \mathbf{l}_i) = \sum \frac{d\mathbf{v}_i}{dt} \cdot \Delta\mathbf{l}_i + \sum \mathbf{v}_i \cdot \Delta\mathbf{v}_i, \tag{4.4}$$

where we use $\mathbf{v}_i = d\mathbf{l}_i/dt$. We then revert to integral forms and note further that $\mathbf{v} \cdot d\mathbf{v} = d(v^2/2)$, which results in

$$\frac{d\Gamma}{dt} = \oint \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} + \oint d(v^2/2). \tag{4.5}$$

The second term on the right is the integral of a perfect differential over a closed path and is therefore zero.

The total time derivative of velocity can be eliminated using the shallow water momentum equation:

$$\frac{d\mathbf{v}}{dt} + g\nabla h + f\hat{\mathbf{z}} \times \mathbf{v} = 0. \tag{4.6}$$

We work for now in an inertial reference frame in which $f = 0$ and introduce a rotating frame at a later stage. In this case equation (4.5) becomes

$$\frac{d\Gamma}{dt} = -\oint g\nabla h \cdot d\mathbf{l} = -\oint d(gh). \tag{4.7}$$

The integral of a perfect differential around a closed loop is zero, so we arrive at the *Kelvin circulation theorem* (see Pedlosky, 1979):

$$\frac{d\Gamma}{dt} = 0. \tag{4.8}$$

In geophysical fluid dynamics we always use the circulation as computed in an inertial reference frame. However, we often have to compute the circulation directly given the fluid velocity in the rotating frame of the earth. Recall that the velocity in the inertial frame $\mathbf{v}_I$ can be related to the velocity in the rotating frame $\mathbf{v}$ by

$$\mathbf{v}_I = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}, \tag{4.9}$$

where $\mathbf{\Omega}$ is the rotation vector of the earth and $\mathbf{r}$ is the position vector relative to the center of the earth. The circulation thus becomes

$$\Gamma = \oint \mathbf{v}_I \cdot d\mathbf{l} = \oint \mathbf{v} \cdot d\mathbf{l} + \int [\nabla \times (\mathbf{\Omega} \times \mathbf{r})] \cdot \mathbf{n} dA, \tag{4.10}$$

where we have used Stokes' theorem to convert the second line integral into an area integral bounded by the circulation loop. A well-known vector identity can be used to reduce the last term: $\nabla \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\Omega}(\nabla \cdot \mathbf{r}) - \mathbf{\Omega} \cdot \nabla \mathbf{r} = 3\mathbf{\Omega} - \mathbf{\Omega} = 2\mathbf{\Omega}$. Substituting this into equation (4.10) results in

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} + \int 2\mathbf{\Omega} \cdot \mathbf{n} dA = \int (\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \cdot \mathbf{n} dA. \tag{4.11}$$

In the context of a rotating reference frame, the curl of the fluid velocity in the inertial frame is called the *absolute vorticity*,

$$\boldsymbol{\zeta}_a = \nabla \times \mathbf{v}_I = \nabla \times \mathbf{v} + 2\mathbf{\Omega}, \tag{4.12}$$

and $\boldsymbol{\zeta} = \nabla \times \mathbf{v}$ is called the *relative vorticity*.

4.2 Two-dimensional homogeneous flow

An important step in understanding geophysical fluid dynamics and how it is used comes from examining the evolution of the flow of a two-dimensional, homogeneous, incompressible fluid. This is a simplification of shallow water flow in which the thickness of the fluid layer is forced to be constant, say, by two horizontal, parallel plates between which the fluid flows. In this case the mass continuity equation becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{4.13}$$

As for the shallow water flow, the vorticity only has a z component,

$$\zeta_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}. \tag{4.14}$$

In this case we can define a *streamfunction* ψ such that

$$v_x = -\frac{\partial \psi}{\partial y} \tag{4.15}$$

and

$$v_y = \frac{\partial \psi}{\partial x}. \tag{4.16}$$

This choice trivially satisfies equation (4.13). Substitution into equation (4.14) results in a Poisson equation for the streamfunction:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \zeta_z. \tag{4.17}$$

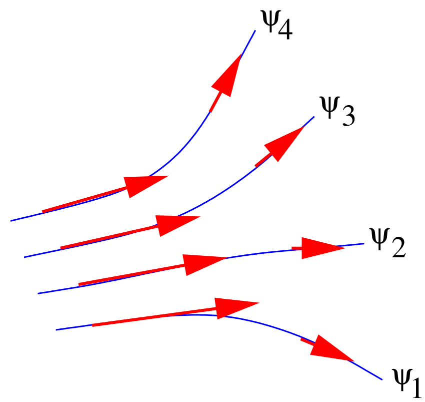

Figure 4.2: Illustration of the relationship between contours of constant streamfunction and the velocity field.

In mathematical physics terms, this is a relatively easy equation to solve, and it shows that the streamfunction, and hence the velocity field, is readily obtained from the vorticity field and the boundary conditions applicable to this equation.

Figure 4.2 illustrates the relationship between contours of constant streamfunction and the velocity field. The velocity vectors are everywhere tangent to the contours of streamfunction and the magnitude of the velocity is inversely proportional to the contour spacing. In a steady flow, parcels traverse the domain following lines of constant streamfunction. Hence these contours are also called *streamlines*. Note however, that if the flow is non-steady, parcel trajectories no longer coincide with streamlines.

The flow adjacent to a stationary wall bounding the fluid is parallel to the wall. The streamfunction along the wall is therefore constant. Specifying the value of the streamfunction on the walls bounding a fluid as well as the vorticity distribution in the interior is sufficient to guarantee a unique solution to equation (4.17).

The governing equation for vorticity may be obtained from the circulation theorem:

$$\frac{d\Gamma}{dt} = 0. \tag{4.18}$$

For two-dimensional homogeneous flow the area of a circulation loop lying in the $x-y$ plane does not change as it advects with the fluid. (Think of a vertical cylinder with end plate area equal to the area of the circulation loop. Since the volume of the cylinder is fixed by the incompressibility condition, and since the height of the cylinder does not change due to the two-dimensional nature of the flow, the end plate area must remain fixed.) For a tiny loop over which the vorticity doesn't vary much, $\Gamma = \zeta_z A$ where A is the area of the loop, and the vorticity of parcels is conserved, i. e.,

$$\frac{d\zeta_z}{dt} = 0. \tag{4.19}$$

Solution to the two-dimensional, incompressible, homogeneous flow problem can now be visualized. Suppose at the initial time the vorticity field is specified. Equation (4.17) is solved to obtain the streamfunction, and hence the velocity field. Equation (4.19) is then

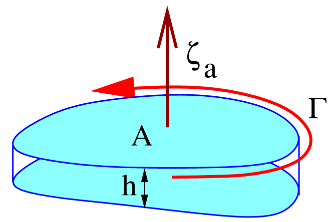

Figure 4.3: Segment of shallow water flow to which we apply the circulation theorem. The two surfaces of area A indicate the upper and lower bounds of the flow. The region between them has volume Ah. The vertical component of the absolute vorticity is ζ_a.

used to move parcels and their associated vorticity to new locations. The process is then repeated.

This is the simplest example of an *advection-inversion* process. The inversion part is the solution of the streamfunction equation given the vorticity. The vorticity is the key dependent variable in this problem, and it obeys a particularly simple evolution equation – it just moves around with the fluid!

4.3 Low Rossby number flow

If both the Coriolis term and the horizontal pressure gradient term in the momentum equation are much larger than the acceleration, then we can ignore the acceleration to zeroth order, resulting in

$$v_x \approx v_{gx} = -\frac{g}{f}\frac{\partial}{\partial y}(h+d) \qquad v_y \approx v_{gy} = \frac{g}{f}\frac{\partial}{\partial x}(h+d), \tag{4.20}$$

where $\mathbf{v}_g = (v_{gx}, v_{gy})$ is called the *geostrophic wind*. (The effects of terrain have been included.) The geostrophic wind is equal to the actual wind when the parcel acceleration is exactly zero. It is thus an approximation to the horizontal momentum equation which is analogous to the hydrostatic approximation in the vertical momentum equation.

Approximating the magnitude of the acceleration as the ratio of a typical velocity V and a typical time scale T, the ratio of the acceleration to the Coriolis term is

$$\mathrm{Ro} \equiv \frac{1}{fT} \tag{4.21}$$

where Ro is the *Rossby number*. Values of $\mathrm{Ro} \ll 1$ indicate that geostrophic balance is approximately satisfied.

4.4 Potential vorticity

We now develop an approximation to shallow water theory for flows in which the Rossby number is very much less than unity. This theory has much in common with the above theory for pure two-dimensional flow.

The circulation theorem yields a variable which is of great use in understanding the dynamics of low Rossby number shallow water flow. Applying the circulation theorem to a segment of the flow as shown in figure 4.3, we conclude from the Kelvin theorem that the circulation around the segment is conserved. We can write the circulation as

$$\Gamma = \zeta_a A = \frac{\zeta_a}{h}(Ah). \tag{4.22}$$

The quantity Ah is the volume of the fluid parcel. Since the fluid is incompressible, the volume of this parcel does not change with time. Since the circulation around the parcel is conserved, the variable

$$q \equiv \frac{\zeta_a}{h} \tag{4.23}$$

which is known as the *potential vorticity,* is also conserved by parcels, i. e.,

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + v_x \frac{\partial q}{\partial x} + v_y \frac{\partial q}{\partial y} = 0. \tag{4.24}$$

The vertical component of absolute vorticity increases as the area of the loop decreases. However, a decrease in the loop area implies an increase in the thickness of the fluid layer. This increase is in proportion to the increase in absolute vorticity, which is why the ratio of the two quantities stays the same.

4.5 Potential vorticity inversion

The absolute vorticity can be written approximately as

$$\zeta_a = f + \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{4.25}$$

where as before we have ignored the horizontal component of the earth's rotation. Replacing the actual flow velocity by the geostrophic wind and substituting into equation (4.23) results in

$$\frac{1}{h}\left[\frac{\partial}{\partial x}\left(\frac{g}{f}\frac{\partial(h+d)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{g}{f}\frac{\partial(h+d)}{\partial y}\right) + f\right] = q. \tag{4.26}$$

This partial differential equation is elliptic when both f and q are uniformly of the same sign in the domain of interest. Unique solutions may be obtained if q is known in the interior and h is specified on the domain boundary.

Equation (4.26) is easiest to understand in the limit of small fractional thickness deviations. Setting $h = h_0(1+\eta)$ as before, and linearizing in η, this becomes

$$q_0\left(1 - \eta + L_R^2 \nabla^2(\eta + d/h_0)\right) = q, \tag{4.27}$$

where $q_0 = f/h_0$ is the *planetary potential vorticity* and where

$$L_R = (gh_0)^{1/2}/f \tag{4.28}$$

is called the *Rossby radius of deformation*. We have assumed as well that f is constant.

Slight further simplification comes from defining a perturbation potential vorticity $q' = q - q_0$:

$$L_R^2 \nabla^2(\eta + d/h_0) - \eta = q'/q_0. \tag{4.29}$$

Solving for the time evolution of a low Rossby number flow is now done (in principle) in three steps:

1. For a given distribution of potential vorticity, find the distribution of thickness using equation (4.26) or in the linear case (4.29).
2. Use equation (4.20) to obtain the geostrophic wind, and hence the approximate flow field.
3. Use equation (4.24) to advect the potential vorticity distribution to the next time level. Repeat these steps as needed.

Notice how similar this procedure is to the solution of the two-dimensional incompressible flow problem described earlier. The main differences are that potential vorticity rather than vorticity is employed, the inversion equation is slightly more complicated, and the solution is approximately valid for low Rossby number, not exact.

4.6 Simple inversion examples

We now explore two simple examples of potential vorticity inversion.

4.6.1 Periodic potential vorticity anomaly

Let us imagine a case in which the potential vorticity perturbation varies periodically in some direction, say the x direction, so that

$$q' = \epsilon q_0 \sin(kx), \tag{4.30}$$

where ϵ and k are constants and $q_0 = f/h_0$ is the ambient planetary potential vorticity. Assuming also a flat-bottomed domain so that $d = 0$, equation (4.29) becomes

$$L_R^2 \nabla^2 \eta - \eta = \epsilon \sin(kx). \tag{4.31}$$

We assume a trial solution $\eta = \eta_0 \sin(kx)$, which upon substitution in equation (4.31) yields the algebraic equation

$$\eta_0 = -\frac{\epsilon}{1 + k^2 L_R^2}. \tag{4.32}$$

Thus, the fluid thickness is

$$h = h_0 \left(1 - \frac{\epsilon \sin(kx)}{1 + k^2 L_R^2}\right), \tag{4.33}$$

and the geostrophic velocity components are

$$v_{gx} = 0 \qquad v_{gy} = -\frac{f k L_R^2 \epsilon \cos(kx)}{1 + k^2 L_R^2}, \tag{4.34}$$

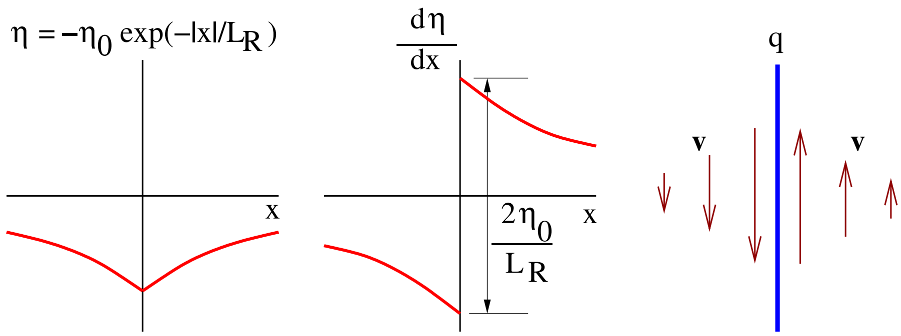

Figure 4.4: Plots of $\eta(x)$, $\partial\eta/\partial x$, and velocity vectors associated with a line of potential vorticity along the y axis, $q = C\delta(x)$.

from which we find the absolute vortcity:

$$\zeta_a = f\left(1 + \frac{k^2 L_R^2 \epsilon \sin(kx)}{1 + k^2 L_R^2}\right) \tag{4.35}$$

Recomputation of the potential vorticity from equations (4.33) and (4.35) yields

$$q = \frac{\zeta_a}{h} = q_0\left[1 + \left(\frac{k^2 L_R^2}{1 + k^2 L_R^2} + \frac{1}{1 + k^2 L_R^2}\right)\epsilon \sin(kx)\right], \tag{4.36}$$

where we have made the approximation $h^{-1} = h_0^{-1}(1+\eta)^{-1} = h_0^{-1}(1-\eta)$ as previously. The two terms inside the large parentheses represent respectively the effects of the vorticity perturbation and the thickness perturbation on the potential vorticity. They clearly add up to unity, which means that we recover the assumed form of the potential vorticity perturbation as expected. However, for $k^2 L_R^2 \ll 1$ the second term dominates, while for $k^2 L_R^2 \gg 1$ the first term dominates. In other words, a nearly balanced potential vorticity perturbation with horizontal scale much greater than the Rossby radius is represented primarily by a thickness perturbation, while one with scale much smaller than the Rossby radius is manifested mainly by a vorticity perturbation.

This result is valid in cases far beyond this particular example, and represents a general characteristic of balanced geophysical flows.

4.6.2 Line of potential vorticity

Let us now invert the flow fields resulting from a concentrated line of potential vorticity along the y axis:

$$q' = q_0 C\delta(x), \tag{4.37}$$

where $q_0 = f/h_0$ as usual, C is a constant, and $\delta(x)$ is the Dirac delta function. The inversion equation thus becomes

$$L_R^2 \frac{\partial^2 \eta}{\partial x^2} - \eta = C\delta(x), \tag{4.38}$$

where the y derivative vanishes for reasons of symmetry. For $x \neq 0$ the right side of this equation is zero and $\eta \propto \exp(\pm x/L_R)$. We desire a solution which does not blow up when $|x| \rightarrow \infty$, so we assume a solution of the form

$$\eta = -\eta_0 \exp(-|x|/L_R), \tag{4.39}$$

where η_0 is a constant. The minus sign is inserted before η_0 on the expectation that a positive potential vorticity anomaly will result in a negative thickness anomaly as usual.

The form of equation (4.39) is illustrated in the left panel of figure 4.4. The center panel shows

$$\frac{\partial \eta}{\partial x} = \frac{x\eta_0}{|x|L_R} \exp(-|x|/L_R), \tag{4.40}$$

which has a discontinuity of magnitude $2\eta_0/L_R$ at the origin. Since the derivative of a step function is a Dirac delta function, we know that the second derivative of η will produce a term $(2\eta_0/L_R)\delta(x)$ in additional to the normal terms arising from this derivative. The full second derivative is

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{2\eta_0}{L_R}\delta(x) - \frac{\eta_0}{L_R^2} \exp(-|x|/L_R), \tag{4.41}$$

which upon substitution into equation (4.38) yields

$$\eta_0 = \frac{C}{2L_R}. \tag{4.42}$$

The y component of the geostrophic velocity is

$$v_{gy} = \frac{g}{f}\frac{\partial h}{\partial x} = \frac{gh_0}{f}\frac{\partial \eta}{\partial x} = \frac{fCx}{2|x|} \exp(-|x|/L_R). \tag{4.43}$$

The geostrophic velocity is illustrated in the right panel of figure 4.4. The delta function potential vorticity anomaly is associated with a "shear line" where the y component of the velocity changes abruptly across the anomaly. The wind decays exponentially away from the potential vorticity anomaly on the scale of the Rossby radius.

4.7 Rossby waves

The so-called Rossby wave is a geophysical phenomenon of great importance. It is a nearly balanced flow pattern which occurs in many contexts in oceanic and atmospheric dynamics. Since the flow is approximately balanced, we can use the mathematical apparatus set up to study low Rossby number flow. However, before doing this, we first make some qualitative arguments which reveal the basic physical mechanism of the Rossby wave.

Let us imagine a fluid initially at rest in a channel with a tilted bottom and level top, as illustrated in figure 4.5. Assuming that the fluid resides in the northern hemisphere of the earth, the Coriolis parameter f is positive and is taken to be constant. The potential vorticity of this fluid varies with y by virtue of the variation in the thickness of the fluid layer with y. Since the fluid layer is thinner for larger y (which we identify as "north"), the potential vorticity, which we recall is equal to $q = f/h$, is larger there.

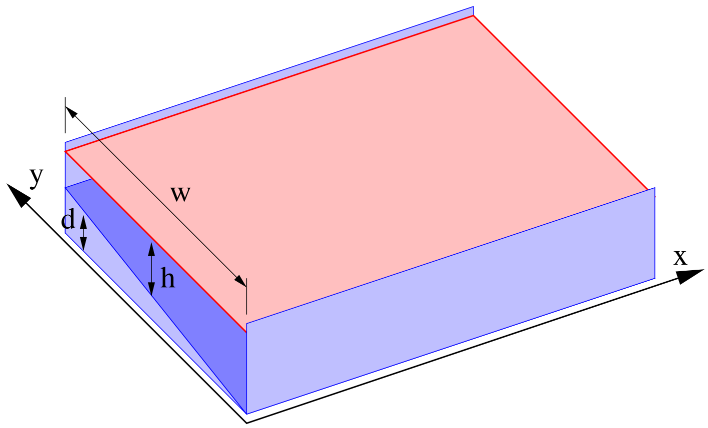

Figure 4.5: Geometry for fluid in a channel of width w with a bottom of variable height $d(y)$.

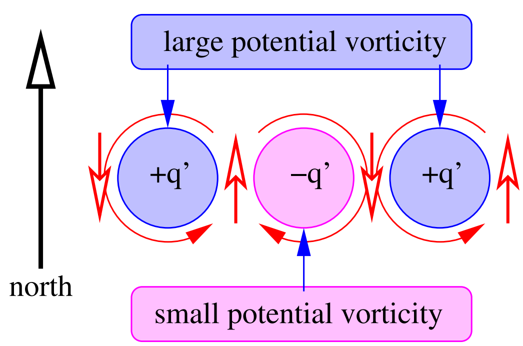

Figure 4.6: Sketch of effect of displacing parcels north and south in a fluid with higher values of potential vorticity to the north.

Figure 4.6 shows how alternating displacements of parcels of fluid north and south from their initial positions affects the potential vorticity distribution. Parcels moved from south to north carry with them potential vorticity values lower than their new environment, and therefore have negative potential vorticity perturbations. The opposite happens with parcels displaced from north to south.

A positive potential vorticity anomaly is associated with positive relative vorticity. A counter-clockwise circulation thus exists about the anomaly in this case. Similarly, a negative anomaly exhibits a clockwise circulation. The net effect of these circulations is to cause northward flow between anomalies where a positive anomaly exists to the west, and southward flow in the opposite case. The northward flow tends to reduce the potential vorticity in the gap between anomalies, whereas the southward flow increases it. Examination of figure 4.6 shows that the net effect of this is to shift all anomalies to the left. We thus have a wave phenomenon in which an east-west train of alternating positive and negative potential vorticity anomalies moves to the west with time. This type of wave is called a *Rossby wave.* Rossby waves play a central role in the large-scale dynamics of the earth's ocean and atmosphere. They depend for their existence on a transverse gradient in potential vorticity. In the present example the potential vorticity gradient is caused by a gradient in the elevation of the lower boundary of the the pool of water. Other mechanisms for producing this gradient exist as well, and we will encounter these as we proceed. Meanwhile, let us analyze this situation more quantitatively.

We divide the fractional thickness perturbation into two components, the first representing the north-south thickness variation due to the tilt of the bottom surface, and the second associated with the Rossby wave structure. We assume that the bottom surface depends on y as

$$d = h_0 \mu y, \tag{4.44}$$

where $y = 0$ at the south wall of the channel. Since the fluid surface at rest must be level, we insist that $h + d = h_0(1 + \eta) + h_0\mu y = h_0$, which means that $\eta = -\mu y$ in the rest case. More generally when there is fluid motion, we postulate that

$$\eta = -\mu y + \eta^*. \tag{4.45}$$

In order to maintain the linearization condition $|\eta| \ll 1$, we must have $\mu w \ll 1$ where w is the channel width, as illustrated in figure 4.5. In the case in which the fluid is in motion, we thus have

$$h + d = h_0(1 + \eta^*), \tag{4.46}$$

which means that the geostrophic velocity is given by

$$v_{gx} = -\frac{gh_0}{f}\frac{\partial \eta^*}{\partial y} \qquad v_{gy} = \frac{gh_0}{f}\frac{\partial \eta^*}{\partial x}. \tag{4.47}$$

At rest the potential vorticity takes the form

$$q = \frac{f}{h_0(1 - \mu y)} \approx q_0(1 + \mu y). \tag{4.48}$$

In the case with motion we add a potential vorticity perturbation q^*:

$$q = q_0(1 + \mu y) + q^*. \tag{4.49}$$

We now substitute equations (4.45), (4.47), and (4.49) into the potential vorticity conservation equation (4.24) and the inversion equation (4.29). We linearize in quantities having to do with motion, i. e., η^*, v_x, v_y, and q^*, and also replace the velocity components with their geostrophic counterparts. Recalling that $q_0 = f/h_0$,

$$\frac{\partial q^*}{\partial t} + v_{gy}\frac{\partial q_0 \mu y}{\partial y} = \frac{\partial q^*}{\partial t} + g\mu\frac{\partial \eta^*}{\partial x} = 0, \tag{4.50}$$

and

$$L_R^2 \nabla^2 \eta^* - \eta^* = q^*/q_0. \tag{4.51}$$

Note how the terms containing μ, the tilt of the bottom surface, cancel out of equation (4.51), leaving only terms involving motion. The effect of the tilt of the bottom surface enters only into the potential vorticity conservation equation (4.50).

Let us now assume a wave moving in the x direction. Since the fluid is confined to an east-west channel, the y velocity must be zero at the north and south boundaries, which occur at $y = 0, w$. As a result, both η^* and q^* must be zero there as well. Trial solutions which satisfy these boundary conditions are

$$\eta^* = \eta_0^* \sin(\pi y/w) \exp[i(kx - \omega t)] \tag{4.52}$$

and

$$q^* = q_0^* \sin(\pi y/w) \exp[i(kx - \omega t)], \tag{4.53}$$

where η_0^* and q_0^* are constants. Substitution into equations (4.50) and (4.51) yields two linear, homogeneous algebraic equations in two unknowns, which can be represented in matrix form as

$$\begin{pmatrix} -\omega & kg\mu \\ 1/q_0 & 1 + L_R^2(k^2 + \pi^2/w^2) \end{pmatrix} \begin{pmatrix} q^* \\ \eta^* \end{pmatrix} = 0. \tag{4.54}$$

Setting the determinant of the matrix of coefficients to zero and solving for the frequency ω results in the dispersion relation for Rossby waves

$$\omega = -\frac{kL_R^2 f\mu}{1 + L_R^2(k^2 + \pi^2/w^2)}. \tag{4.55}$$

As predicted by the qualitative arguments outlined above, the phase speed ω/k is negative, i. e., the wave moves in the $-x$ direction, or to the west. Furthermore, the presence of the wavenumber k in the denominator makes this wave dispersive.

It is perhaps easiest to understand this dispersion relation by adjusting the length and time scales so that length is measured in units of the Rossby radius L_R and time is measured in terms of the inverse Coriolis parameter f^{-1}. With this rescaling, the dispersion relation simplifies to

$$\omega = -\frac{\mu k}{1 + k^2 + \pi^2/w^2}. \tag{4.56}$$

Figure 4.7 shows how this dispersion relation behaves for $\mu = 0.1$ and $w = 3$. The magnitude of the frequency peaks for $k = k_c \approx 1.5$. For smaller wavenumbers the group velocity of the wave, $\partial\omega/\partial k$, is negative (i. e., westward) for $k < k_c$ and positive (eastward) for $k > k_c$. Thus, in the short wavelength limit, the group velocity moves in the direction opposite the phase speed.

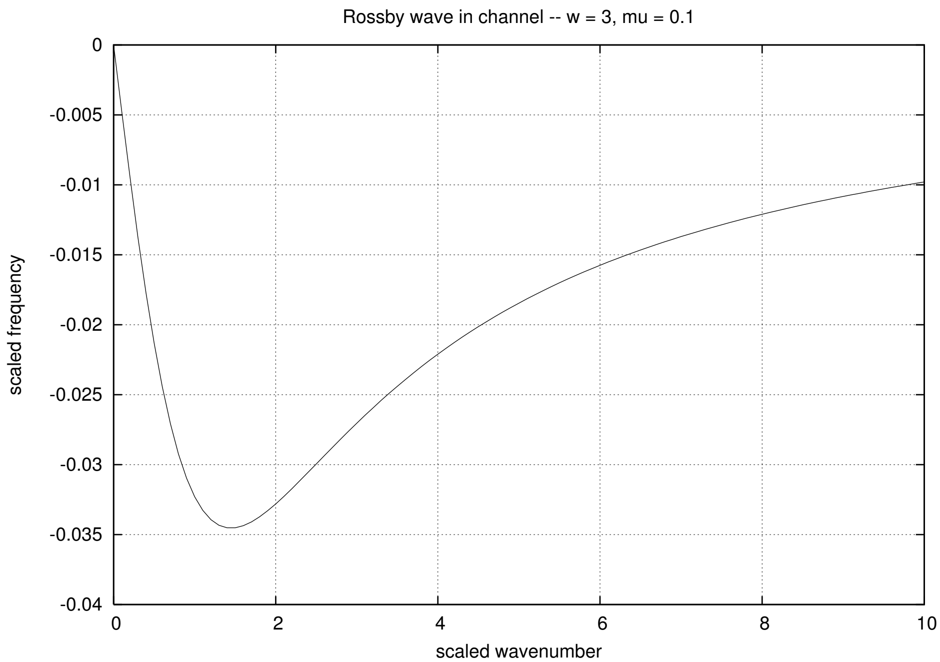

Figure 4.7: Dispersion relation for shallow water Rossby waves in a channel with tilted bottom as represented by equation (4.56) with scaled parameter values $\mu = 0.1$ and $w = 3$.

4.8 References

Pedlosky, J., 1979: *Geophysical Fluid Dynamics.* Springer-Verlag, 624 pp.

Vallis, G. K., 2006: *Atmospheric and Oceanic Fluid Dynamics.* Cambridge University Press, 745 pp.

4.9 Problems

1. Suppose we add an additional force per unit mass $\mathbf{F}$ to the shallow water momentum equation, such as might be caused by friction. The momentum equation thus takes the form (in a non-rotating reference frame)

$$\frac{d\mathbf{v}}{dt} + g\nabla h = \mathbf{F}. \tag{4.57}$$

   Derive the Kelvin circulation theorem in this extended case. If $\mathbf{F}$ is conservative, does it enter?

2. Imagine a point vortex in two-dimensional homogeneous flow where the vorticity field is given by $\zeta_z = C\delta(x)\delta(y)$ where C is a constant equal to the strength of the vortex and $\delta()$ is the Dirac delta function. (The Dirac delta function has an integral of one, but is only non-zero where the argument is zero.) Solve for the streamfunction on an infinite domain. Hint: Use cylindrical symmetry and the Kelvin theorem applied to a circular

loop centered on the vortex to obtain the velocity field. From this the streamfunction can be obtained by integration. If you have experience with electromagnetism, think of the problem of the magnetic field surrounding an infinite wire carrying a current.

3. Consider a two-dimensional flow which is stationary except for the flows associated with two point vortices of equal but opposite strength $\pm C$ separated by a distance d. Describe the speed and direction of motion of the two vortices.

4. Repeat the above problem for the case in which the two vortices have strength of the same sign and magnitude.

5. Imagine a shallow water basin of uniform ambient depth h_0 in which the Coriolis parameter f is constant, with $f > 0$.

   (a) Using the linearized shallow water equations from the previous chapter, show that for a gravity-inertia wave of form $(\eta, v_x, v_y) \propto \exp[i(kx - \omega t)]$, one has $v_x = (\omega/k)\eta$ and $v_y = -(if/k)\eta$. The physical solutions are obtained by taking the real part of these equations.

   (b) Compute the potential vorticity distribution produced by the wave. Use the linearized form for the potential vorticity $q = q_0(1 - \eta + \zeta/f)$ where ζ is the relative vorticity and where the reference potential vorticity is $q_0 = f/h_0$.

   (c) Determine whether the flow field of this wave is in geostrophic balance.

6. Repeat the analysis of section 4.6.1 for the case $q' = \epsilon q_0 \sin(kx) \sin(ky)$.

7. Consider an east-west channel with flat bottom of width w with a flow moving uniformly in geostrophic balance to the east at speed U. Assume constant f.

   (a) For this steady flow, show that the fractional thickness perturbation takes the form $\eta = -Uy/(fL_R^2) \equiv -\mu y$.

   (b) Assume a wave of the form $(q^*, \eta^*) \propto \sin(\pi y/w) \exp[i(kx - \omega t)]$ on this basic flow, where $\eta = -\mu y + \eta^*$, and where the potential vorticity $q = q_0(1 + \mu y) + q^*$ as in section 4.7, and find the dispersion relation $\omega = \omega(k)$. Hint: Linearize the potential vorticity evolution equation about a state of uniform motion rather than a state of rest.