Chapter 12

Mid-latitude Atmospheric Dynamics

We now use the tools we have developed to understand the behavior of the earth's atmosphere at middle latitudes. This is a region dominated by Rossby wave dynamics acting on a complex basic state. We first describe this state and then show how the various types of Rossby waves produced the observed flows.

12.1 Basic state of atmosphere

Figure 12.1 shows an idealized sketch of the north-south vertical structure in middle latitudes. This is the region in which the equator-pole temperature gradient is concentrated. As a result, the surface slopes downward toward the pole in isentropic coordinates. The tropopause also slopes downward in the same sense – the coldest temperatures in the atmosphere are at the tropopause over the equator, but at the same time the warmest potential temperatures in the troposphere exist there.

Ambient potential vorticity in the troposphere tends to be nearly constant on isentropic surfaces, i. e., along horizontal lines in isentropic coordinates, in spite of the increase in planetary vorticity with latitude. This is related to the fact that potential vorticity tends to decrease somewhat with elevation within the troposphere in middle latitudes due to an increase in isentropic density with height. Isentropic surfaces increase in geometric height with latitude, and the resulting increase in isentropic density approximately compensates for the increase in planetary vorticity.

The reason for the near-constancy of potential vorticity on isentropic surfaces is unknown, but it is speculated to be the result of north-south mixing of parcels on isentropic surfaces, which would tend to homogenize the potential vorticity on these surfaces. However it comes about, it has one important consequence: The free troposphere in middle latitudes does not support Rossby waves to any great extent! As we shall see, two other types of Rossby waves dominate in middle latitudes, those based on the surface temperature gradient, and those rooted in the tropopause.

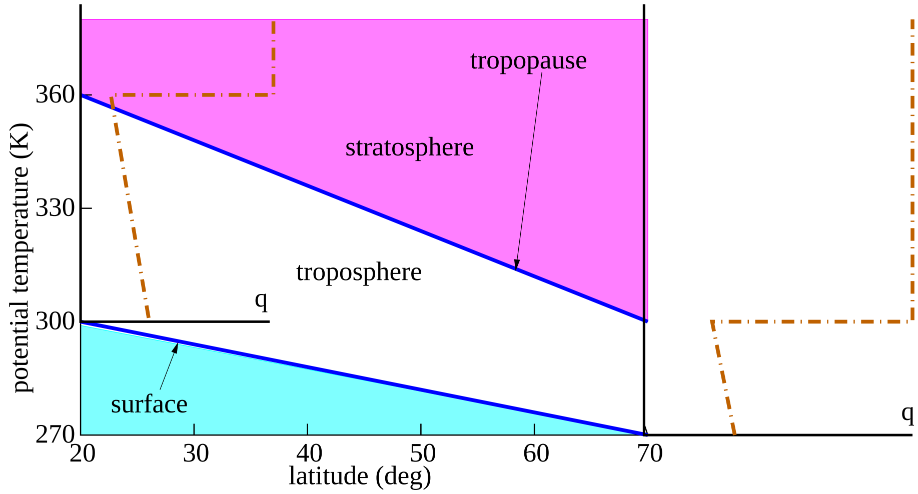

Figure 12.1: Schematic of north-south structure of atmosphere in middle latitudes. The surface tilts down to the north in isentropic coordinates, as does the tropopause. The dash-dotted lines show profiles of ambient potential vorticity as a function of potential temperature at two latitudes.

12.2 Simplifying approximations

The approximations made to simplify to a tractable form the governing equations away from the equator while retaining their essence are a bit complex, involving two levels of linearization. Let us assume a constant reference value of the isentropic density η_R. Corresponding to this we define reference profiles of Montgomery potential $M_R(\theta)$ and geopotential Φ_R. In the Boussinesq context, we easily calculate that

$$M_R = -\frac{L^2(\theta - \theta_R)^2}{2} \tag{12.1}$$

and

$$\Phi_R = \theta_R L^2(\theta - \theta_R) \tag{12.2}$$

where the combination

$$L^2 \equiv \frac{g\eta_R}{\rho_R \theta_R} \tag{12.3}$$

occurs frequently enough to warrant its own symbol.

We postulate for simplicity an ambient state with a zonal (east-west) wind U which takes the form of constant shears in the meridional (north-south) and θ directions:

$$U = U_0 + S_\theta(\theta - \theta_R) + S_y y. \tag{12.4}$$

The full wind is then a combination of this and a disturbance part, $\mathbf{v} = U\hat{\mathbf{x}} + \mathbf{v}_g^*$, where the disturbance wind is approximated by the geostrophic wind. The condition of geostrophic balance then requires the Montgomery potential and geopotential to be

$$\begin{aligned} M &= M_R(\theta) - f_0 \int U dy + M^* \\ &= -L^2(\theta - \theta_R)^2/2 - f_0[U_0 y + S_\theta(\theta - \theta_R)y + S_y y^2/2] + M^* \end{aligned} \tag{12.5}$$

and

$$\Phi = \theta_R \left[L^2(\theta - \theta_R) + f_0 S_\theta y - \frac{\partial M^*}{\partial \theta} \right] \tag{12.6}$$

where the disturbance part of the Montgomery potential M^* is related to the disturbance geostrophic wind by

$$\mathbf{v}_g^* = \frac{1}{f_0}\left(-\frac{\partial M^*}{\partial y}, \frac{\partial M^*}{\partial x} \right). \tag{12.7}$$

The potential vorticity inversion equation comes as usual from the potential vorticity definition $q = (f + \zeta)/\eta$, which we linearize to

$$\frac{q}{q_0} = \frac{1 + \beta y/f_0 + \zeta/f_0}{1 + \eta^*/\eta_R} \approx 1 + \frac{\beta y}{f_0} - \frac{\eta^*}{\eta_R} + \frac{\zeta}{f_0}, \tag{12.8}$$

where $q_0 = f_0/\eta_R$. The isentropic density is split into a constant reference part and a disturbance part for this linearization, $\eta = \eta_R + \eta^*$, with the assumption that $|\eta^*| \ll \eta_R$. The relative vorticity equals

$$\zeta = \frac{\partial v_{gy}}{\partial x} - \frac{\partial v_{gx}}{\partial y} = \frac{\partial v_{gy}^*}{\partial x} - \frac{\partial v_{gx}^*}{\partial y} - S_y \tag{12.9}$$

in this case, from which we deduce that the potential vorticity divides naturally into three pieces,

$$q = q_0 + \frac{\beta y - S_y}{\eta_R} + q^*. \tag{12.10}$$

The first term is the planetary vorticity at $y = 0$, the second has to do with latitudinal variation of the ambient absolute vorticity, and the third is the disturbance part of the potential vorticity, which takes the form

$$q^* = -\frac{\eta^*}{\eta_R} + \frac{1}{\eta_R}\left(\frac{\partial v_{gy}^*}{\partial x} - \frac{\partial v_{gx}^*}{\partial y} \right). \tag{12.11}$$

Using equation (12.7) as well as the relation

$$\frac{\eta^*}{\eta_R} = -\frac{1}{L^2}\frac{\partial^2 M^*}{\partial \theta^2}, \tag{12.12}$$

the linearized inversion equation in terms of disturbance quantities only finally becomes

$$\frac{q^*}{q_0} = \frac{1}{L^2}\frac{\partial^2 M^*}{\partial \theta^2} + \frac{1}{f_0^2}\nabla^2 M^*. \tag{12.13}$$

The linearized advection equation in this case becomes

$$\frac{\partial q^*}{\partial t} + U\frac{\partial q^*}{\partial x} + \frac{\beta}{f_0 \eta_R}\frac{\partial M^*}{\partial x} = 0 \tag{12.14}$$

where we have assumed that $|\mathbf{v}_g^*| \ll U$ and that $|q^*| \ll |(\beta y - S_y)/\eta_R|$.

We often desire U to be independent of y in $y - z$ space, i. e., $U = U(z)$. Taking the partial derivative of equation (12.6) while holding Φ constant and assuming $M^* = 0$ implies that the ambient flow satisfies $(\partial\theta/\partial y)_\Phi = -f_0 S_\theta / L^2$. A similar y derivative of equation (12.4) combined with the desired condition that $(\partial U/\partial y)_\Phi = 0$ results in

$$S_y = f_0 S_\theta^2 / L^2 \tag{12.15}$$

for U to be independent of y in geometrical coordinates.

12.3 Mid-latitude Rossby waves

In this section we work out the dynamics of three types of Rossby waves, those related to the beta effect, those supported by a north-south temperature gradient at the surface, and those which are linked to the tropopause.

12.3.1 Internal waves

As noted above, internal Rossby waves dependent on a north-south gradient of potential vorticity are unlikely to occur in the middle latitudes. We nevertheless investigate their properties. In the interests of simplicity, let us imagine that the troposphere is bounded above and below by (geometrically) horizontal rigid surfaces at geopotential values of $\Phi_L = 0$ and $\Phi_U = gh$, corresponding to constant potential temperature reference values θ_L and θ_U, imagining the upper surface to correspond to the tropopause. We thus ignore the tilt of the surface and the tropopause in middle latitudes. We also postulate a background environment at rest.

The potential vorticity inversion equation is

$$\frac{1}{f_0^2}\nabla^2 M^* + \frac{1}{L^2}\frac{\partial^2 M^*}{\partial \theta^2} = \frac{q^*}{q_0} \equiv \epsilon \tag{12.16}$$

with lower and upper boundary conditions

$$\left(\frac{\partial M^*}{\partial \theta}\right)_{L,U} = -\Phi^*_{L,U}/\theta_R = 0. \tag{12.17}$$

The potential vorticity evolution equation is

$$\frac{\partial \epsilon}{\partial t} + \frac{\beta}{f_0^2}\frac{\partial M^*}{\partial x} = 0. \tag{12.18}$$

A trial plane wave solution is

$$M^* = M_0 \cos[m(\theta - \theta_L)] \sin(kx - \omega t), \tag{12.19}$$

$$\epsilon = \epsilon_0 \cos[m(\theta - \theta_L)] \sin(kx - \omega t), \tag{12.20}$$

where M_0 and ϵ_0 are constants and m is the vertical wave number in isentropic coordinates. This trial solution satisfies the equations subject to the conditions

$$\epsilon_0 = -\left(\frac{m^2}{L^2} + \frac{k^2}{f_0^2}\right) M_0 \tag{12.21}$$

and

$$m = \frac{\pi}{\theta_U - \theta_L} \tag{12.22}$$

with the dispersion relation

$$\omega = -\frac{k\beta}{k^2 + f_0^2 m^2/L^2}. \tag{12.23}$$

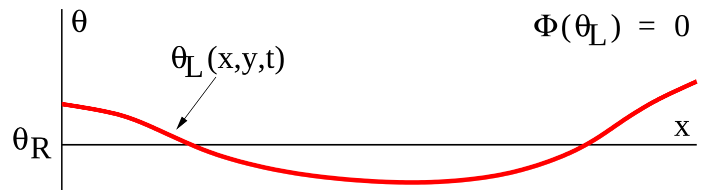

Figure 12.2: Illustration of lower boundary surface $\theta_L(x,y,t)$ where the geopotential is assumed to equal zero.

The family resemblance with the Rossby waves on a beta plane in shallow water flow is clear. The wave moves to the west with frequency increasing in magnitude until a critical wavenumber is reached. Beyond this point the frequency decreases with wavenumber and the group velocity is to the east.

Constant values of potential vorticity along isentropic surfaces corresponds to effectively having $\beta = 0$. In this case wave propagation does not occur.

12.3.2 Surface waves

A more interesting type of wave is one rooted in the north-south temperature gradient at the surface. A consequence of the surface potential temperature getting colder to the north is that a geostrophically balanced ambient wind shear $S_\theta > 0$ exists in the troposphere, with westerlies increasing with potential temperature. We assume that the surface wind is zero so that the Montgomery potential is constrained by equations (12.5) and (12.15):

$$M = -L^2(\theta - \theta_R)^2/2 - f_0[S_\theta(\theta - \theta_R)y + f_0 S_\theta^2 y^2/(2L^2)] + M^*. \tag{12.24}$$

The resulting inversion equation for the disturbance Montgomery potential is

$$\frac{1}{f_0^2}\nabla^2 M^* + \frac{1}{L^2}\frac{\partial^2 M^*}{\partial \theta^2} = 0 \tag{12.25}$$

subject to the lower boundary condition (see figure 12.2)

$$\Phi_L = -\theta_R\left(\frac{\partial M}{\partial \theta}\right)_L \approx -\theta_R\left[\left(\frac{\partial M}{\partial \theta}\right)_R + \left(\frac{\partial^2 M}{\partial \theta^2}\right)_R(\theta_L - \theta_R)\right] = 0. \tag{12.26}$$

The awkward condition on the θ derivative of M at the free lower boundary has been approximated by a condition on M at the constant reference level θ_R. Realizing that $\theta_L - \theta_R$ is small in magnitude and dropping the term $(\partial^2 M^*/\partial\theta^2)_R(\theta_L - \theta_R)$ since it is quadratic in small quantities, this can be further approximated using equation (12.24)

$$-f_0 S_\theta y + \left(\frac{\partial M^*}{\partial \theta}\right)_R - L^2(\theta_L - \theta_R) = 0. \tag{12.27}$$

Dividing the surface potential temperature perturbation into ambient and disturbance parts, $\theta_L = \theta_{L0} + \theta_L^*$, we note that the ambient potential temperature distribution at the surface satisfies

$$\theta_{L0} = \theta_R - f_0 S_\theta y/L^2 \tag{12.28}$$

while the surface boundary condition for the disturbance is

$$\left(\frac{\partial M^*}{\partial \theta}\right)_R = L^2\theta_L^*. \tag{12.29}$$

The lower boundary potential temperature governing equation evolves the system in this case:

$$\frac{\partial \theta_L^*}{\partial t} + \frac{1}{f_0}\frac{d\theta_{L0}}{dy}\left(\frac{\partial M^*}{\partial x}\right)_R = 0. \tag{12.30}$$

A trial plane wave solution which decays exponentially with height is

$$M^* = M_0 \exp[-\mu(\theta - \theta_R)] \sin(kx - \omega t) \tag{12.31}$$

$$\theta_L^* = \theta_0 \sin(kx - \omega t). \tag{12.32}$$

This trial solution works if

$$\mu = \frac{kL}{f_0}. \tag{12.33}$$

It has the dispersion relation

$$\omega = \frac{kS_\theta}{\mu} = f_0 S_\theta / L. \tag{12.34}$$

The ambient surface temperature anomaly (relative to θ_R) is equivalent in this case to a thin surface layer of potential vorticity which is positive for $\theta_{L0} > \theta_R$ and negative otherwise. It thus decreases to the north, and the resulting gradient in potential vorticity supports Rossby wave propagation. The sign of the gradient causes the wave propagation to be toward the east rather than toward the west as in previous cases. Notice that the frequency is a constant independent of wavenumber. The group velocity of this wave is thus $u_g = \partial\omega/\partial k = 0$, whereas the phase speed $c = f_0 S_\theta/(kL)$. Thus, waves of longer wavelength move more rapidly.

12.3.3 Tropopause waves

The potential vorticity increases drastically across the tropopause, due to the higher static stability in the stratosphere relative to the tropopsphere. If we define a horizontal reference level θ_R near the mean level of the tropopause, the variability in the height of the tropopause means that some regions above the reference level have tropospheric values of potential vorticity, while other regions below the reference level have stratospheric values, as illustrated in figure 12.3. If the deviations of the tropopause elevations from the reference level are not too great, we can consider these deviations to be concentrated at the reference level, resulting in a potential vorticity anomaly at this level given by

$$q' = -\Delta\theta(q_S - q_T)\delta(\theta - \theta_R), \tag{12.35}$$

where $\Delta\theta = \theta_T - \theta_R$, $\theta_T(x, y, t)$ being the height (in isentropic coordinates) of the tropopause as a function of position and time. We approximate $q_S = f_0/\eta_S$ and $q_T = f_0/\eta_T$ to be the planetary values of stratospheric and tropospheric potential vorticity at the reference

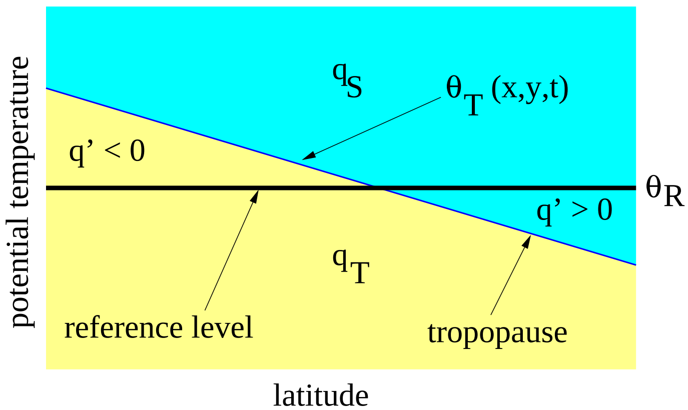

Figure 12.3: Sketch of tilted tropopause in isentropic coordinates relative to a reference level θ_R. The potential vorticity anomaly Δq in the triangular region above the reference level is negative, whereas it is positive in the triangular region below the reference level. This is because the stratospheric potential vorticity exceeds the tropospheric value, $q_S > q_T$.

latitude where $f = f_0$ and assume constant values of isentropic density η above and below the tropopause with $\eta_T > \eta_S$.

Since the potential vorticity thus takes on uniform, constant values above and below the tropopause, the evolution of the system is governed solely by the evolution of the potential vorticity perturbation at the tropopause, or in the above approximation, at the reference level. This reduces to an evolution equation for $\Delta\theta$, since all other factors in equation (12.35) are constant and drop out:

$$\frac{\partial \Delta\theta}{\partial t} + \mathbf{v}_T \cdot \nabla \Delta\theta = 0, \tag{12.36}$$

where $\mathbf{v}_T$ is the wind at the tropopause.

Just as surface Rossby waves are supported by the north-south gradient in surface potential temperature, tropopause Rossby waves are supported by the north-south gradient in tropopause potential temperature. However, the details differ slightly because the structure of the waves both above and below the tropopause is important.

Both the Montgomery potential and its θ derivative are continuous across the tropopause. This is because both the geopotential (proportional $\partial M/\partial\theta$) and isentropic density (proportional to $\partial^2 M/\partial\theta^2$) are expected to remain finite there. To make the analysis tractable, we wish to convert these conditions into interface conditions at the reference height θ_R, as in the surface case. We do this using Taylor series expansions of M and $\partial M/\partial\theta$ about the reference level. The condition that M be continuous at θ_T becomes

$$M_S(\theta_R) + \left(\frac{\partial M_S}{\partial \theta}\right)_R \Delta\theta = M_T(\theta_R) + \left(\frac{\partial M_T}{\partial \theta}\right)_R \Delta\theta, \tag{12.37}$$

where the subscripted S and T indicate stratospheric and tropospheric values. Similarly, the condition that $\partial M/\partial\theta$ be continuous there reduces to

$$\left(\frac{\partial M_S}{\partial \theta}\right)_R + \left(\frac{\partial^2 M_S}{\partial \theta^2}\right)_R \Delta\theta = \left(\frac{\partial M_T}{\partial \theta}\right)_R + \left(\frac{\partial^2 M_T}{\partial \theta^2}\right)_R \Delta\theta. \tag{12.38}$$

We the Montgomery potential into ambient and disturbance parts, with the latter being denoted $M^*_{S,T}$.

Taking the second condition first, we note that the ambient Montgomery potential in the stratosphere satisfies $(\partial^2 M_S/\partial\theta^2)_R = -g\eta_S/(\rho_R\theta_R) \equiv -L_S^2$ with a similar equation for the troposphere yielding L_T^2. Solving for the difference in $\partial M/\partial\theta$ values at the reference level and dropping disturbances contributions to $(\partial^2 M/\partial\theta^2)_R$ in this equation as was done for surface waves, we get

$$\left(\frac{\partial M_S}{\partial\theta}\right)_R - \left(\frac{\partial M_T}{\partial\theta}\right)_R = -(L_T^2 - L_S^2)\Delta\theta. \tag{12.39}$$

Thus, this difference is proportional to $\Delta\theta$, which in our usual linearization we shall treat as a small quantity. We further split $\Delta\theta$ into ambient and disturbance parts, $\Delta\theta = \Delta\theta_0 + \Delta\theta^*$.

Equation (12.37) can be solved for $M_S - M_T$, from which we find that this quantity is proportional to $\Delta\theta^2$. In the linearization we set such terms to zero, so we have

$$M_S(\theta_R) - M_T(\theta_R) = 0 \tag{12.40}$$

to first order.

We now postulate trial solutions for the Montgomery potential in the troposphere and the stratosphere,

$$M_T = -L_T^2(\theta-\theta_R)^2/2 - f_0\int U_T dy + M_T^*, \quad \theta < \theta_T \tag{12.41}$$

$$M_S = -L_S^2(\theta-\theta_R)^2/2 - f_0\int U_S dy + M_S^*, \quad \theta > \theta_T \tag{12.42}$$

where $U_T(\theta, y)$ and $U_S(\theta, y)$ represent ambient wind patterns in the troposphere and the stratosphere. We assume that the ambient wind in each case has linear shears in the θ and y directions:

$$U_T = U_R + S_{\theta T}(\theta-\theta_R) + S_{yT}y, \tag{12.43}$$

$$U_S = U_R + S_{\theta S}(\theta-\theta_R) + S_{yS}y. \tag{12.44}$$

In order to make equation (12.36) solvable, we insist that the ambient flow at the tropopause be independent of y. Letting $\theta - \theta_R$ equal the ambient potential temperature perturbation at the tropopause in equations (12.43) and (12.44) and setting the y derivatives of U_T and U_S to zero results in

$$\frac{\partial\Delta\theta_0}{\partial y} = -\frac{S_{yT}}{S_{\theta T}} = -\frac{S_{yS}}{S_{\theta S}} = -\frac{\Delta S_y}{\Delta S_\theta} \tag{12.45}$$

where $\Delta\theta_0$ is the ambient part of $\Delta\theta$ and where

$$\Delta S_y = S_{yT} - S_{yS} \qquad \Delta S_\theta = S_{\theta T} - S_{\theta S}. \tag{12.46}$$

As in the case of the surface wave, we assume that the wave amplitude decays exponentially away from the tropopause in both the troposphere and the stratosphere, giving us the following form for the disturbance trial solution:

$$M_T^* = M_0 \sin(kx - \omega t)\exp[\mu_T(\theta-\theta_R)], \quad \theta < \theta_R \tag{12.47}$$

$$M_S^* = M_0 \sin(kx - \omega t) \exp[-\mu_S(\theta - \theta_R)], \quad \theta > \theta_R. \tag{12.48}$$

Substituting equations (12.41) and (12.42) plus these results into equation (12.39) and solving for $\Delta\theta$ yields

$$\Delta\theta = \Delta\theta_0 + \Delta\theta^* = \frac{-f_0 \Delta S_\theta y + (\mu_S + \mu_T) M^*}{L_T^2 - L_S^2}, \tag{12.49}$$

where M^* is the common value of M_S^* and M_T^* evaluated at $\theta = \theta_R$. Since $L_T^2 > L_S^2$, the ambient north-south gradient in $\Delta\theta_0$ is negative, with $\partial\Delta\theta_0/\partial y = -f_0\Delta S_\theta/(L_T^2 - L_S^2)$. Combining this with equation (12.45) results in a constraint on ΔS_y:

$$\Delta S_y = \frac{f_0(\Delta S_\theta)^2}{L_T^2 - L_S^2}. \tag{12.50}$$

The disturbance-related parts of $\Delta\theta$ and M are related by

$$\Delta\theta^* = \frac{(\mu_S + \mu_T)M^*}{L_T^2 - L_S^2}. \tag{12.51}$$

The values of μ_S and μ_T are determined by substituting the trial solutions for M_S^* and M_T^* into the inversion equation (12.25) with η_R replaced respectively by η_S and η_T for the stratospheric and tropospheric cases. The results are

$$\mu_S = \frac{kL_S}{f_0} \qquad \mu_T = \frac{kL_T}{f_0}. \tag{12.52}$$

Finally we substitute $\Delta\theta$ from equation (12.49) into equation (12.36) and linearizing in starred quantities:

$$\frac{\partial\Delta\theta^*}{\partial t} + U_R\frac{\partial\Delta\theta^*}{\partial x} - \frac{\Delta S_\theta}{L_T^2 - L_S^2}\frac{\partial M^*}{\partial x} = 0, \tag{12.53}$$

where $U_R = U_T(\theta_R) = U_S(\theta_R)$. Using equations (12.47), (12.48), and (12.49) and assuming space and time dependence $\sin(kx - \omega t)$ finally results in the dispersion relation for these waves:

$$\omega = kU_R - \frac{f_0\Delta S_\theta}{L_S + L_T}. \tag{12.54}$$

This Rossby wave has very similar characteristics to the surface wave, in that the group velocity of the wave equals the speed of the wind at the surface, in this case the tropopause. The main difference is that the phase speed of the wave is to the west relative to the mean flow at the tropopause, whereas the phase speed is to the east in the case of the surface wave. The phase speeds relative to the wind are quantitatively the same (except for sign) as in the surface wave if $L_T \gg L_S$ and $|S_{\theta T}| \gg |S_{\theta S}|$.

12.4 Baroclinic instability

In the last two sections, the interaction of the surface wave with the tropopause and the interaction of the tropopause wave with the surface were ignored. In the atmosphere these

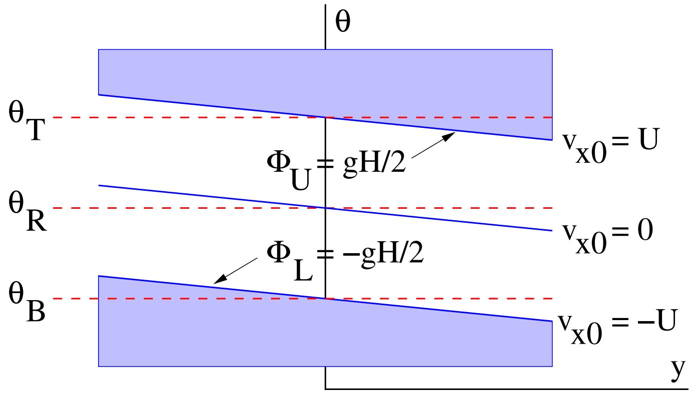

Figure 12.4: Idealized baroclinic environment in the Boussinesq approximation. The shaded areas indicate the bounding surfaces. The tilt downward with y of the geopotential surfaces in isentropic coordinates is geostrophically related to the increase in the x component of the anbient wind with height.

interactions are quite important, and we now investigate a model in which they are considered. In order to make the analysis easier, the tropopause is replaced by an upper rigid boundary, since the tropopause acts like such a boundary in the limit of a highly stable stratosphere.

In the ambient flow the two boundaries are assumed to have the same isentropic gradient of geopotential with latitude, as illustrated in figure 12.4. We further assume an f-plane with uniform vertical S_θ and horizontal S_y shear in isentropic coordinates, so that the ambient flow takes the form

$$U = S_\theta(\theta - \theta_R) + S_y y. \tag{12.55}$$

Note that the ambient flow is taken to be zero at $\theta = \theta_R$ and $y = 0$. Further assuming a constant ambient isentropic density η_R, the ambient Montgomery potential consistent with this wind is

$$M_0 = -L^2(\theta - \theta_R)^2/2 - fS_\theta(\theta - \theta_R)y - fS_y y^2/2 \tag{12.56}$$

where $L^2 = g\eta_R/(\rho_r\theta_R)$ as usual. The ambient Boussinesq geopotential is easily obtained as well:

$$\Phi_0 = -\theta_R\frac{\partial M_0}{\partial\theta} = \theta_R L^2(\theta - \theta_R) + f\theta_R S_\theta y. \tag{12.57}$$

The ambient potential vorticity takes the simple form

$$q_0 = \frac{f - S_y}{\eta_R}. \tag{12.58}$$

Since this is constant, the potential vorticity advection equation is trivially satisfied, which means that system evolution governed by the advection of the potential temperature on the upper and lower surfaces.

The ambient y-gradient of potential temperature on these surfaces (which have constant geopotential) may be obtained by differentiating equation (12.57) with respect to y: $\partial\theta/\partial y =$

$-fS_\theta/L^2$. The ambient potential temperature distributions on the upper and lower surfaces are therefore

$$\theta_{U0} = \theta_T - (fS_\theta/L^2)y \tag{12.59}$$

$$\theta_{L0} = \theta_B - (fS_\theta/L^2)y \tag{12.60}$$

where θ_T and θ_B are constant reference potential temperatures for the top and bottom surfaces (see figure 12.4). We demand that the ambient wind at the top surface be $U_0 = S_\theta(\theta_T - \theta_B)/2$ and $-U_0$ at the bottom surface for all y. Thus, in geometrical coordinates the ambient wind profile is independent of height. As shown in equation (12.15), this results in the constraint $S_y = fS_\theta^2/L^2$.

As in the last two sections, potential vorticity inversion is governed by equation (12.25), which yields real exponential behavior of the disturbance Montgomery potential M^* in θ when space and time dependence of the form $\exp[i(kx - \omega t)]$ is assumed. The form

$$M^* = M_T \exp[\mu(\theta - \theta_T)] + M_B \exp[-\mu(\theta - \theta_B)] \tag{12.61}$$

includes both upper and lower surface waves, where $\mu = kL/f$ as before, and is subject to the boundary conditions

$$\left(\frac{\partial M^*}{\partial \theta}\right)_T = L^2\theta_U^* \qquad \left(\frac{\partial M^*}{\partial \theta}\right)_B = L^2\theta_L^* \tag{12.62}$$

as in the single surface wave case. The quantities θ_U^* and θ_L^* are the disturbance potential temperature perturbations at the upper and lower surfaces. Applying these boundary conditions yields

$$\mu[M_T - M_B \exp(-\mu\Delta\theta)] = L^2\theta_U^* \tag{12.63}$$

and

$$\mu[M_T \exp(-\mu\Delta\theta) - M_B] = L^2\theta_L^* \tag{12.64}$$

where $\Delta\theta = \theta_T - \theta_B$.

The linearized temperature advection equations on the upper and lower surfaces are easily found to be

$$\frac{\partial \theta_U^*}{\partial t} + U_0\frac{\partial \theta_U^*}{\partial x} - \frac{2U_0}{L^2\Delta\theta}\left(\frac{\partial M^*}{\partial x}\right)_T = 0 \tag{12.65}$$

and

$$\frac{\partial \theta_L^*}{\partial t} - U_0\frac{\partial \theta_L^*}{\partial x} - \frac{2U_0}{L^2\Delta\theta}\left(\frac{\partial M^*}{\partial x}\right)_B = 0. \tag{12.66}$$

Substituting $\exp[i(kx - \omega t)]$ as before yields

$$(c - U_0)\theta_U^* + \frac{2U_0}{L^2\Delta\theta}[M_T + M_B \exp(-\mu\Delta\theta)] = 0 \tag{12.67}$$

and

$$(c + U_0)\theta_L^* + \frac{2U_0}{L^2\Delta\theta}[M_T \exp(-\mu\Delta\theta) + M_B] = 0 \tag{12.68}$$

where we define the disturbance's phase speed $c = \omega/k$.

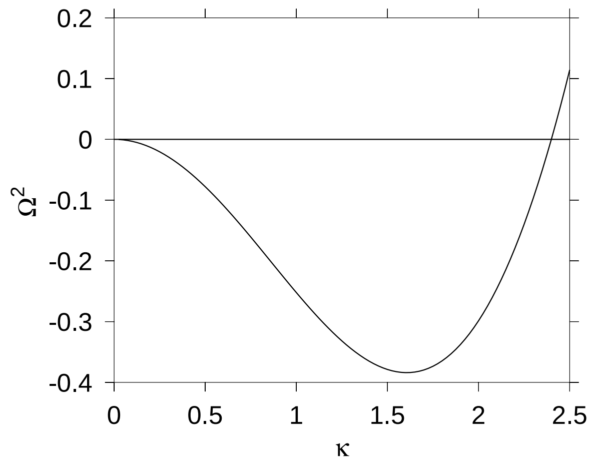

Figure 12.5: Plot of equation (12.69), dispersion relation for baroclinic instability. Instability occurs when $\Omega^2 < 0$, which is confined to the approximate range $0 < \kappa < 2.4$. The maximum growth rate occurs near $\kappa = 1.6$, at which point $|\Omega| = 0.62$. For $\kappa > 2.4$, a steady propagating wave occurs.

Equations (12.63), (12.64), (12.67), and (12.68) constitute a set of four linear, homogeneous equations in the four unknowns θ_U^*, θ_L^*, M_T, and M_B. Setting the determinant of the coefficients to zero to obtain the dispersion relation yields

$$\Omega^2 = -\frac{(2+\kappa)^2 \exp(-2\kappa) - (2-\kappa)^2}{1 - \exp(-2\kappa)} \tag{12.69}$$

where the frequency and wavenumber have been non-dimensionalized:

$$\Omega = \frac{L\Delta\theta}{fU_0}\omega \qquad \kappa = \frac{L\Delta\theta}{f}k. \tag{12.70}$$

With typical mid-latitude values $L = 3.7$ m s^{-1} K^{-1}, $\Delta\theta = 45$ K, $f = 10^{-4}$ s^{-1}, and $U_0 = 20$ m s^{-1}, we calculate a scaling length $L\Delta\theta/f = 1700$ km and a scaling time $L\Delta\theta/(fU_0) = 1$ d.

Figure 12.5 shows the dispersion relation for the baroclinic mode we are studying. In the region where $\Omega^2 < 0$, we can write $\omega = \pm i\sigma$ where σ is real. In this range the time evolution of the mode takes the form $\exp(-i\omega t) = \exp(\pm\sigma t)$. The mode is stationary in the reference frame we have used, and it either grows or decays in amplitude exponentially with time, depending on the sign chosen. Modes of this type are called instabilities, as small perturbations on the ambient flow will ultimately grow in amplitude until their further growth is limited by factors not accounted for in the linear analysis. The fastest growing mode should eventually dominate if the initial perturbations are small. As figure 12.5 shows, this occurs when $\kappa = 1.6$, which for the case of the example discussed above corresponds to a wavelength near 6700 km.

12.5 Problems

1. Show that if the ambient zonal wind has the form $U = C(\theta - \theta_R)^2$ rather than the form shown in equation (12.4), the ambient isentropic density cannot be a constant independent of y.

2. Redo the analysis of section 12.3.1 for the case in which the wave is confined to a east-west channel of width w, as was done in the case of shallow water Rossby waves caused by the beta effect. Compare the two solutions.