Chapter 5

Sound Waves and Vortices

In this chapter we explore a set of characteristic solutions to the fluid equations with the goal of familiarizing the reader with typical behaviors in fluid dynamics. Sound waves form the basis of compressible fluid flow, while vortex dynamics is characteristic of incompressible flow.

5.1 Sound waves

Sound waves are the small-amplitude solutions to the equations for a homogeneous, compressible fluid which is initially at rest and subject to no external forces. Sound waves exist in more complicated circumstances, but they appear in their purest form in this context.

"Small amplitude" means that deviations in dependent variables from the rest state are sufficiently small that approximate equations obtained by retaining only terms linear in these deviations are valid. This process of *linearization* is used frequently in geophysical fluid dynamics.

We analyze in particular sound waves in an ideal gas. For this situation we assume that the (dry) entropy is constant, a trivial solution to the entropy governing equation in the absence of moisture and radiation effects:

$$\frac{ds_d}{dt} = 0. \tag{5.1}$$

Since the density and pressure, but not the temperature, appear in the momentum and continuity equations, it is useful to rewrite the entropy in terms of these variables, eliminating the temperature using the ideal gas law:

$$s_d = C_{vd} \ln(p/p_R) - C_{pd} \ln(\rho/\rho_R), \tag{5.2}$$

where $C_{vd} = C_{pd} - R_d$ is the specific heat of dry air at constant volume, and where the reference density ρ_R is consistent with the reference pressure and temperature in the ideal gas law.

Let us now assume a rest state for the atmosphere in which $p = p_R$ and $\rho = \rho_R$. From equation (5.2) we see that the entropy must take on the value $s_d = 0$ under these conditions.

If we now let $p = p_R + p'$ and $\rho = \rho_R + \rho'$, where $|p'| \ll p_R$ and $|\rho'| \ll \rho_R$, then equation (5.2) becomes, upon solving for p',

$$\frac{p'}{p_R} = \gamma \frac{\rho'}{\rho_R}, \tag{5.3}$$

where $\gamma = C_{pd}/C_{vd} = 1.4$, and where we have used $\ln(1 + p'/p_R) \approx p'/p_R$ for $|p'|/p_R \ll 1$, etc.

We now linearize the mass continuity and momentum equations. Since the initial state is one of rest, the velocity vector itself is a small quantity, which means that terms like $\mathbf{vv}$, $\mathbf{v} \cdot \nabla \rho'$, and $\rho' \nabla \cdot \mathbf{v}$ must be ignored when we linearize these equations. Further realizing that space and time derivatives of ρ_R and p_R vanish due to their constancy, we find that the mass continuity and momentum equations reduce to

$$\frac{\partial \rho'}{\partial t} + \rho_R \nabla \cdot \mathbf{v} = 0 \tag{5.4}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\nabla p'}{\rho_R} = 0. \tag{5.5}$$

Equations (5.4), (5.4), and (5.5) together govern the behavior of sound waves. Let us substitute a trial plane wave solution of the form $\rho' = \rho_0 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, $p' = p_0 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, $\mathbf{v} = \mathbf{v}_0 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, where $\mathbf{k}$ is the wave vector of the wave, ω is its frequency, and ρ_0, p_0, and $\mathbf{v}_0$ are constants. After eliminating p' in favor of ρ' in equation (5.5), we get

$$-i\omega \rho_0 + i\rho_R \mathbf{k} \cdot \mathbf{v}_0 = 0 \tag{5.6}$$

and

$$-i\omega \mathbf{v}_0 + i\gamma \mathbf{k} p_R \rho_0 / \rho_R^2 = 0. \tag{5.7}$$

Equation (5.7) implies that $\mathbf{v}_0$ must be parallel to $\mathbf{k}$ as long as $p_0 \neq 0$, which means that sound waves are longitudinal waves. Thus, we can write $\mathbf{k} \cdot \mathbf{v_0}$ in terms of the vector magnitudes as kv_0.

Equations (5.6) and (5.7) constitute a set of linear, homogeneous equations in ρ_0 and v_0, and can be rewritten in matrix form as

$$\begin{pmatrix} -\omega & k\rho_R \\ \gamma k p_R / \rho_R^2 & -\omega \end{pmatrix} \begin{pmatrix} \rho_0 \\ v_0 \end{pmatrix} = 0. \tag{5.8}$$

This has a non-trivial solution only when the determinant of the coefficients of the matrix is zero. This condition results in the *dispersion relation* for sound waves:

$$\omega^2 = (\gamma p_R / \rho_R) k^2 = (\gamma R_d T_R) k^2. \tag{5.9}$$

From the dispersion relation we see that the phase speed of sound waves in dry air is $c = \omega/k = (\gamma R_d T_R)^{1/2}$. At $T_R = 300$ K, this yields $c = 347$ m s^{-1}. Interestingly, the speed of sound in an ideal gas depends only on the temperature of the gas. Equation (5.6) tells us that the density perturbation is related to the velocity perturbation by $\rho_0 = \rho_R v_0 / c$, while equation (5.3) shows that $p_0 = c^2 \rho_0$.

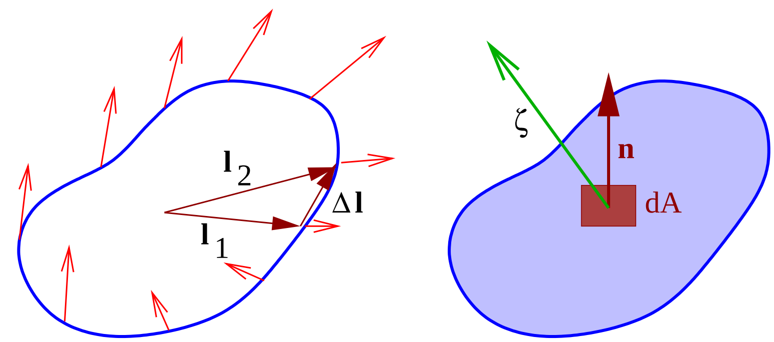

Figure 5.1: Sketch of a circulation loop which advects with the fluid flow, symbolized by the arrows on the left. Stokes' theorem relates the circulation to an area integral of the vorticity, as shown on the right.

The speed of sound forms a dividing line in fluid dynamics. Even for flows in a compressible fluid like air, the fluid acts much like an incompressible fluid if the characteristic velocities are much less than the speed of sound. Only for fluid velocities comparable to or greater than c does essentially incompressible behavior enter. The ratio of the characteristic velocity of a flow to the speed of sound is called the *Mach number*:

$$\mathrm{M} = \frac{V_{typical}}{c}. \tag{5.10}$$

Most flows in geophysical contexts have $\mathrm{M} \ll 1$, and thus behave in essentially incompressible fashion.

The analysis in this section is presented in detail to provide a clear example of how linearized perturbation solutions are obtained. One point worth emphasizing is that the initial or base state itself must be a solution to the full equations of motion for the perturbation analysis to be valid. The reader is left to verify that the base state in this case, $\mathbf{v} = 0$, $\rho = \rho_R$, $p = p_R$, does indeed satisfy the full governing equations.

5.2 Vorticity and the Kelvin theorem

The *vorticity* of a fluid flow is defined as the curl of the velocity field:

$$\zeta \equiv \nabla \times \mathbf{v}. \tag{5.11}$$

The vorticity plays a key role in the dynamics of an incompressible fluid, as well as in the low Mach number flow of a compressible fluid.

The vorticity is related to a quantity called the *circulation*, defined as the closed line integral of the fluid velocity component parallel to the path:

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \int \nabla \times \mathbf{v} \cdot \mathbf{n} dA = \int \zeta \cdot \mathbf{n} dA. \tag{5.12}$$

The second form of the circulation involving the vorticity is obtained using Stokes' theorem. Hence the area integral is over the region bounded by the circulation path. Figure 5.1 illustrates the circulation loop.

Of particular interest is the circulation loop which moves and deforms with the fluid flow. The area, shape, and orientation of this loop evolve with time. However, the time rate of change of the circulation around such a loop obeys a surprisingly simple law, as we now show.

We wish to take the time derivative of Γ. However, the fact that the circulation loop evolves with time complicates this calculation. It is simplest to write the circulation integral in finite sum form while taking the derivative in which $d\mathbf{l} \to \Delta\mathbf{l}_i = \mathbf{l}_{i+1} - \mathbf{l}_i$ as illustrated in the left panel of figure 5.1:

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \sum \mathbf{v}_i \cdot (\mathbf{l}_{i+1} - \mathbf{l}_i) = \sum \frac{d\mathbf{v}_i}{dt} \cdot \Delta\mathbf{l}_i + \sum \mathbf{v}_i \cdot \Delta\mathbf{v}_i, \tag{5.13}$$

where we use $\mathbf{v}_i = d\mathbf{l}_i/dt$. We then revert to integral forms and note further that $\mathbf{v} \cdot d\mathbf{v} = d(v^2/2)$, which results in

$$\frac{d\Gamma}{dt} = \oint \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} + \oint d(v^2/2). \tag{5.14}$$

The second term is the integral of a perfect differential over a closed path and is therefore zero.

The total time derivative of velocity can be eliminated using the momentum equation. We work for now in an inertial reference frame and introduce a rotating frame at a later stage. In this case equation (5.14) becomes

$$\frac{d\Gamma}{dt} = \oint \left(-\frac{\nabla p}{\rho} - \nabla\Phi \right) \cdot d\mathbf{l}. \tag{5.15}$$

The second term can be written $-\nabla\Phi \cdot d\mathbf{l} = -d\Phi$, and is therefore also a perfect differential which integrates to zero. However, the first term $-\rho^{-1}\nabla p \cdot d\mathbf{l} = -\rho^{-1} dp$ in general is not, and therefore must be retained. Thus we arrive at the *Kelvin circulation theorem*[1] (see Pedlosky, 1979):

$$\frac{d\Gamma}{dt} = -\oint \frac{dp}{\rho}. \tag{5.16}$$

In geophysical fluid dynamics we always use the circulation as computed in an inertial reference frame. However, we often have to compute the circulation directly given the fluid velocity in the rotating frame of the earth. Recall that the velocity in the inertial frame $\mathbf{v}_I$ can be related to the velocity in the rotating frame $\mathbf{v}$ by

$$\mathbf{v}_I = \mathbf{v} + \Omega \times \mathbf{r}, \tag{5.17}$$

where Ω is the rotation vector of the earth and $\mathbf{r}$ is the position vector relative to the center of the earth. The circulation thus becomes

$$\Gamma = \oint \mathbf{v}_I \cdot d\mathbf{l} = \oint \mathbf{v} \cdot d\mathbf{l} + \int [\nabla \times (\Omega \times \mathbf{r})] \cdot \mathbf{n} dA, \tag{5.18}$$

where we have used Stokes' theorem to convert the second line integral into an area integral bounded by the circulation loop. A well-known vector identity can be used to reduce the

[1] Technically, Kelvin's theorem refers to the special case of a homogeneous, incompressible fluid, in which case (as we shall show) the right side of equation (5.16) is zero.

last term: $\nabla \times (\Omega \times \mathbf{r}) = \Omega(\nabla \cdot \mathbf{r}) - \Omega \cdot \nabla \mathbf{r} = 3\Omega - \Omega = 2\Omega$. Substituting this into equation (5.18) results in

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} + \int 2\Omega \cdot \mathbf{n} dA = \int (\nabla \times \mathbf{v} + 2\Omega) \cdot \mathbf{n} dA. \tag{5.19}$$

In the context of a rotating reference frame, the curl of the fluid velocity in the inertial frame is called the *absolute vorticity*,

$$\zeta_a = \nabla \times \mathbf{v}_I = \nabla \times \mathbf{v} + 2\Omega, \tag{5.20}$$

and $\zeta = \nabla \times \mathbf{v}$ is called the *relative vorticity*.

5.2.1 Homogeneous, incompressible fluid

For a homogeneous, incompressible fluid, we have constant density ρ. In this case the pressure term in equation (5.16) is also a perfect differential, so the Kelvin circulation theorem becomes

$$\frac{d\Gamma}{dt} = 0. \tag{5.21}$$

For a very small loop over which the absolute vorticity is essentially constant, $\Gamma = \zeta_a \cdot \mathbf{n} A$, where A is the area of the loop and $\mathbf{n}$ is the unit normal to the loop. Thus, if the loop associated with a parcel of fluid expands in area, the component of the absolute vorticity normal to the loop must decrease correspondingly.

In the special case in which the absolute vorticity in a homogeneous, incompressible fluid is everywhere zero, the circulation theorem shows that it will always remain zero. This is a profound result which we will exploit later.

5.2.2 Inhomogeneous, incompressible fluid

If an incompressible fluid is inhomogeneous, i. e., if the density varies with position, then equation (5.21) doesn't generally apply. However, in the limited case in which the circulation loop is confined to a surface of constant density, this equation is valid, since ρ is constant over the circulation loop. Furthermore, by virtue of the incompressibility condition, a circulation loop which advects with the fluid (as it does in the case of the Kelvin theorem) will always remain embedded in a surface of constant density if it starts out that way. Thus, equation (5.21) remains valid for all time in this case.

5.2.3 Diabatic ideal gas

In the case of an ideal gas in which no latent heat release, radiative heating or cooling, or other heat source exists, changes in the specific enthalpy h can be related solely to changes in the dry entropy and pressure:

$$dh = T ds_d + dp/\rho. \tag{5.22}$$

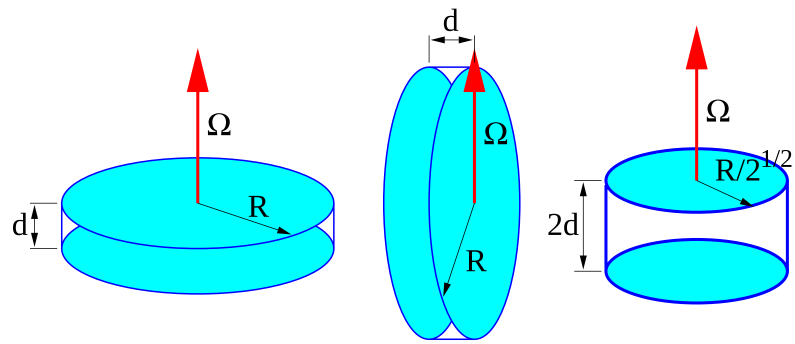

Figure 5.2: Sketch of a homogeneous slab of ocean water. The rotation vector Ω of the earth is shown in each of three cases. In the left panel the slab is at the north pole. In the center panel it has moved to the equator without change of shape. In the right panel it remains at the north pole, but deforms so as to reduce its radius and increase its thickness.

In this case the circulation theorem can be written

$$\frac{d\Gamma}{dt} = \oint T ds_d. \tag{5.23}$$

The enthalpy term drops out as it appears in the form of a perfect differential. In this case equation (5.21) applies to circulation loops embedded in surfaces of constant dry entropy. The condition of no heating means that parcels conserve dry entropy, so analogously to the case of an inhomogeneous, incompressible fluid, equation (5.21) will apply for all time in this special case.

An alternate way to represent the right side of the Kelvin theorem equation in the case of an ideal gas is to rewrite dp/ρ in terms of the potential temperature and the Exner function:

$$\frac{d\Gamma}{dt} = -\oint \theta d\Pi = \oint \Pi d\theta. \tag{5.24}$$

The last step results from $d(\theta\Pi)$ being a perfect differential. This result is equivalent to that of equation (5.23) since $ds_d = C_{pd} d\theta/\theta$ and $\Pi = C_{pd}T/\theta$. The latter equation comes from combining the definition of potential temperature with the definition of Exner function.

5.3 Ocean example

As an example of the use of the Kelvin theorem, let us consider a circular slab of ocean water, assumed to be homogeneous in temperature and salinity. Suppose the slab starts out at the north pole, as illustrated in the left panel of figure 5.2. Let us further suppose that the slab is at rest in this location. The circulation around the periphery of the slab is totally due to planetary rotation in this case, and has the value $\Gamma = 2\pi R^2\Omega$.

If the slab moves to the equator without change of shape, the orientation is such that the planetary contribution to the circulation is zero. However, since circulation is conserved in this case, the slab must be rotating with tangential velocity at the rim V determined by $\Gamma = 2\pi RV$. Solving for V results in $V = R\Omega$.

If the slab remains at the north pole, but is deformed such that the thickness is doubled, the radius necessarily decreases by a factor $2^{1/2}$ so that the volume of the slab remains the same. This deformation decreases the planetary part of the circulation to $2\pi(R/2^{1/2})^2\Omega = \pi R^2\Omega$. The balance has to be taken up by rotation relative to the earth with tangential velocity at the rim determined by conservation of circulation: $2\pi R^2\Omega = \pi R^2\Omega + 2\pi(R/2^{1/2})V$ implies $V = R\Omega/2^{1/2}$.

Let us attach some numbers to these estimates. The rotation rate of the earth is 2π divided by the period of rotation, which is about 4 min less than 24 h – dynamically, the *sidereal* day rather than the *solar* day should be used. This results in $\Omega = 7.29 \times 10^{-5}$ s^{-1}. If $R = 100$ km, then we find $V = 7.29$ m s^{-1} in the case in which the slab moves undeformed to the equator.

5.4 Irrotational, incompressible flow

Suppose we have an incompressible, homogeneous (i. e., constant density) fluid which has zero vorticity everywhere. Thus, we have the conditions $\nabla \times \mathbf{v} = 0$ and $\nabla \cdot \mathbf{v} = 0$ everywhere in the fluid for all time, according to arguments based on the Kelvin circulation theorem. Solutions for flows of this type are particularly easy to obtain.

Zero vorticity implies that we can represent the velocity as the gradient of some potential, χ, which we call the *velocity potential*:

$$\mathbf{v} = -\nabla\chi. \tag{5.25}$$

However, zero divergence further implies that the velocity potential obeys a very simple equation

$$\nabla^2\chi = 0. \tag{5.26}$$

Let us imagine that the outward normal component of the velocity $v_n = \mathbf{v} \cdot \mathbf{n} = -\nabla\chi \cdot \mathbf{n}$ is specified on the boundary of some region. The quantity $\mathbf{n}$ is the unit outward normal on this boundary. Assuming that there are no sources or sinks of fluid inside the region, then the net flow out of the region must be zero:

$$\oint \nabla\chi \cdot \mathbf{n} dA = 0. \tag{5.27}$$

Given this condition, equation (5.26) uniquely determines χ up to a trivial additive constant.

Physically, this implies that the fluid flow in this case is passive – there are no interesting internal dynamics, as the entire flow is determined by what is imposed at the boundary. The pressure distribution as well as the velocity distribution is fixed as well. To show this, we note that the term $\mathbf{v} \cdot \nabla\mathbf{v} = \nabla(v^2/2) - \mathbf{v} \times \zeta$. (This follows from expanding $\mathbf{v} \times (\nabla \times \mathbf{v})$ according to the rules of vector analysis.) Since $\zeta = 0$ by hypothesis in this case, and since the density is constant, the momentum equation becomes

$$\nabla(-\partial\chi/\partial t + v^2/2 + p/\rho + \Phi) = 0. \tag{5.28}$$

It follows that

$$-\partial\chi/\partial t + v^2/2 + p/\rho + \Phi = \text{constant}. \tag{5.29}$$

In the special case in which the flow is time-independent, this reduces to Bernoulli's equation:

$$v^2/2 + p/\rho + \Phi = \text{constant}. \tag{5.30}$$

5.5 References

Pedlosky, J., 1979: *Geophysical Fluid Dynamics.* Springer-Verlag, New York, 624 pp.

5.6 Problems

1. Consider perturbations on a rotating, homogeneous (i. e., constant entropy), compressible fluid at rest in its rotating environment, with constant geopotential. Assume a rotation vector of magnitude Ω pointing in the z direction.

   (a) Linearize the governing equations using the treatment of sound waves as guide and assume all dependent variables to be proportional to $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Write the governing equations in matrix form as in equation (5.8). The dispersion relation is obtained by taking the determinant of the matrix of coefficients.

   (b) Examine the special case in which $\mathbf{k}$ points in the z direction. Solving the dispersion relation for ω^2 shows that two solutions exist, since this equation is quadratic in ω^2. For each of these solutions determine the relationship between the various components of the velocity, and between the velocity and the density perturbation.

   (c) Repeat for the special case in which $\mathbf{k}$ points in the x direction.

2. Suppose we add an additional force per unit mass $\mathbf{F}$ to the momentum equation, such as might be caused by friction. The momentum equation thus takes the form (in a non-rotating reference frame)

$$\frac{d\mathbf{v}}{dt} + \frac{\nabla p}{\rho} + \nabla\Phi = \mathbf{F}. \tag{5.31}$$

   Derive the Kelvin circulation theorem in this extended case. If $\mathbf{F}$ is conservative, does it enter?

3. Imagine that convection forms in an initially stationary atmosphere at latitude 45° S. The convection draws air in horizontally at low levels, lifts it, and ejects it horizontally at high levels, as shown in figure 5.3.

   (a) Assuming cylindrical symmetry, find the tangential velocity of the inflowing air at radius $a/2$, assuming that it started at radius $a = 500$ km.

   (b) Find the tangential velocity of outflow air at radius a, assuming that it started at radius $a/2$.

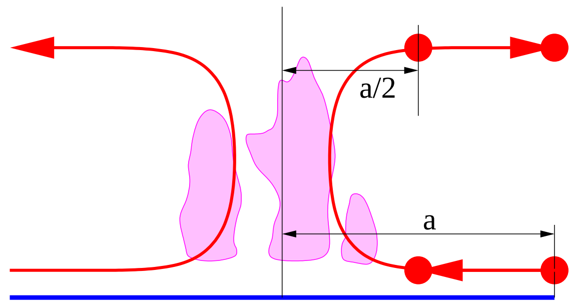

Figure 5.3: Sketch of airflow through a convective system.

4. One can think of a hurricane as an axially symmetric vortex with inflow at low levels and outflow at high levels. The earth-relative velocities in a hurricane are so large that the planetary contribution to the circulation can be ignored to a reasonable approximation inside the hurricane.

   (a) If the tangential velocity in the hurricane inflow at the outer radius b of a hurricane is V_b, what does the circulation theorem tell us the tangential velocity should be at radius $a < b$?

   (b) The actual hurricane tangential velocity in the inflow is observed to be equal to $Cr^{-1/2}$ where r is the distance from the center of the hurricane and C is a constant. Determine how the circulation varies with r in this case.

   (c) Assume that the tangential frictional force in a hurricane takes the form $F = -KV^2$ where V is the tangential velocity and K is a constant. Given this and the fact that $d\Gamma/dt = (d\Gamma/dr)(dr/dt) = U(d\Gamma/dr)$ in a steady hurricane, determine how the radial velocity U varies as a function of radius. Hint: Consult the answer to problem 2.

5. Consider the flow with the velocity potential

$$\chi = -Ux + A\sin(kx)\exp(-mz) \tag{5.32}$$

   where U, A, k, and m are constants.

   (a) Determine the condition needed to satisfy $\nabla^2\chi = 0$.

   (b) Compute the velocity vector as a function of x and z and make a sketch of the velocity field, showing velocity vectors on an $x - z$ grid.

   (c) On the above plot sketch in the trajectories of a representative set of air parcels as they move through this flow pattern.