Chapter 6

Two-dimensional Homogeneous Flow

6.1 Vorticity in two dimensions

The first step in understanding geophysical fluid dynamics and how it is used comes from examining the evolution of the flow of a two-dimensional, incompressible fluid. In this case we assume that the flow is confined to the $x-y$ plane, with no dependence on z. The incompressibility condition in this case becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{6.1}$$

The vorticity only has a z component in this case,

$$\zeta_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}. \tag{6.2}$$

In the two-dimensional, incompressible case, we can define a *streamfunction* ψ such that

$$v_x = -\frac{\partial \psi}{\partial y} \tag{6.3}$$

and

$$v_y = \frac{\partial \psi}{\partial x}. \tag{6.4}$$

This choice trivially satisfies equation (6.1). Substitution into equation (6.2) results in a Poisson equation for the streamfunction:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \zeta_z. \tag{6.5}$$

In mathematical physics terms, this is a relatively easy equation to solve, and it shows that the streamfunction, and hence the velocity field, is readily obtained from the vorticity field and the boundary conditions applicable to this equation.

Figure 6.1 illustrates the relationship between contours of constant streamfunction and the velocity field. The velocity vectors are everywhere tangent to the contours of streamfunction and the magnitude of the velocity is inversely proportional to the contour spacing. In

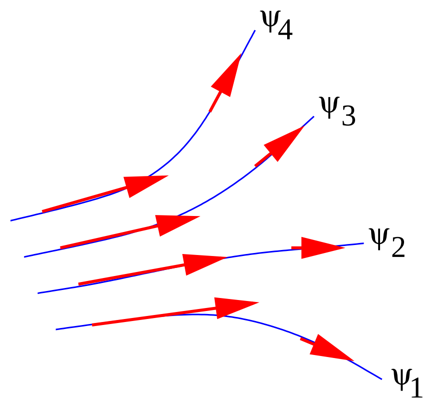

Figure 6.1: Illustration of the relationship between contours of constant streamfunction and the velocity field.

a steady flow, parcels traverse the domain following lines of constant streamfunction. Hence these contours are also called *streamlines*. Note however, that if the flow is non-steady, parcel trajectories no longer coincide with streamlines.

The flow adjacent to a stationary wall bounding the fluid is parallel to the wall. The streamfunction along the wall is therefore constant. Specifying the value of the streamfunction on the walls bounding a fluid as well as the vorticity distribution in the interior is sufficient to guarantee a unique solution to equation (6.5).

The governing equation for vorticity may be obtained from the circulation theorem in this case:

$$\frac{d\Gamma}{dt} = 0. \tag{6.6}$$

In the two-dimensional case the area of a circulation loop lying in the $x - y$ plane does not change in area as it advects with the fluid in this case. (Think of a vertical cylinder with end plate area equal to the area of the circulation loop. Since the volume of the cylinder is fixed by the incompressibility condition, and since the height of the cylinder does not change due to the two-dimensional nature of the flow, the end plate area must remain fixed.) For such a loop, $\Gamma = \zeta_z A$ where A is the area of the loop, and the vorticity of parcels is conserved, i. e.,

$$\frac{d\zeta_z}{dt} = 0. \tag{6.7}$$

Solution to the two-dimensional, incompressible, homogeneous flow problem can now be visualized. Suppose at the initial time the vorticity field is specified. Equation (6.5) is solved to obtain the streamfunction, and hence the velocity field. Equation (6.7) is then used to advect parcels and their associated vorticity to new locations. The process is then repeated.

This is the simplest example of an *advection-inversion* process. The inversion part is the solution of the streamfunction equation given the vorticity. The vorticity is the key dependent variable in this problem, and it obeys a particularly simple evolution equation – it just moves around with the fluid!

6.2 Problems

1. Imagine a point vortex where the vorticity field is given by $\zeta_z = C\delta(x)\delta(y)$ where C is a constant equal to the strength of the vortex and $\delta()$ is the Dirac delta function. (The Dirac delta function has an integral of one, but is only non-zero where the argument is zero.) Solve for the streamfunction on an infinite domain on which $\psi \to 0$ at infinity. Hint: Use cylindrical symmetry and the Kelvin theorem applied to a circular loop centered on the vortex to obtain the velocity field. From this the streamfunction can be obtained by integration. If you have experience with electromagnetism, think of the problem of the magnetic field surrounding an infinite wire carrying a current.

2. Consider a two-dimensional, incompressible fluid which is irrotational except for two point vortices of equal but opposite strength $\pm C$ separated by a distance d. Describe the speed and direction of motion of the two vortices.

3. Repeat the above problem for the case in which the two vortices have strength of the same sign and magnitude.