

# Chapter 2

## Buoyancy and Coriolis forces

In this chapter we address several topics that we need to understand before starting on our study of geophysical fluid dynamics.

### 2.1 Hydrostatic approximation

Consider the control volume shown in figure 2.1. The mass in this element of fluid equals its density  $\rho$  times the volume  $A\Delta z$ ;  $M = \rho A\Delta z$ , which means that a downward gravitational force  $\rho A\Delta z g$  exists, where  $g$  is the gravitational field. Given pressure values  $p_1$  and  $p_2$  on the lower and upper faces of the volume, an upward pressure force  $A(p_1 - p_2)$  exists on the volume. If these are the only two forces acting on the volume, then the vertical component of Newton's second law states that

$$\rho A\Delta z \frac{dw}{dt} = A(p_1 - p_2) - \rho A\Delta z g \quad (2.1)$$

where  $w$  is the vertical velocity of the fluid element. Dividing by  $\rho A\Delta z$  and replacing  $(p_1 - p_2)/\Delta z$  by  $-dp/dz$ , we obtain

$$\frac{dw}{dt} = -\frac{1}{\rho} \frac{dp}{dz} - g. \quad (2.2)$$

Figure 2.2 illustrates a typical large-scale atmospheric or oceanic circulation in which the vertical dimension  $h$  might be 5 – 10 km and the horizontal dimension  $d$  would be of order several thousand kilometers. Assuming constant density (excellent approximation in the ocean; not so good in the atmosphere, but adequate for our purposes here), then

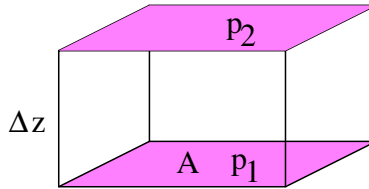


Figure 2.1: Control volume for hydrostatic equation with horizontal area  $A$  and vertical dimension  $\Delta z$ . The pressures on the bottom and top of the box are  $p_1$  and  $p_2$ .

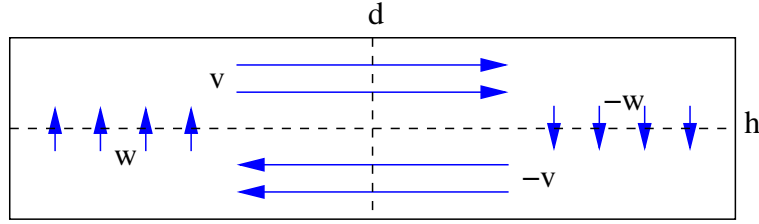


Figure 2.2: Schematic illustration of a large-scale atmospheric or oceanic flow with horizontal dimension  $d$  and vertical dimension  $h$  in which the aspect ratio  $d/h \gg 1$ .

mass continuity in, say, the upper left quadrant of the circulation in figure 2.2 requires that  $vh/2 \approx wd/2$ , which implies that  $w \ll v$ . Typical atmospheric circulations might have  $v$  equal to several tens of meters per second, which would make  $w$  of order a few centimeters per second. Given the time required for the fluid element to ascend through, say, 10 km, given this value of  $w$ , it is safe to assume that the acceleration term in equation (2.2) is small compared to the gravity and pressure gradient terms. thus, to a good approximation we can ignore it, resulting in the hydrostatic equation

$$\frac{dp}{dz} = -g\rho. \quad (2.3)$$

This applies to both the atmosphere and the ocean, and it shows that the pressure decreases with height with the rate of decrease being in proportion to the density.

## 2.2 Buoyancy force

When a parcel is displaced up or down in a fluid in hydrostatic equilibrium, the resulting buoyancy force (sum of gravity and vertical pressure gradient forces) can act either to reinforce the initial displacement or act against it. The details are different depending on whether we are dealing with the ocean, a dry atmosphere, or a moist atmosphere.

### 2.2.1 Ocean

The ocean is nearly, but not quite incompressible. In an incompressible fluid the density of fluid parcels doesn't change. However, different parts of the fluid may have different densities. Imagine, for instance, ocean water, with variable distributions of temperature and salinity (salt content). Both of these factors affect the density of ocean water. However, to a good approximation, a parcel's density is maintained as it moves up and down to different pressure levels.

Figure 2.3 shows what happens to a fluid parcel displaced vertically for two different cases of density stratification in an incompressible fluid in a gravitational field. In the left panel the density of the displaced parcel, which doesn't change as a result of the displacement, is greater than that of the surrounding environment if the displacement is upward. As a result, the buoyancy force is downward, accelerating the parcel back toward its initial level. A downward displacement results in an upward force, which also tends to return the parcel

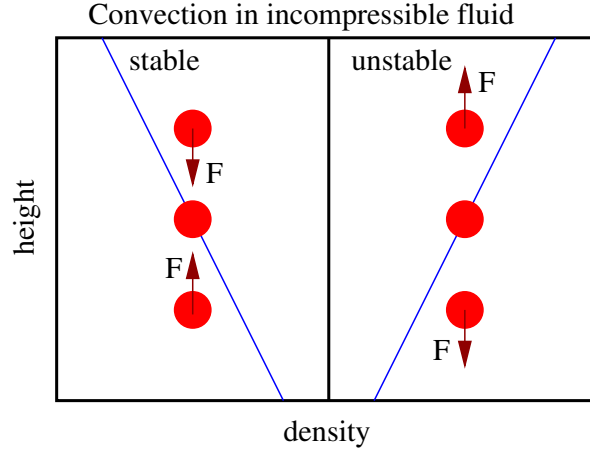


Figure 2.3: Stability and instability of a vertically displaced parcel in a stratified, incompressible fluid.

to its initial level. The fluid in this case is *statically stable*. The opposite holds in the right panel, where a vertical displacement, either upward or downward, results in a buoyancy force which tends to reinforce the initial displacement. This is the *statically unstable* case. It is clear from figure 2.3 that static stability occurs when the density decreases with height, whereas static instability is associated with density increasing with height.

In the stable case the fluid parcel oscillates up and down about the equilibrium point. If the density profile has the form  $\rho = \rho_0 - \Gamma z$ , where  $\rho_0$  is the ambient fluid density at height  $z = 0$ , then displacing a parcel from this level results in a buoyancy force per unit mass of  $F_g = -g\delta\rho/\rho_0 = -g(\rho_0 - \rho)/\rho_0 = -g\Gamma z/\rho_0$ . If no other forces are acting on the parcel, then the parcel displacement is governed by

$$\frac{d^2 z}{dt^2} = -\frac{g\Gamma}{\rho_0} z. \quad (2.4)$$

If the density changes only weakly with height, then we can consider  $\rho$  in the denominator of the above equation to be essentially constant, and we have a harmonic oscillator equation with oscillation frequency

$$N = \left( \frac{g\Gamma}{\rho_0} \right)^{1/2}. \quad (2.5)$$

This is called the *Brunt-Väisälä* or *buoyancy* frequency, and it is the oscillation frequency of a parcel displaced vertically in an incompressible fluid and released.

The case of unstable stratification is represented by negative values of  $\Gamma$ . In this case equation (2.4) has solutions of the form

$$z = a \exp(\sigma t) + b \exp(-\sigma t) \quad (2.6)$$

where  $\sigma = (-g\Gamma/\rho_0)^{1/2}$  and  $a$  and  $b$  are constants. The exponential growth in vertical displacement represents the onset of convective overturning. This solution is only valid for very small displacements. When displacements become large, nonlinear effects become important and the behavior of the fluid becomes much more complex.

### 2.2.2 Properties of sea water

The density of sea water is given by a complex series of equations blessed officially by the UNESCO Joint Panel on Oceanographic Tables and Standards (Gill 1982). The density of pure water at atmospheric pressure is

$$\begin{aligned}\rho_w = & 999.842594 + 6.793952 \times 10^2 T - 9.095290 \times 10^{-3} T^2 \\ & + 1.001685 \times 10^{-4} T^3 - 1.120083 \times 10^{-6} T^4 + 6.536332 \times 10^{-9} T^5\end{aligned}\quad (2.7)$$

where  $T$  is the temperature in degrees Celsius. The ocean temperature ranges typically over  $0 - 30^\circ$  C. The density of sea water with salinity  $S$  at atmospheric pressure is

$$\begin{aligned}\rho_0 = & \rho_w + (0.824493 - 4.0899 \times 10^{-3} T + 7.6438 \times 10^{-5} T^2 \\ & - 8.2467 \times 10^{-7} T^3 + 5.3875 \times 10^{-9} T^4) S \\ & + (-5.72466 \times 10^{-3} + 1.0227 \times 10^{-4} T \\ & - 1.6546 \times 10^{-6} T^2) S^{3/2} + 4.8314 \times 10^{-4} S^2.\end{aligned}\quad (2.8)$$

Salinity has the units of practical salinity units (psu), which is very close to the concentration in parts per thousand (ppt). Salinity ranges over  $27 - 37$  psu in the world's oceans. The pressure dependence of density is given by

$$\rho = \rho_0 / (1 - p/K) \quad (2.9)$$

where  $p$  is the pressure in bars ( $1 \text{ bar} = 10^5 \text{ Pa}$ ) and  $K \approx 20000 \text{ bar}$  for pure water at freezing.  $K$  is a weak function of temperature, pressure, and salinity. See Gill (1982) for details. The specific heat of sea water is roughly  $C_l = 4000 \text{ J kg}^{-1} \text{ K}^{-1}$ .

### 2.2.3 Dry atmosphere

Air is compressible, which makes atmospheric convection somewhat more complex than that which occurs in an incompressible fluid such as water. In particular, the density of an air parcel is not conserved in vertical displacements. Thus, in computing the buoyancy of a displaced air parcel, the change in buoyancy of the air parcel with height must be considered along with the vertical profile of the density of ambient air. This is best done using the dry entropy.

The dry entropy is defined

$$s_d = C_p \ln(T/T_R) - R \ln(p/p_R), \quad (2.10)$$

where  $T_R = 273.15 \text{ K}$  and  $p_R = 1000 \text{ hPa}$  are constant reference values of the temperature and pressure,  $R$  is the universal gas constant divided by the molecular weight of air, and  $C_p$  is the specific heat of air at constant pressure. The dry entropy is conserved in non-condensing, slow vertical displacements parcels to which no heat is added. One must simply convert a difference in entropy between the parcel and its environment to a difference in density.

Eliminating  $T$  in favor of  $p$  in the dry entropy equation by means of the ideal gas law

$$\frac{p}{\rho} = RT \quad (2.11)$$

results in

$$s_d = C_v \ln(p/p_R) - C_p \ln(\rho/\rho_R) \quad (2.12)$$

where  $\rho_R = p_R/(RT_R)$  is a constant reference density and  $C_v = C_p - R$  is the specific heat of air at constant volume. A small difference between the parcel and the environmental density at a fixed pressure is obtained by taking the differential of this equation with  $\delta p = 0$ :

$$\delta s_d = -\frac{C_p \delta \rho}{\rho_0} \quad (2.13)$$

where the “ $\delta$ ” indicates parcel minus environment values. However, this difference can be related to the vertical parcel displacement  $z$  by

$$\delta s_d = -\frac{ds_d}{dz} z \quad (2.14)$$

where  $ds_d/dz$  is the vertical gradient of environmental entropy. Combining these equations one may compute the buoyancy force per unit mass as previously, resulting in

$$F_B = -\frac{g\delta\rho}{\rho_0} = \frac{g}{C_p} \delta s_d = -\frac{g}{C_p} \frac{ds_d}{dz} z \quad (2.15)$$

and a value for the buoyancy frequency of

$$N = \left( \frac{g}{C_p} \frac{ds_d}{dz} \right)^{1/2}. \quad (2.16)$$

Meteorologists often use the potential temperature

$$\theta = T_R \exp(s_d/C_p) = T \left( \frac{p_R}{p} \right)^\kappa \quad (2.17)$$

instead of the dry entropy, where  $\kappa = R/C_p$ . Physically, the potential temperature is the temperature a parcel would have if it were compressed isentropically to the reference pressure  $p_R$ . The buoyancy frequency written in terms of the potential temperature is easily shown to be

$$N = \left( \frac{g}{\theta} \frac{d\theta}{dz} \right)^{1/2}. \quad (2.18)$$

### 2.2.4 Atmosphere with water vapor

Water vapor in the atmosphere obeys the ideal gas law closely:

$$\frac{e}{\rho_v} = R_v T, \quad (2.19)$$

where  $e$  is the partial pressure of water vapor,  $\rho_v$  is the density of water vapor,  $R_v$ , the ratio of the universal gas constant and the molecular weight of water, is the gas constant for water vapor. The mixing ratio for water vapor is the ratio of the densities of water vapor and dry air:

$$r = \rho_v / \rho. \quad (2.20)$$

For an atmosphere containing water vapor, the moist entropy (referred to hereafter as the “entropy”), takes on the approximate value

$$s = s_d + \frac{Lr}{T_R} \quad (2.21)$$

where  $L$ , assumed constant, is the latent heat of condensation. Meteorologists define a quantity called the equivalent potential temperature  $\theta_e$  which is related to the moist entropy in the same fashion that the potential temperature is related to dry entropy:

$$\theta_e = T_R \exp(s/C_p) = \theta \exp[Lr/(C_p T_R)]. \quad (2.22)$$

The saturation vapor pressure of water vapor  $e_s(T)$  is a function of temperature only, and is given by the Clausius-Clapeyron equation:

$$\frac{d \ln e_s}{dT} = \frac{L}{R_v T^2}. \quad (2.23)$$

Consistent with the approximation made in the definition of entropy, we take the latent heat of condensation to be constant, allowing this equation to be integrated:

$$e_s(T) = e_F \exp \left[ \frac{L}{R_v} \left( \frac{1}{T_F} - \frac{1}{T} \right) \right] \quad (2.24)$$

where  $T_F$  is the freezing temperature and  $e_F$  is the saturation vapor pressure at freezing. The saturation mixing ratio  $r_s$  may be obtained by taking the ratio of the saturation vapor pressure and the air pressure formulas from the respective ideal gas laws:

$$\frac{e_s}{p} = \frac{m_d}{m_v} \frac{\rho_{vs}}{\rho} = \frac{r_s}{\epsilon} \quad (2.25)$$

where  $\rho_{vs}$  is the saturated water vapor density and  $\epsilon = m_v/m_d$  is a dimensionless constant.

It must be emphasized that this treatment of moist thermodynamics is significantly simplified, and is useful for qualitative analyses only. For a more accurate and complete treatment see Emanuel (1994) or Iribarne and Godson (1981).

Imagine now a completely saturated environment. In this environment the entropy  $s$  is conserved by parcels, and furthermore  $s = s_s$ , the saturated moist entropy, obtained by replacing  $r$  by  $r_s$  in the moist entropy equation. Proceeding as before and using the approximate formula for saturated entropy, the difference between parcel and environmental values of saturated entropy at constant pressure is

$$\delta s_s \approx \left( \frac{C_p}{T} + \frac{L}{T_R} \frac{\partial r_s}{\partial T} \right) \delta T. \quad (2.26)$$

Using  $r_s = \epsilon e_s(T)/p$  and the Clausius-Clapeyron equation, this becomes

$$\delta s_s \approx \left( C_p + \frac{L^2 r_s}{R_v T_R^2} \right) \frac{\delta T}{T_R} \approx - \left( C_p + \frac{L^2 r_s}{R_v T_R^2} \right) \frac{\delta \rho}{\rho}, \quad (2.27)$$

where consistent with earlier approximations  $T$  has been set to  $T_R$ .

Constant	Value	Meaning
$R_{univ}$	$8.314\text{JK}^{-1}$	Universal gas constant
$m_d$	$28.9 \times 10^{-3}\text{kgmol}^{-1}$	Molecular weight of dry air
$m_v$	$18.0 \times 10^{-3}\text{kgmol}^{-1}$	Molecular weight of water
$\epsilon$	0.623	Ratio of molecular weights
$R$	$287\text{JK}^{-1}\text{kg}^{-1}$	Gas constant for dry air
$R_v$	$461\text{JK}^{-1}\text{kg}^{-1}$	Gas constant for water vapor
$C_p$	$1005\text{JK}^{-1}\text{kg}^{-1}$	Specific heat of dry air at const pres
$C_v$	$718\text{JK}^{-1}\text{kg}^{-1}$	Specific heat of dry air at const vol
$L$	$2.50 \times 10^6\text{Jkg}^{-1}$	Latent heat of condensation
$e_F$	6.11hPa	Saturation vapor pressure at freezing
$T_F$	273.15K	Freezing point
$p_R$	$10^3\text{hPa}$	Reference pressure
$T_R = T_F$	273.15K	Reference temperature

Table 2.1: Thermodynamic constants.

The buoyancy force per unit mass on a parcel with a saturated entropy differing from that of the environment by  $\delta s_s$  is therefore

$$F = -\frac{g\delta\rho}{\rho} \approx g \left( C_p + \frac{L^2 r_s}{R_v T_R^2} \right)^{-1} \delta s_s = -g \left( C_p + \frac{L^2 r_s}{R_v T_R^2} \right)^{-1} \frac{ds_s}{dz} z \quad (2.28)$$

and the buoyancy frequency in the statically stable case is

$$N \approx \left[ g \left( C_p + \frac{L^2 r_s}{R_v T_R^2} \right)^{-1} \frac{ds_s}{dz} \right]^{1/2}. \quad (2.29)$$

Table 2.1 gives values of thermodynamic constants.

## 2.3 Alternate vertical coordinates

So far we have taken the geometrical height  $z$  as the vertical coordinate, writing the hydrostatic equation as

$$\frac{dp}{dz} = -g\rho. \quad (2.30)$$

The density  $\rho$  in this coordinate system is just the mass per unit geometrical volume and the vertical velocity is

$$w = \frac{dz}{dt}. \quad (2.31)$$

There are alternative possibilities that are useful in different situations, as we now see.

### 2.3.1 Pressure coordinates

Sometimes it is useful to use the pressure as a vertical coordinate. Using  $dp/dz = (dz/dp)^{-1}$ , the hydrostatic equation can be rewritten

$$\frac{d\Phi}{dp} = -\frac{1}{\rho} \quad (2.32)$$

where the *geopotential* is defined

$$\Phi = gz. \quad (2.33)$$

It is important to realize that the pressure now becomes an independent variable and the height and geopotential dependent variables. Thus, in this coordinate system we consider  $\Phi$  to be a function of  $p$  (as well as horizontal coordinates and time) rather than vice versa.

The density in pressure coordinates is defined as the mass per unit area per unit vertical increment of the pressure coordinate. It is related to the ordinary density by  $\rho dz = \rho_p dp$  where  $dz$  and  $dp$  refer to thicknesses of the same layer in geometrical and pressure coordinates. Combining this with the hydrostatic equation results in

$$\rho_p = \rho \frac{dz}{dp} = -\frac{1}{g}. \quad (2.34)$$

The density is negative because pressure decreases upward. The principal advantage of pressure coordinates is that this density is constant, making the representation of the flow of a geophysical fluid in pressure coordinates incompressible.

The vertical velocity in pressure coordinates is

$$\omega = \frac{dp}{dt} \quad (2.35)$$

in analogy with the vertical velocity in geometrical coordinates. Note that positive  $\omega$  is downward. Unlike the geometrical vertical velocity, which is zero at horizontal surfaces, the pressure vertical velocity can be non-zero there, since the surface pressure following a surface parcel may change with time. However, this is usually not a large effect and the surface value of  $\omega$  is often approximated to be zero in simplified models.

### 2.3.2 Isentropic coordinates

In the atmosphere a particularly useful vertical coordinate is the potential temperature  $\theta$ , which increases monotonically upward in a stably stratified atmosphere. The vertical velocity in this coordinate is

$$S_\theta = \frac{d\theta}{dt}. \quad (2.36)$$

The main advantage of isentropic coordinates is that this vertical velocity is zero except where heating or cooling occurs.

We derive the mass density  $\eta$  in isentropic coordinates from the condition  $\rho dz = \eta d\theta$ :

$$\eta = \rho \frac{dz}{d\theta} = \frac{1}{g} \frac{d\Phi}{d\theta}. \quad (2.37)$$



The hydrostatic equation is easily shown to become

$$\frac{dp}{d\theta} = -g\eta. \quad (2.38)$$

The ordinary density may be written in terms of the pressure and potential temperature using equations (2.11) and (2.17) as

$$\rho = \frac{p^{1-\kappa} p_R^\kappa}{R\theta}. \quad (2.39)$$

This is useful because we can rewrite the pressure force term  $\rho^{-1}\nabla p$  as

$$\frac{1}{\rho}\nabla p = \frac{R\theta\nabla p}{p^{1-\kappa} p_R^\kappa} = \theta\nabla[C_p(p/p_R)^\kappa] \equiv \theta\nabla\Pi \quad (2.40)$$

where  $\Pi = C_p(p/p_R)^\kappa$  is called the Exner function, thus eliminating any reference to the ordinary density. Using the definition of potential temperature (2.17), we can also write the Exner function as  $\Pi = C_p\theta/T$ . An alternate way to write the hydrostatic equation using the Exner function is

$$\theta \frac{\partial \Pi}{\partial z} = -g. \quad (2.41)$$

The main disadvantages of isentropic coordinates are that they become singular when the static stability goes to zero and that the lower boundary condition is difficult to implement computationally. However, the equations of atmospheric dynamics are perhaps most simply expressed in isentropic coordinates.

### 2.3.3 Isopycnal coordinates

In the ocean isopycnal coordinates play the same role as isentropic coordinates in the atmosphere, and they have the same advantages and disadvantages. Since the density is (nearly) conserved in the absence of heating and evaporation, and since it decreases monotonically upwards in stably stratified regions, it provides a useful coordinate in the ocean in analogy to the potential temperature in the atmosphere.

The isopycnal coordinate density is

$$\eta = \rho \frac{dz}{d\rho} = \frac{dz}{d \ln \rho}, \quad (2.42)$$

the vertical velocity is

$$S_\rho = \frac{d\rho}{dt}, \quad (2.43)$$

and the hydrostatic equation is

$$\frac{dp}{d\rho} = -g\eta. \quad (2.44)$$

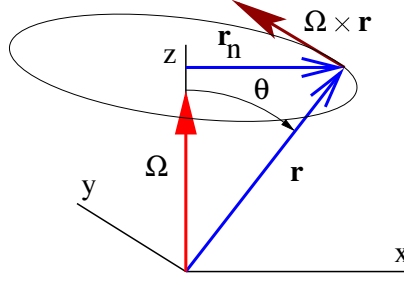


Figure 2.4: Sketch for understanding a rotating reference frame.

## 2.4 Coriolis force

The inertial force is somewhat more tricky to derive. Figure 2.4 shows a Cartesian coordinate system with the rotation vector of the earth  $\Omega$  aligned with the  $z$  axis. A point  $\mathbf{r}$  fixed to the earth moves with velocity  $\Omega \times \mathbf{r}$ . That the direction of this vector is correct can be verified by examination of figure 2.4 –  $\Omega \times \mathbf{r}$  is tangent to a circle centered on the  $z$  axis. The magnitude of this velocity is  $|\Omega \times \mathbf{r}| = \Omega r \sin \theta = \Omega r_n$  as expected, where  $r_n$  is the radius of this circle. Thus, the velocity in the inertial frame  $\mathbf{v}_I$  of something moving relative to the earth's reference frame is

$$\mathbf{v}_I = \mathbf{v}_R + \Omega \times \mathbf{r}, \quad (2.45)$$

where  $\mathbf{v}_R$  is the velocity relative to the earth. A similar relationship can be verified for the time derivative of any vector  $\mathbf{B}$ :

$$\left( \frac{d\mathbf{B}}{dt} \right)_I = \left( \frac{d\mathbf{B}}{dt} \right)_R + \Omega \times \mathbf{B}. \quad (2.46)$$

To verify this, simply replace  $\mathbf{r}$  by  $\mathbf{B}$  in figure 2.4.

Let us now consider the acceleration of an object at position  $\mathbf{r}$ . From equation 2.46 the acceleration in the inertial reference frame may be written

$$\mathbf{a}_I = \left( \frac{d\mathbf{v}_I}{dt} \right)_I = \left( \frac{d\mathbf{v}_I}{dt} \right)_R + \Omega \times \mathbf{v}_I. \quad (2.47)$$

Substitution of equation 2.45 yields after rearrangement

$$\mathbf{a}_I = \mathbf{a}_R + 2\Omega \times \mathbf{v}_R + \Omega \times (\Omega \times \mathbf{r}), \quad (2.48)$$

where  $\mathbf{a}_R = (d\mathbf{v}_R/dt)_R$  is the acceleration of the object relative to the rotating earth. The quantity  $2\Omega \times \mathbf{v}_R + \Omega \times (\Omega \times \mathbf{r})$  is therefore the acceleration of the reference frame of the object. The term  $\Omega \times (\Omega \times \mathbf{r})$  may also be written (using a standard vector identity) as  $-\Omega^2 \mathbf{r}_n$ , where  $\mathbf{r}_n$  is defined in figure 2.4. This is nothing more than the standard centripetal acceleration of an object rotating with the earth. The other term accounts for the fact that the object may be moving relative to the rotating earth. The inertial force per unit mass is simply minus the additional terms beyond  $\mathbf{a}_R$  on the right side of equation (2.48):

$$\mathbf{F}_i = -(2\Omega \times \mathbf{v}_R - \Omega^2 \mathbf{r}_n). \quad (2.49)$$

We drop the subscripted  $R$  from the velocity for brevity of notation since this is generally the only velocity we deal with in geophysical fluid dynamics.

The gravitational force and the second term in equation (2.49) are both derivable from a scalar potential. It is customary to combine these two terms into an “effective gravity” with a potential  $\Phi$  called the *geopotential*. The net force per unit mass from these two terms is therefore

$$\mathbf{F}_e = -\nabla\Phi = (\mathbf{g} + \Omega^2\mathbf{r}_n) = \mathbf{g}_e. \quad (2.50)$$

The centrifugal force term is small compared to the gravitational force, so for most work it is generally ignored and  $\mathbf{g}_e$  is simply replaced by  $\mathbf{g}$ .

## 2.5 References

- Emanuel**, K. A., 1994: *Atmospheric convection*. Oxford University Press, New York, 580 pp.
- Gill**, A. E., 1982: *Atmosphere-ocean dynamics*. Academic Press, New York, 662 pp.
- Iribarne**, J. V., and W. L. Godson, 1981: *Atmospheric thermodynamics, 2nd Ed.* D. Reidel, Dordrecht, 259 pp.

## 2.6 Laboratory

1. Make plots of the density  $\rho$  of water as a function of temperature over the range  $0 - 30^\circ \text{C}$  at atmospheric pressure for salinities  $S = 0, 30, 40$  psu. Use a tool such as Octave or Python to do this. You may also wish to plot  $\rho - \rho_0$  instead of  $\rho$ , where  $\rho_0 = 1000 \text{ kg m}^{-3}$ , since the range in variation of the density of sea water is so small.
2. Get a time-depth series of sub-surface temperature and salinity from the TAO Mooring site for a particular buoy and compute the time-depth pattern of density and display it.

## 2.7 Problems

1. If a parcel of ocean water is cooled by evaporation, the salinity is increased because the evaporated water leaves behind its salt content. Compute the ratio of final to initial salinity of a parcel of ocean water cooled by evaporation from  $30^\circ\text{C}$  to  $0^\circ\text{C}$ .
2. Roughly how deep (in meters) does one have to go to find ocean water that has density 1% higher than that of the same parcel at the surface? Ignore the possible change in temperature of the parcel due to compression.
3. Derive  $dT/dp$  for a non-condensing parcel under the condition that it is expanding or contracting dry adiabatically. Hint: Hold the dry entropy constant.

4. Derive  $dT/dp$  for a saturated parcel under the condition that it is expanding or contracting moist adiabatically. Hint: Hold the saturated moist entropy constant. (There is no need to evaluate  $de_s(T)/dT$ .)
5. One could construct an extended vertical momentum equation for a moving fluid parcel in an environment at rest on the equator which includes the vertical component of the Coriolis force on the parcel:

$$\frac{d^2 z}{dt^2} = -\frac{1}{\rho} \frac{dp}{dz} - g + 2\Omega v$$

where  $\Omega$  is the earth's rotation rate and  $v$  is the (constant) eastward parcel velocity. Further assume an incompressible fluid environment in which we approximate the vertical pressure gradient by its hydrostatic environmental value

$$\frac{dp}{dz} = -g\rho_{env} = -g(\rho_0 - \Gamma z),$$

where  $\Gamma$  and  $\rho_0$  are constants. Finally, assume that the parcel density  $\rho = \rho_0$ , which means that it is in buoyancy equilibrium with the environment at level  $z = 0$ .

- (a) Obtain the general solution for  $z(t)$ .
- (b) Assuming a value of  $\Gamma/\rho_0$  which produces a Brunt-Väisälä frequency of  $10^{-2}\text{s}^{-1}$ , find the parcel equilibrium levels for  $v = 0, \pm 10\text{ms}^{-1}$ .