Chapter 8

Atmospheric Models

An accurate model of the atmosphere requires the representation of continuous vertical profiles, leading to a fully three-dimensional treatment. However, many aspects of atmospheric flow can be represented qualitatively by a small number of layers. In this chapter we consider single and two layer models of the atmosphere.

8.1 Atmospheric thermodynamics

In order to define our models, we need to learn a bit about atmospheric thermodynamics. The earth's atmosphere obeys the ideal gas law, which can be written

$$\frac{p}{\rho} = RT, \tag{8.1}$$

where p is the pressure, ρ is the air density, T is the absolute temperature, and $R = 287\ \mathrm{J\ kg^{-1}\ K^{-1}}$ is the universal gas constant divided by the molecular weight of air. Ignoring the effects of moisture, the specific dry entropy (i. e., the entropy per unit mass of air) is

$$s_d = C_p \ln(T/T_R) - R\ln(p/p_R), \tag{8.2}$$

where $T_R = 300$ K and $p_R = 1000$ hPa are constant reference values of temperature and pressure and $C_p = 1005\ \mathrm{J\ kg^{-1}\ K^{-1}}$ is the specific heat of air at constant pressure. The dry entropy is useful because the specific entropy of a parcel does not change under reversible expansions and contractions in which heat is neither added nor removed from the parcel. Meteorologists use a variable related to the entropy called the *potential temperature* θ:

$$\theta = T_R \exp(s_d/C_p) = T(p_R/p)^\kappa, \tag{8.3}$$

where $\kappa = R/C_p$. Since the potential temperature is a function of the entropy alone, it is also conserved in expansions and contractions. The potential temperature of a parcel is the temperature it would acquire upon reversible adiabatic compression or expansion to the reference pressure p_R.

Combining equations (8.1) and (8.3) yields an equation for the atmospheric density as a function of potential temperature:

$$\rho = \frac{p^{1-\kappa}p_R^\kappa}{R\theta}. \tag{8.4}$$

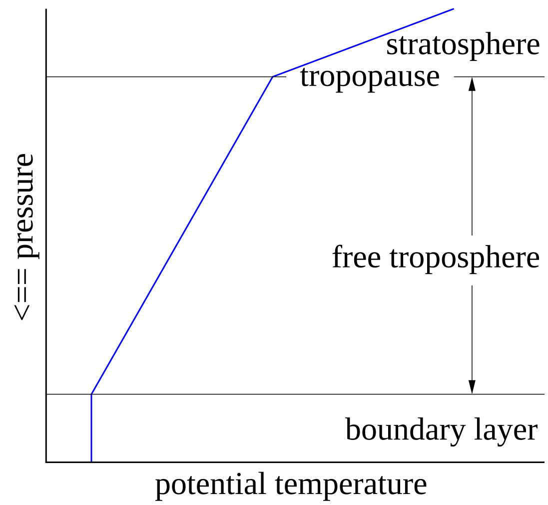

Figure 8.1: Schematic profile of potential temperature as a function of height in the earth's atmosphere.

In order to simplify the momentum equation in the atmospheric case, we wish to eliminate the density ρ in the expression dp/ρ in favor of the potential temperature. Using equation (8.4), we note that

$$\frac{dp}{\rho} = \frac{R\theta dp}{p^{1-\kappa}p_R^{\kappa}} = \frac{R\theta}{\kappa} d\left(\frac{p}{p_R}\right)^{\kappa}. \tag{8.5}$$

Using $R/\kappa = C_p$ and defining the *Exner function* $\Pi = C_p(p/p_R)^{\kappa}$, we write

$$\frac{dp}{\rho} = \theta d\Pi. \tag{8.6}$$

Combining the definition of the Exner function with that of potential temperature in equation (8.3), we can easily show that

$$\Pi = C_p(p/p_R)^{\kappa} = C_p T/\theta. \tag{8.7}$$

We now rewrite the hydrostatic equation $\partial p/\partial z = -g\rho$ in terms of potential temperature and Exner function as

$$\theta\frac{\partial \Pi}{\partial z} = -g \tag{8.8}$$

where g is the acceleration of gravity. This is useful for layer models of the atmosphere in which the potential temperature is constant in each layer, since the layer thickness h is proportional to the change in Exner function across the layer:

$$h = -\theta\Delta\Pi/g. \tag{8.9}$$

Figure 8.1 shows a highly schematic profile of the potential temperature in the earth's atmosphere. Typically, a boundary layer exists next to the earth's surface which has nearly constant potential temperature through its depth. Above the tropopause in the stratosphere the potential temperature increases strongly with height. The free troposphere between the top of the boundary layer and the tropopause exhibits a less strong increase in potential temperature with height than the stratosphere. The boundary layer thickness ranges typically from 500 m over the ocean to $2-3$ km over land, while the tropopause ranges from 8 km above sea level in polar regions to 16 km in the tropics.

8.2 Single layer model of atmosphere

Though of limited applicability, we could in principle define a single layer model of the earth's atmosphere with a uniformly constant potential temperature equal to the average potential temperature θ of the atmosphere. From the hydrostatic equation (8.8) the Exner function as a function of height would be

$$\Pi = \frac{g(h+d-z)}{\theta} \tag{8.10}$$

where $d(x, y)$ is the terrain height. The horizontal Exner function gradient in this case is $\nabla\Pi = g\nabla h/\theta$, and the momentum equation is therefore

$$\frac{d\mathbf{v}}{dt} + g\nabla(h+d) + f\hat{\mathbf{z}} \times \mathbf{v} = \mathbf{F} \tag{8.11}$$

where $\mathbf{F}$ is an externally applied force, typically surface friction. Notice that this equation is identical to the momentum equation for the shallow water flow of an incompressible fluid.

The mass per unit area in a layer of fluid of thickness h is $\overline{\rho}h$ where $\overline{\rho}$ is the vertical average of the density over the layer. The mass continuity equation thus becomes

$$\frac{\partial\overline{\rho}h}{\partial t} + \nabla\cdot(\overline{\rho}h\mathbf{v}) = \overline{\rho}M \tag{8.12}$$

where a mass source term $\overline{\rho}M$ has been added to this equation.

For a nearly incompressible fluid of almost uniform density such as ocean water, the average density $\overline{\rho}$ can be accurately approximated by a constant value, which as we saw earlier can then be extracted from the space and time derivatives. A similar approximation, called the *Boussinesq approximation* is sometimes used for the atmosphere. This has the effect of making the atmospheric governing equations identical to those for the ocean, but is much less justified in the case of the atmosphere than it is in the ocean. In the Boussinesq approximation we equate $\overline{\rho}$ to the mean density of the atmospheric layer in its unperturbed state, ρ_m. This could be obtained by dividing the mass per unit area in the layer $\Delta p_0/g$ by the layer thickness:

$$\rho_m = \frac{\Delta p_0}{gh}. \tag{8.13}$$

The quantity Δp_0 is the constant pressure thickness of the layer in the reference state. Since the density is now taken to be constant, the mass continuity equation can then be approximated by

$$\frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{v}) = M, \tag{8.14}$$

which is identical to the mass continuity equation for an incompressible fluid.

8.3 Boundary layer and surface friction

Recall that the force per unit area of the atmosphere on the ocean is given by the so-called bulk flux formula. By Newton's third law, the force of the ocean on the atmosphere is equal and opposite to the force of the atmosphere on the ocean resulting in a frictional force per unit area on the atmosphere of

$$\mathbf{T} = -\rho C_D |\mathbf{v}|\mathbf{v} \tag{8.15}$$

where ρ is the atmospheric density at the surface, $\mathbf{v}$ is the atmospheric surface wind (actually the wind minus the surface ocean current), and $C_D \approx 1 - 2 \times 10^{-3}$ is the drag coefficient.

The atmosphere generally has a turbulent, neutrally stratified layer next to the surface known as the *boundary layer* in which surface friction is thought to be distributed more or less uniformly. The force per unit mass acting on the air in the boundary layer is thus $\mathbf{F} = \mathbf{T}/(\rho h)$ where h is the thickness of the boundary layer. In the free atmosphere above the boundary layer we often approximate the flow by the geostrophic wind, which is a result of geostrophic balance, i. e., a balance between the pressure gradient force and the Coriolis force. In the boundary layer a better approximation is a three-way balance between the pressure gradient force, the Coriolis force, and surface friction. This balance is called *Ekman balance*, and as with geostrophic balance, it is obtained by ignoring parcel accelerations.

In a single layer model we can write the two components of the momentum equation absent the acceleration terms as

$$g\frac{\partial h}{\partial x} - f v_y + (C_D v/h) v_x = 0 \tag{8.16}$$

$$g\frac{\partial h}{\partial y} + f v_x + (C_D v/h) v_y = 0 \tag{8.17}$$

where $v = (v_x^2 + v_y^2)^{1/2}$.

Let us specialize to the case in which $\partial h/\partial x = 0$, which constitutes no loss of generality since we can orient the coordinate axes any way we like. We divide equations (8.16) and (8.17) by f and recognize $-(g/f)(\partial h/\partial y)$ as the geostrophic wind in the x direction, v_{gx}. Further defining $\epsilon = C_D/(hf)$ as a measure of the strength of friction, (8.16) and (8.17) simplify to

$$\epsilon v v_x - v_y = 0 \tag{8.18}$$

$$v_x + \epsilon v v_y = v_{gx} \tag{8.19}$$

with the resulting solutions

$$v_x = \frac{v_{gx}}{1 + \epsilon^2 v^2} \qquad v_y = \frac{\epsilon v v_{gx}}{1 + \epsilon^2 v^2}. \tag{8.20}$$

These solutions are not completely explicit, because v remains undetermined. However, squaring and adding the equations for v_x and v_y results in a quadratic equation for v^2 which has the solution

$$v^2 = \frac{(1 + 4\epsilon^2 v_{gx}^2)^{1/2} - 1}{2\epsilon^2}. \tag{8.21}$$

A not very accurate approximation to equation (8.20) is to assume that v is constant, presumably taking on a value determined by the mean geostrophic wind in equation (8.21).

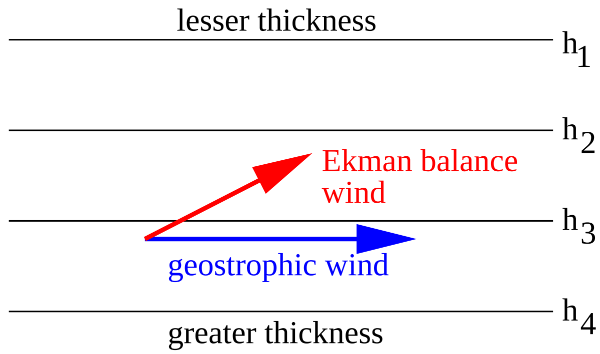

Figure 8.2: Illustration of geostrophic wind and Ekman balance wind in the atmospheric boundary layer.

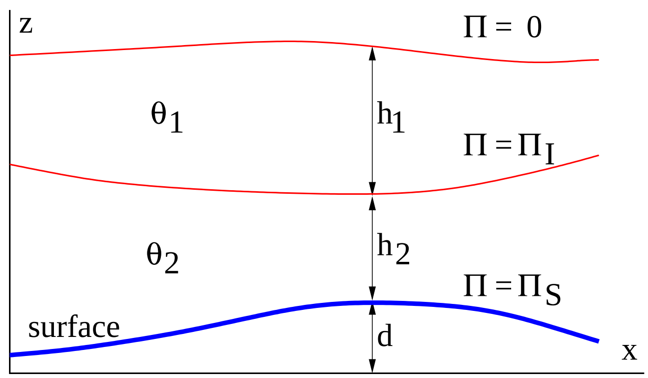

Figure 8.3: Two layer model of the atmosphere. The effect of terrain is represented by the terrain height $d(x, y)$.

This *linear Ekman balance* approximation is used when a linear relationship between the boundary layer wind and the geostrophic wind is needed to simplify computations.

Figure 8.2 provides a schematic illustration of the boundary layer wind resulting from Ekman balance. In this figure the thickness decreases to the north, resulting in the illustrated geostrophic wind (assuming $f > 0$). The Ekman balance wind is smaller in magnitude and is rotated in direction down the thickness or pressure gradient.

The single layer model of the boundary layer ignores the effect of the overlying atmosphere, which is a major approximation. If the free troposphere is approximated as the upper layer in a two-layer model, the flow in the boundary layer responds to the thickness gradient in this layer as well as in the boundary layer.

8.4 Two-layer model

The single layer model of the atmosphere is of limited validity, and as in the ocean, a two layer model describes a much wider range of observed phenomena. Figure 8.3 shows a model of the atmosphere containing two homogeneous layers. The upper layer has potential temperature θ_1 and thickness h_1, while θ_2 and h_2 represent the corresponding variables for the lower layer.

We compute the Exner function in layer 1 to be

$$\Pi_1 = \frac{g}{\theta_1}(h_1 + h_2 + d - z), \tag{8.22}$$

where we have assumed that $\Pi = 0$ at the top of layer 1. The Exner function at the interface between the layers is

$$\Pi_I = \frac{g}{\theta_1}h_1 \tag{8.23}$$

and in layer 2 is

$$\Pi_2 = \Pi_I + \frac{g}{\theta_2}(h_2 + d - z) = g[h_1/\theta_1 + (h_2 + d - z)/\theta_2]. \tag{8.24}$$

The surface Exner function ($z = d$) is thus

$$\Pi_S = g(h_1/\theta_1 + h_2/\theta_2). \tag{8.25}$$

Proceeding as in the single layer model, the momentum equations for the two layers are therefore

$$\frac{d\mathbf{v}_1}{dt} + g\nabla(h_1 + h_2 + d) + f\hat{\mathbf{z}} \times \mathbf{v}_1 = \mathbf{F}_1, \tag{8.26}$$

$$\frac{d\mathbf{v}_2}{dt} + g\nabla[(\theta_2/\theta_1)h_1 + h_2 + d] + f\hat{\mathbf{z}} \times \mathbf{v}_2 = \mathbf{F}_2. \tag{8.27}$$

These look a lot like the corresponding momentum equations for the two layer ocean, the only difference being the replacement of ρ_1/ρ_2 by θ_2/θ_1. For generality an arbitrary external force per unit mass is included for each level.

The mass continuity equations for the Boussinesq approximation are derived as in the single layer case, resulting in

$$\frac{\partial h_1}{\partial t} + \nabla \cdot (h_1\mathbf{v}_1) = M_1 \tag{8.28}$$

$$\frac{\partial h_2}{\partial t} + \nabla \cdot (h_2\mathbf{v}_2) = M_2, \tag{8.29}$$

where as in the single layer case we have added source terms M_1 and M_2. The quantities $\rho_{m1}M_1$ and $\rho_{m2}M_2$ represent the mass of air per unit area added to each layer as a result of heating or cooling associated with convection or radiation. The quantities ρ_{m1} and ρ_{m2} are the (constant) mean densities in each layer in analogy with ρ_m defined above for the single layer model. Conservation of mass implies that mass lost in one layer reappears in the other layer, i. e.,

$$\rho_{m1}M_1 + \rho_{m2}M_2 = 0. \tag{8.30}$$

We think of the two layer model as approximating an atmosphere with a constant gradient in potential temperature with respect to pressure, as illustrated in figure 8.4. The level separating the upper and lower layers is adjusted so that the average potential temperature in each layer of the actual atmosphere is the same as the potential temperature of the layer. In this way an atmosphere with horizontal variability in pressure-averaged potential temperature (but no variation in vertical structure) can be represented approximately by

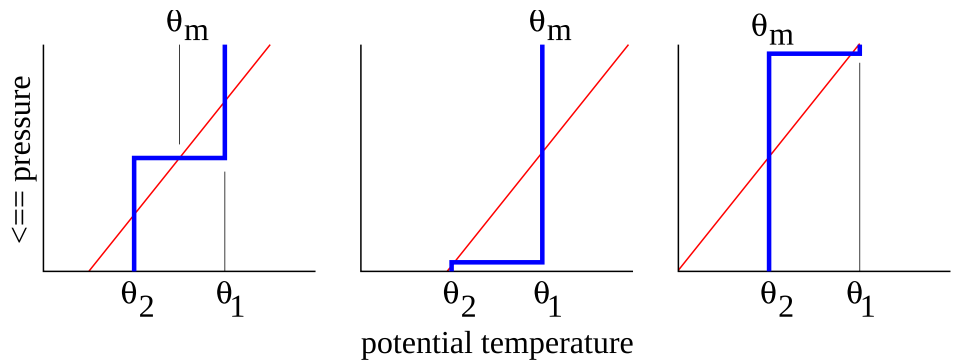

Figure 8.4: Sketch of the potential temperature as a function of pressure in the two layer model (thick lines) and the constant gradient profile it is assumed to approximate (slanted thin lines).

the two-layer model. The mean potential temperature of the atmosphere in this model is given by

$$\theta_m = \frac{\rho_{m1} h_1 \theta_1 + \rho_{m2} h_2 \theta_2}{\rho_{m1} h_1 + \rho_{m2} h_2}. \tag{8.31}$$

The surface potential temperature is sometimes needed for calculating surface heat fluxes. Examination of figure 8.4 shows that the actual surface potential temperature, as opposed to θ_2 the potential temperature of the lower layer, is given by

$$\theta_S = \theta_m - (\theta_1 - \theta_2). \tag{8.32}$$

8.5 Effects of heating

In our two-layer model, heating increases the mean temperature of the atmospheric column, not by increasing θ_1 or θ_2 , but by transferring mass from the lower layer to the upper layer. This is accomplished in the model by assigning a positive value of M_1 and a negative value of M_2, with the ratio of the two source terms adjusted to satisfy equation (8.30).

If ΔQ is the heat added to the atmosphere per unit area in time interval Δt, we can relate ΔQ to M_1 and M_2. We approximate $\Delta Q = T\Delta S$ by $T_R \Delta S$, where T_R is a constant reference temperature and ΔS is the entropy per unit area added to the column. We can relate ΔS to the (constant) specific entropy in each layer and the change in mass per unit area in each layer, $\rho_{m1} M_1$ and $\rho_{m2} M_2$:

$$\Delta Q = T_R \Delta S = T_R (s_{d1} \rho_{m1} M_1 + s_{d2} \rho_{m2} M_2) \Delta t. \tag{8.33}$$

Using equation (8.30) and the relationship between potential temperature and dry entropy (8.3) simplifies this to

$$\frac{dQ}{dt} = C_p T_R \ln(\theta_1/\theta_2) \rho_{m1} M_1, \tag{8.34}$$

from which we can infer M_1. Using equation (8.30) we also find that $M_2 = -(\rho_{m1}/\rho_{m2}) M_1$.

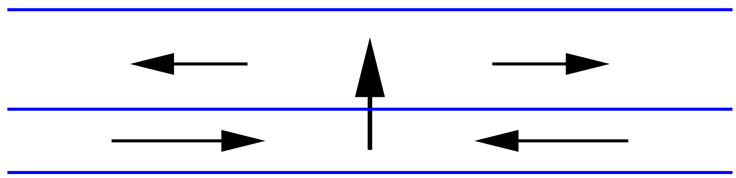

Figure 8.5: Schematic illustration of the flow that occurs in response to steady, isolated heating in a non-rotating environment.

8.5.1 Steady heating in non-rotating environment

As a simple example of the response of the two-layer model to heating, we consider a linearized, two-dimensional case with steady, isolated heating which produces a source term in the upper layer of the form

$$M_1(x) = M_0\delta(x) \tag{8.35}$$

where M_0 is a constant and $\delta(x)$ is the Dirac delta function. The time-independent, linearized, two-dimensional momentum equations in the absence of an applied force are

$$g\frac{\partial}{\partial x}(h_{01}\eta_1 + h_{02}\eta_2) = 0 \qquad g\frac{\partial}{\partial x}(\nu h_{01}\eta_1 + h_{02}\eta_2) = 0 \tag{8.36}$$

which simply tells us that the fractional thickness perturbations $\eta_1 = \eta_2 = 0$ in this case. (We could take them as being constant, but this is pointless, as it would be tantamount to needlessly redefining the ambient thicknesses h_{01} and h_{02}.) The mass continuity equations in the linearized, time-independent case simplify to

$$\frac{\partial v_{1x}}{\partial x} = \frac{M_1}{h_{01}} \qquad \frac{\partial v_{2x}}{\partial x} = \frac{M_2}{h_{02}} = -\frac{\rho_{m1}M_1}{\rho_{m2}h_{02}} = -\frac{\rho_{m1}h_{01}}{\rho_{m2}h_{02}}\frac{\partial v_{1x}}{\partial x}, \tag{8.37}$$

with the resulting solutions

$$v_{1x} = \frac{M_0}{2h_{01}}\frac{x}{|x|} \qquad v_{2x} = -\frac{\rho_{m1}M_0}{2\rho_{m2}h_{02}}\frac{x}{|x|}. \tag{8.38}$$

The character of this flow is illustrated in figure 8.5. Upward mass transfer occurs between layers in the isolated heating region at $x = 0$. There is flow away from the heated region in the upper layer (layer 1) while the reverse occurs in the lower layer (layer 2). In the steady, linearized case the layer thicknesses do not change. It is noteworthy that the outflow in the upper layer and the inflow in the lower layer continue undiminished to $\pm\infty$ in the time-independent case.

8.5.2 Response to a heating pulse

How does the above response to steady heating become established? We approach this question by determining the response of a non-rotating, two layer system to a pulse of heat at the origin at zero time. In order to simplify the calculation we assume a special case

$h_{01} = h_{02}$ and define $c^2 = gh_{01}$. We make the further idealization that $\rho_{m1} = \rho_{m2}$, which we call the *strict Boussinesq approximation.* This assumption is not defensible in a quantitative sense, but the results nevertheless qualitatively represent what happens in the more general case.

The two-dimensional governing equations linearized about a resting base state are

$$\frac{\partial v_1}{\partial t} + c^2 \frac{\partial}{\partial x}(\eta_1 + \eta_2) = 0 \tag{8.39}$$

$$\frac{\partial v_2}{\partial t} + c^2 \frac{\partial}{\partial x}(\nu\eta_1 + \eta_2) = 0 \tag{8.40}$$

$$\frac{\partial \eta_1}{\partial t} + \frac{\partial v_1}{\partial x} = M_0 \delta(x)\delta(t) \tag{8.41}$$

$$\frac{\partial \eta_2}{\partial t} + \frac{\partial v_2}{\partial x} = -M_0 \delta(x)\delta(t) \tag{8.42}$$

where v_1 and η_1 are the velocity and fractional thickness perturbation in the upper layer and v_2 and η_2 are the corresponding variables in the lower layer. The constant M_0 represents the strength of the heating pulse and the ratio of the potential temperatures in the two layers is given by $\nu = \theta_2/\theta_1$. As can be verified by direct substitution, the solution to these equations for $x > 0$ is

$$\eta_1 = [\eta_{1x}\delta(x - c_x t) + \eta_{1i}\delta(x - c_i t)]H(t) \tag{8.43}$$

$$\eta_2 = [\eta_{2x}\delta(x - c_x t) + \eta_{2i}\delta(x - c_i t)]H(t) \tag{8.44}$$

$$v_1 = [v_{1x}\delta(x - c_x t) + v_{1i}\delta(x - c_i t)]H(t) \tag{8.45}$$

$$v_2 = [v_{2x}\delta(x - c_x t) + v_{2i}\delta(x - c_i t)]H(t) \tag{8.46}$$

where the Heaviside function $H(t)$ is defined

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} . \tag{8.47}$$

The solution is in the form of two non-dispersive waves originating at the heating pulse and propagating to the right at the speeds of the external and internal gravity modes,

$$c_x = c(1 + \nu^{1/2})^{1/2} \qquad c_i = c(1 - \nu^{1/2})^{1/2}. \tag{8.48}$$

The thickness perturbations are symmetric in x while the velocities are antisymmetric, which allows the solutions for $x < 0$ to be obtained from the $x > 0$ solutions.

The constant coefficients in equations (8.43) - (8.46) are

$$\eta_{1x} = -M_0(\nu^{-1/2} - 1)/4 \qquad v_{1x} = c_x \eta_{1x} \tag{8.49}$$

$$\eta_{2x} = -M_0(1 - \nu^{1/2})/4 \qquad v_{2x} = c_x \eta_{2x} \tag{8.50}$$

$$\eta_{1i} = M_0(\nu^{-1/2} + 1)/4 \qquad v_{1i} = c_i \eta_{1i} \tag{8.51}$$

$$\eta_{2i} = -M_0(1 + \nu^{1/2})/4 \qquad v_{2i} = c_i \eta_{2i}. \tag{8.52}$$

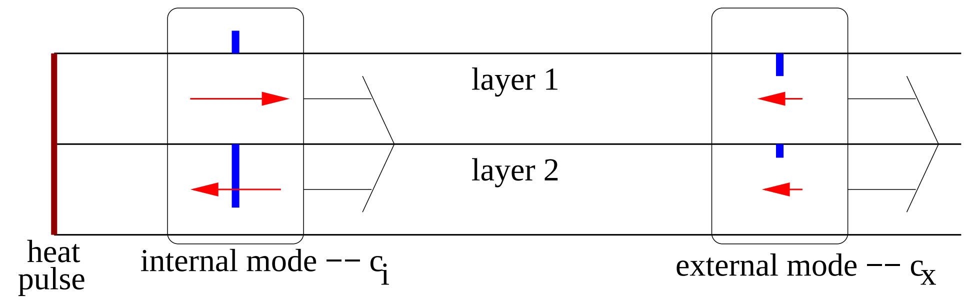

Figure 8.6: Schematic diagram of the rightward-moving external and internal waves in thickness and velocity resulting from a heat pulse at the origin. The thick vertical line segments represent the displacements of the layer tops while the arrows illustrate the directions and relative magnitudes of the wind pulses in the waves.

In verifying these solutions, first note that $\partial H(t)/\partial t = \delta(t)$. Second, derivatives of the Dirac delta function are subject to the chain rule as with any other function, e. g., $\partial\delta(x-ct)/\partial t = -c\delta'(x-ct)$. Third, note that a product of delta functions of the form $\delta(t)\delta(x-ct)$ simplifies to $\delta(t)\delta(x)$ since this product is only non-zero when $t=0$ by virtue of the first delta function.

Figure 8.6 illustrates qualitatively the results of this calculation. At time $t=0$ a pulse of heating occurs at the origin, resulting in the formation of wave pulses which propagate away from the origin. There are two types of pulses, those associated with the external mode of the two layer system, and those associated with the internal mode. The former propagates more rapidly than the latter. The external mode pulse consists of a depression in both the top of the upper layer and the interface between the layers and instantaneous flows toward the origin in both layers. The internal mode pulse exhibits elevation in the top of the upper layer of the same magnitude as the depression in the upper layer associated with the external mode. However, the displacement of the interface is downward and of much larger magnitude. The flow velocities in the upper and lower layer are respectively away from and toward the origin, and have much larger magnitudes than in the external mode.

The solution we have obtained for this problem is special in the sense that the solution for any heating distribution may be derived from it. In particular, defining $K_{\eta 1,2}(x,t)$ as $\eta_{1,2}$ in equations (8.43) and (8.44) with $M_0 = 1$, the distribution of fractional thickness perturbation in the upper layer can be obtained for any space-time distribution of heating, as represented by the upper and lower layer source functions $M_{1,2}(x,t)$, can be obtained via the double integral

$$\eta_{1,2}(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' K_{\eta 1,2}(x-x', t-t') M_{1,2}(x',t'). \tag{8.53}$$

The function $K_{\eta 1,2}(x,t)$ is called the *Green's function* for $\eta_{1,2}$. Similar solutions can be obtained for the velocity in the upper and lower layers as well by defining the Green's function $K_{v1,2}(x,t)$ for the velocities in the same manner.

8.6 Potential vorticity in the atmosphere

Hoskins et al. (1987) provides an extensive discussion of the usefulness of analyzing potential vorticity on isentropic (constant potential temperature) surfaces in the atmosphere. The effects of friction and heating on the potential vorticity distribution are considered by Haynes and McIntyre (1987, 1990) and Raymond (1992).

The compressibility of the atmosphere forces a slight change in the definition of potential vorticity q of a layer:

$$q = \frac{\zeta_a}{\overline{\rho} h} \tag{8.54}$$

where ζ_a and h are the absolute vorticity and layer depth as usual and $\overline{\rho}$ is the vertically averaged air density in the layer. The reason for this change becomes evident upon taking the total (or parcel) time derivative of the potential vorticity:

$$\frac{dq}{dt} = \frac{d}{dt}\left(\frac{\zeta_a}{\overline{\rho} h}\right) = \frac{d}{dt}\left(\frac{\zeta_a A}{\overline{\rho} h A}\right) = \frac{d}{dt}\left(\frac{\Gamma_a}{m}\right) \tag{8.55}$$

where A is a tiny area of a column in the air layer of interest. In the last step we equate the absolute vorticity times this area to the absolute circulation Γ_a about the column and the product $\overline{\rho} h A$ to the mass m of the column. Since the column moves with the flow, the circulation is conserved in the absence of external forces, and the mass is similarly conserved if there are no mass sources or sinks. Thus, the potential vorticity obeys $dq/dt = 0$ in this case as before. This condition would not have held for air had the mean air density not been added to the definition of potential vorticity.

In order to obtain the governing equation for potential vorticity in the case of mass sources and sinks, we take a slightly different approach than before, starting with the components of the momentum equation for a single layer with no topography but with an externally applied force:

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + g\frac{\partial h}{\partial x} - f v_y = F_x \tag{8.56}$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + g\frac{\partial h}{\partial y} + f v_x = F_y \tag{8.57}$$

We now use a trick which simplifies the derivation. Taking the advection terms in equation (8.56), we add and subtract an additional term:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_y \frac{\partial v_y}{\partial x} - v_y \frac{\partial v_y}{\partial x}. \tag{8.58}$$

The first and third terms on the right can be written $\partial[(v_x^2 + v_y^2)/2]/\partial x$ while the second and fourth combine to make $-v_y \zeta$ where $\zeta = \partial v_y/\partial x - \partial v_x/\partial y$ is the relative vorticity. Applying the same trick to equation (8.57), we write these equations as

$$\frac{\partial v_x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v_x^2 + v_y^2}{2}\right) - v_y \zeta + g\frac{\partial h}{\partial x} - f v_y = F_x \tag{8.59}$$

$$\frac{\partial v_y}{\partial t} + \frac{\partial}{\partial y}\left(\frac{v_x^2 + v_y^2}{2}\right) + v_x \zeta + g\frac{\partial h}{\partial y} + f v_x = F_y \tag{8.60}$$

Finally, we take the y derivative of equation (8.59) and subtract it from the x derivative of equation (8.60), resulting in an equation for the absolute vorticity $\zeta_a = \zeta + f$:

$$\frac{\partial \zeta_a}{\partial t} + \frac{\partial \zeta_a v_x}{\partial x} + \frac{\partial \zeta_a v_y}{\partial y} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}. \tag{8.61}$$

This is called the *vorticity equation* for obvious reasons. The absolute vorticity appears in the time derivative because $\partial f/\partial t = 0$, and therefore f can be added to ζ in this term without changing the equation. Note that this equation also applies to individual layers in a fluid represented by two or more layers, as the only change in this case is the replacement of the $g\nabla h$ term with g times the gradient of something more complicated. As long as this term is in the form of a gradient, it cancels out in the derivation of the vorticity equation.

The next step involves use of the mass continuity equation, which for single or multi-layer systems takes the form

$$\frac{\partial \overline{\rho} h}{\partial t} + \nabla \cdot (\overline{\rho} h \mathbf{v}) = \overline{\rho} M \tag{8.62}$$

for the layer under consideration, as demonstrated above. Since $q = \zeta_a/(\overline{\rho} h)$, we can solve this for absolute vorticity $\zeta_a = q\overline{\rho} h$ and substitute into equation (8.61). Product rule expansions of derivatives yield

$$\overline{\rho} h \left(\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q \right) + q \left(\frac{\partial \overline{\rho} h}{\partial t} + \nabla \cdot (\overline{\rho} h \mathbf{v}) \right) = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{8.63}$$

and use of equation (8.62) simplifies this to

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = -\frac{q}{h} M + \frac{1}{\overline{\rho} h} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \tag{8.64}$$

which is the governing equation for potential vorticity in the presence of mass sources and sinks as well as frictional or other nonconservative forces. Each layer in a multi-layer system will have a governing equation for potential vorticity in that layer.

Note that no approximation has yet been made in the derivation of equation (8.64). However, potential vorticity inversion is most easily discussed in the context of the Boussinesq approximation. Below we investigate inversion in single and two-layer models.

8.6.1 Single layer model

It remains only to derive the potential vorticity inversion equations for the models of interest. The linearized inversion equation on an f-plane for a single layer model is almost identical to that used for the ocean in the Boussinesq case:

$$q = \frac{\zeta_{ag}}{\rho_m h} = \frac{f}{\rho_m h_0 (1 + \eta)} \left(1 + \frac{1}{f} \frac{\partial v_{gy}}{\partial x} - \frac{1}{f} \frac{\partial v_{gx}}{\partial y} \right). \tag{8.65}$$

The only difference is that $q_0 = f/(\rho_m h_0)$; the Rossby radius $L_R^2 = g h_0 / f^2$ has its usual definition and the geostrophic wind as before is

$$v_{gx} = -\frac{g h_0}{f} \frac{\partial \eta}{\partial y} \qquad v_{gy} = \frac{g h_0}{f} \frac{\partial \eta}{\partial x}. \tag{8.66}$$

Defining $q^* = q - q_0$, equation (8.65) becomes after linearization

$$L_R^2 \nabla^2 \eta - \eta = q^*/q_0. \tag{8.67}$$

The velocity potential is derived as before from the mass continuity equation, but with the mass source term M included:

$$\nabla^2 \chi = \frac{\partial \eta}{\partial t} - \frac{M}{h_0}. \tag{8.68}$$

The ageostrophic wind is $\mathbf{v}_a = -\nabla \chi$ as before.

8.6.2 Two layer model

We now investigate a two layer model with equal ambient layer thicknesses h_0 and no topography, i.e., $d = 0$. In this model the inversion equations for the two layers become coupled together. Deriving the geostrophic wind from equations (8.26) (8.27) yields

$$v_{1gx} = -\frac{gh_0}{f}\frac{\partial(\eta_1 + \eta_2)}{\partial y} \qquad v_{1gy} = \frac{gh_0}{f}\frac{\partial(\eta_1 + \eta_2)}{\partial x} \tag{8.69}$$

$$v_{2gx} = -\frac{gh_0}{f}\frac{\partial(\nu\eta_1 + \eta_2)}{\partial y} \qquad v_{2gy} = \frac{gh_0}{f}\frac{\partial(\nu\eta_1 + \eta_2)}{\partial x} \tag{8.70}$$

where $\nu = \theta_2/\theta_1$ is the ratio of lower to upper level potential temperature as before. In the Boussinesq approximation the inversion equations become

$$L_R^2 \nabla^2 (\eta_1 + \eta_2) - \eta_1 = q_1^*/q_0 \tag{8.71}$$

$$L_R^2 \nabla^2 (\nu\eta_1 + \eta_2) - \eta_2 = q_2^*/q_0 \tag{8.72}$$

$q_1^* = q_1 - q_0$, $q_2^* = q_2 - q_0$. The quantities q_0 and L_R are defined as above. The velocity potential in each layer is obtained from

$$\nabla^2 \chi_1 = \frac{\partial \eta_1}{\partial t} - \frac{M_1}{h_0} \qquad \nabla^2 \chi_2 = \frac{\partial \eta_2}{\partial t} - \frac{M_2}{h_0} \tag{8.73}$$

with the ageostrophic velocities in each layer being $\mathbf{v}_{1a} = -\nabla \chi_1$ and $\mathbf{v}_{2a} = -\nabla \chi_2$.

8.7 Balanced response to heating

Let us now investigate the balanced response to heating in a two-layer atmosphere with equal ambient layer thicknesses h_0, constant Coriolis parameter f, and steady, delta function convection in two dimensions starting at time $t = 0$. For simplicity we make the strict Boussinesq approximation of equal densities in the two layers, i. e., $\rho_{m1} = \rho_{m2} \equiv \rho_m$. Heating is represented by mass sources in the upper (subscripted 1) and lower (subscripted 2) layers of

$$\rho_m M_1 = -\rho_m M_2 = \rho_m M_0 \delta(x) H(t) \tag{8.74}$$

where M_0 is a constant representing the strength of the convection and $H(t)$ is the Heaviside function as before. Starting from a uniform state of rest at $t = 0$, the resulting potential vorticity perturbation in the upper layer as computed by equation (8.64) is

$$q_1^* = -\frac{q_0 M_0 \delta(x) t}{h_0}, \tag{8.75}$$

with $q_2^* = -q_1^*$ in the lower layer. In deriving this equation we have linearized in the sense that we have set $q = q_0$ and $h = h_0$ on the right side of equation (8.64).

Given our experience with the solution for a single layer problem of this type, we hypothesize that

$$\eta_1 = \eta_{10} \exp(-|x|/L_R) \qquad \eta_2 = \eta_{20} \exp(-|x|/L_R) \tag{8.76}$$

where $L_R = (gh_0)^{1/2}/f$ is the Rossby radius as before and η_{10} and η_{20} are constants (actually functions of time). Substitution of these trial solutions into equations (8.71) and (8.72) verifies their validity subject to the conditions

$$\eta_{10} + \eta_{20} = \frac{M_0 t}{h_0 L_R} \qquad \nu\eta_{10} + \eta_{20} = -\frac{M_0 t}{h_0 L_R}. \tag{8.77}$$

Solution for η_{10} and η_{20} and substitution into equation (8.76) results in

$$\eta_1 = \frac{2M_0 t}{(1-\nu)h_0 L_R} \exp(-|x|/L_R) \qquad \eta_2 = -\frac{(1+\nu)M_0 t}{(1-\nu)h_0 L_R} \exp(-|x|/L_R). \tag{8.78}$$

The fractional height perturbation for the top of the upper layer is given by

$$\eta_1 + \eta_2 = \frac{M_0 t}{h_0 L_R} \exp(-|x|/L_R). \tag{8.79}$$

The geostrophic wind has a zero x component in this case whereas the y component of the ageostrophic wind is zero. From equations (8.69) and (8.70) we find the y component of the geostrophic wind in the two layers to be

$$v_{1gy} = -\frac{2fM_0 tx}{(1-\nu)h_0|x|} \exp(-|x|/L_R) \qquad v_{2gy} = \frac{(1+\nu)fM_0 tx}{(1-\nu)h_0|x|} \exp(-|x|/L_R), \tag{8.80}$$

while from equation (8.73) we derive the x component of the ageostrophic wind in each layer:

$$v_{1ax} = \frac{2M_0 x}{(1-\nu)h_0|x|} \exp(-|x|/L_R) \qquad v_{2ax} = -\frac{(1+\nu)M_0 x}{(1-\nu)h_0|x|} \exp(-|x|/L_R). \tag{8.81}$$

Figure 8.7 shows the response of the two-layer atmosphere to the convection, which is represented by the upward arrow in the center, illustrating the upward flow of mass from layer 2 to layer 1. The steady ageostrophic wind is given by the horizontal arrows, which flow away from the convection in the upper layer and toward the convection in the lower layer. This flow diminishes on the scale of the Rossby radius L_R as one moves away from the convection. The flow thus differs from the nonrotating case where the inflow and outflow extend laterally to infinity. Potential vorticity inversion implies that the interface between the

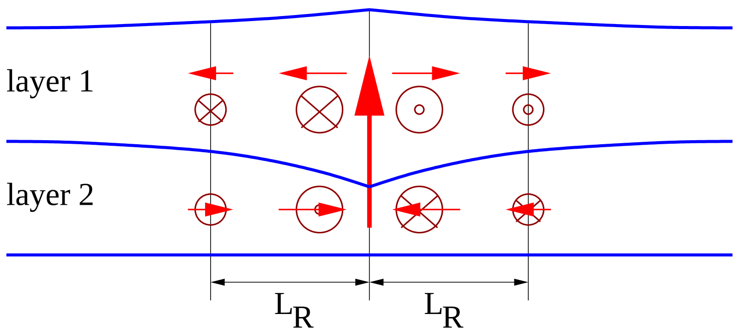

Figure 8.7: Response of a two-layer model to a line of heating normal to the page, which causes a transfer of mass from the lower to the upper layer. The ageostrophic response is shown by the arrows while the geostrophic wind normal to the page is represented by circles with crosses (into the page) and dots (out of the page).

two layers deflects downward increasingly with time, with the biggest deflection occurring at the location of the convection. In contrast, the top of the upper layer deflects weakly upward. The resulting geostrophic flow is normal to the page as illustrated by the circles with either a cross (into the page) or a dot (out of the page). The flow is anticyclonic around the convection in the upper layer and cyclonic in the lower layer. The geostrophic flow increases with time as the deflection of the surfaces increases. Note that this flow will soon intensify enough to invalidate the linearity assumption used at various points in the analysis.

8.8 References

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8.9 Problems

1. More on Ekman balance.

(a) Generalize equation (8.20) to the case where both v_{gx} and v_{gy} are nonzero.

(b) Using the linear Ekman balance approximation (holding ϵ, v, and h constant) and assuming also that f is constant, show that the divergence

$$D = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

of the Ekman balance wind in the boundary layer is proportional to the geostrophic relative vorticity and find the constant of proportionality.

(c) In the steady state case use the linearized mass continuity equation in the boundary layer to relate the geostrophic relative vorticity to the mass source M in the boundary layer. When $M < 0$, it is often considered that boundary layer air is being exported aloft, thus acting as a source for deep convective updrafts. This process is called *Ekman pumping.*

2. Obtain $\eta_1(x,t)$ and $v_1(x,t)$ for the response of the two-layer model to the mass source $M_1 = -M_2 = \delta(x)H(t)$. Hint: Use equation (8.53).

3. Consider a single layer Boussinesq model with rotation but no friction.

(a) Linearize and solve for the fractional thickness perturbation and fluid velocity in response to the mass forcing periodic in space and time

$$M = M_0 \cos(kx)\sin(\omega t)$$

where M_0, k, and ω are externally specified constants. Hint: Try solutions of the form $v_x \propto \sin(kx)\sin(\omega t)$, $v_y \propto \sin(kx)\cos(\omega t)$, and $\eta \propto \cos(kx)\cos(\omega t)$.

(b) For fixed k, determine how the layer thickness and the wind components respond to the mass forcing as a function of ω. Note particularly the value of ω for which the solution blows up. Give a physical interpretation of this blowup.

(c) Compare your results with those of Robinson et al. (2008).

4. Repeat the above problem except consider the balanced response to the mass forcing. In particular:

(a) Use the linearized potential vorticity advection equation to obtain q^* from M.

(b) Invert the linearized potential vorticity perturbation q^* equation to obtain the fractional thickness perturbation η.

(c) From η obtain the geostrophic wind.

(d) Also from η, obtain the ageostrophic wind. Combine with the geostrophic wind to obtain the total wind.

(e) Determine the range of ω values for which the linearized balanced response is in reasonable agreement with the linearized full response.