Chapter 7 Isopycnal and Isentropic Coordinates

The two-dimensional shallow water model can carry us only so far in geophysical fluid dynamics. In this chapter we begin to investigate fully three-dimensional phenomena in geophysical fluids using a type of model which builds on the insights obtained using the shallow water model. This is done by treating a three-dimensional fluid as a stack of layers, each of constant density (the ocean) or constant potential temperature (the atmosphere). Equations similar to the shallow water equations apply to each layer, and the layer variable (density or potential temperature) becomes the vertical coordinate of the model. This is feasible because the requirement of convective stability requires this variable to be monotonic with geometric height, decreasing with height in the case of water density in the ocean, and increasing with height with potential temperature in the atmosphere. As long as the slope of model layers remains small compared to unity, the coordinate axes remain close enough to orthogonal to ignore the complex correction terms which otherwise appear in non-orthogonal coordinate systems.

7.1 Isopycnal model for the ocean

The word *isopycnal* means "constant density". Recall that an assumption behind the shallow water equations is that the water have uniform density. For layered models of a threedimensional, incompressible fluid, we similarly assume that each layer is of uniform density. We now see how the momentum, continuity, and hydrostatic equations appear in the context of an isopycnal model.

7.1.1 Momentum equation

Recall that the horizontal (in terms of z) pressure gradient must be calculated, since it appears in the horizontal momentum equations. Figure 7.1 shows how to calculate the horizontal pressure gradient in terms of the pressure gradient taken parallel to the sloping isopycnal layer. In particular, note that

$$\left(\frac{\partial p}{\partial x}\right)_z = \frac{p_2 - p_1}{\Delta x},\tag{7.1}$$



Figure 7.1: Sketch to determine isopycnal pressure gradient. Points 1 and 3 have the same density, whereas points 1 and 2 have the same value of z.

whereas

$$\left(\frac{\partial p}{\partial x}\right)_{\rho} = \frac{p_3 - p_1}{\Delta x},\tag{7.2}$$

where the subscripted z and ρ indicate respectively that the geometric height and the density are held constant.

The hydrostatic equation tells us that

$$p_2 - p_3 = g\rho\Delta z. \tag{7.3}$$

Combining equations (7.1), (7.2), and (7.3) results in

$$\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)_z = \frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)_\rho + \frac{\partial \Phi}{\partial x},\tag{7.4}$$

where

$$\Phi = gz \tag{7.5}$$

is called the *geopotential*.

We note that the density can be moved inside the x derivative in the first term on the right side of equation (7.4), since the derivative is taken while holding ρ constant. Equation (7.4) can therefore be rewritten

$$\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)_z = \left(\frac{\partial M}{\partial x} \right)_\rho, \tag{7.6}$$

where the Montgomery potential is defined

$$M = p/\rho + \Phi. \tag{7.7}$$

Henceforth we drop the subscripted ρ , with the understanding that unlabelled x and y partial derivatives implicitly hold the density rather than the geometrical height constant. Based on our derivation of the momentum equation in the shallow water case, we have the momentum equation for the isopycnal coordinate system

$$\frac{d\mathbf{v}}{dt} + \boldsymbol{\nabla}M + f\hat{\mathbf{z}} \times \mathbf{v} = \mathbf{F}$$
(7.8)

where \mathbf{F} is an externally imposed force per unit mass.

In isopycnal coordinates the total time derivative has the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + S_{\rho} \frac{\partial}{\partial \rho}$$
(7.9)

where $S_{\rho} = d\rho/dt$ is the vertical velocity of isopycnal coordinates. This is time rate of change of density of a parcel, which is mainly related to changes in temperature and salinity of the parcel. When these changes are absent, there is no vertical motion in isopycnal coordinates. (Vertical motion still occurs in geometric or pressure coordinates as isopycnal surfaces move up and down or as parcels slide up and down tilted isopycnal surfaces.)

7.1.2 Continuity equation

As derived previously in the shallow water case, the mass continuity equation for a layer of finite thickness h is

$$\frac{\partial \rho h}{\partial t} + \boldsymbol{\nabla} \cdot (\rho h \mathbf{v}) = 0. \tag{7.10}$$

In the ocean the layer refers to water between upper and lower surfaces of constant density ρ_1 and ρ_2 , with density difference between the lower and upper surfaces equal to $\Delta \rho = \rho_2 - \rho_1$. In a stably stratified ocean, the upper surface density is less than the lower surface density and hence $\Delta \rho > 0$. This equation comes from relating the flow of mass in and out of a stationary test volume to the time rate of change of mass in the volume.

The mean density ρ in the layer, which was originally included in the derivation of the mass continuity equation but canceled out, has been reinserted. Its variability becomes important when we generalize to the atmosphere, where the upper and lower layers have constant potential temperature rather than density. The density in such a layer in the atmosphere does not remain constant, and therefore cannot be factored out and canceled.

As we intend ultimately to let the layer thickness tend to zero and increase the number of layers to infinity, expressing mass continuity in terms of the thickness of the layer is not satisfactory. However, the layer thickness can be represented as $h = -\Delta\rho(\partial z/\partial\rho)$ where $\Delta\rho$ is the constant density difference across each layer. The minus sign comes from the realization that $\partial z/\partial\rho$ is negative in a stably stratified ocean. Substituting this in equation (7.10) and canceling the constant factor $\Delta\rho$ results in

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla} \cdot (\eta \mathbf{v}) = 0 \tag{7.11}$$

where the density in isopycnal space is

$$\eta = -\rho \frac{\partial z}{\partial \rho} = -\frac{\rho}{g} \frac{\partial \Phi}{\partial \rho}.$$
(7.12)

However, this is not the complete story. If S_{ρ} is non-zero, then there is a vertical flux of mass in and out of the test volume in isopycnal coordinates. The vertical mass flux is ηS_{ρ} , and incorporating the fluxes through the top and bottom surfaces of the test volume results in the extended equation

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla} \cdot (\eta \mathbf{v}) + \frac{\partial \eta S_{\rho}}{\partial \rho} = 0.$$
(7.13)

7.1.3 Density-potential relation

We lack only a relationship between the density in isopycnal space and the Montgomery potential to complete our system of governing equations for the ocean in isopycnal coordinates.

We start with the standard hydrostatic equation

$$\frac{\partial p}{\partial z} = -g\rho \tag{7.14}$$

and change independent variables via the chain rule: $\partial p/\partial z = (\partial p/\partial \rho)(\partial \rho/\partial z)$. Using $\partial \rho/\partial z = (\partial z/\partial \rho)^{-1}$, we then get

$$\frac{\partial p}{\partial \rho} = -g\rho \frac{\partial z}{\partial \rho} = -\rho \frac{\partial \Phi}{\partial \rho} = -\frac{\partial \rho \Phi}{\partial \rho} + \Phi.$$
(7.15)

Moving the derivative of $\rho\Phi$ to the left side of the equation and using the definition of Mongomery potential (7.7), we note that the hydrostatic equation takes the simple form

$$\Phi = \frac{\partial \rho M}{\partial \rho} \tag{7.16}$$

in isopycnal coordinates. Combining this with equation (7.12) yields

$$\eta = -\frac{\rho}{g} \frac{\partial^2 \rho M}{\partial \rho^2},\tag{7.17}$$

which is the desired result. Equations (7.8), (7.11), and (7.17) form the complete set of governing equations for isopycnal coordinates, with auxiliary definitions and relations (7.5), (7.7), (7.12), and (7.16).

7.2 Isentropic model for the atmosphere

In the ocean the density is used as the vertical coordinate because water is (almost) incompressible and the density is therefore effectively conserved for many purposes. In the atmosphere the dry entropy or potential temperature plays a similar role. In the atmosphere we use the potential temperature as the vertical coordinate, giving us *isentropic coordinates*. This is by far the simplest coordinate system in which to investigate large-scale atmospheric phenomena.

7.2.1 Momentum equation

Recalling that $dp/\rho = \theta d\Pi$ where $\Pi = C_p (p/p_R)^{\kappa}$ is the Exner function and θ is the potential temperature, and using the same trick employed to get the horizontal pressure gradient in the isopycnal coordinates of the ocean, we find that

$$\theta \left(\frac{\partial \Pi}{\partial x}\right)_{z} = \theta \left(\frac{\partial \Pi}{\partial x}\right)_{\theta} + \left(\frac{\partial \Phi}{\partial x}\right)_{\theta} = \frac{\partial}{\partial x}(\theta \Pi + \Phi)_{\theta} = \left(\frac{\partial M}{\partial x}\right)_{\theta}$$
(7.18)

where

$$M = \theta \Pi + \Phi \tag{7.19}$$

is the isentropic coordinate Montgomery potential and

$$\Phi = gz \tag{7.20}$$

is the geopotential as before. The momentum equation thus becomes

$$\frac{d\mathbf{v}}{dt} + \boldsymbol{\nabla}M + f\hat{\mathbf{z}} \times \mathbf{v} = \mathbf{F}$$
(7.21)

as in isopycnal coordinates.

The total time derivative in isentropic coordinates is similar to that in isopycnal coordinates

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + S_{\theta} \frac{\partial}{\partial \theta}$$
(7.22)

where $S_{\theta} = d\theta/dt$ is the heating rate of a parcel. This heating is generally the result of solar or thermal infrared radiation or latent heat release. As in isopycnal coordinates, there is no vertical motion in isentropic coordinates if no parcel heating occurs.

7.2.2 Continuity equation

The isentropic density in the atmosphere is defined

$$\eta = \rho \frac{\partial z}{\partial \theta} = \frac{\rho}{g} \frac{\partial \Phi}{\partial \theta}.$$
(7.23)

By analogy with the isopycnal case, the mass continuity equation becomes

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla} \cdot (\eta \mathbf{v}) + \frac{\partial \eta S_{\theta}}{\partial \theta} = 0.$$
(7.24)

7.2.3 Density-potential relation

Given the definition of isentropic density by equation (7.23), we can rewrite the standard hydrostatic equation $\partial p/\partial z = -g\rho$ in isentropic coordinates as

$$\frac{\partial p}{\partial \theta} = -g\eta. \tag{7.25}$$

Taking the θ derivative of the Montgomery potential and using the Exner function form of the hydrostatic equation $\theta(\partial \Pi/\partial z) = -g$ gives us the identity

$$\frac{\partial M}{\partial \theta} = \Pi. \tag{7.26}$$

Finally, writing the pressure in terms of the Exner function $p = p_R (\Pi/C_p)^{1/\kappa}$ and combining equations (7.25) and (7.26) results in a relation between the isentropic density and the Montgomery potential:

$$\eta = -\frac{\partial}{\partial\theta} \left[\frac{p_R}{g} \left(\frac{1}{C_p} \frac{\partial M}{\partial\theta} \right)^{1/\kappa} \right].$$
(7.27)

In contrast to the isopycnal coordinate case, this relationship is nonlinear.

Equation (7.26) and the definition of Montgomery potential may also be used to derive an equation for the geopotential:

$$\Phi = M - \theta \frac{\partial M}{\partial \theta} = -\theta^2 \frac{\partial}{\partial \theta} \left(\frac{M}{\theta}\right).$$
(7.28)

The definition of Exner function $\Pi = C_p (p/p_R)^{\kappa}$ plus that of potential temperature $\theta = T(p_R/p)^{\kappa}$ tells us that the Exner function can also be written $\Pi = C_p T/\theta$, from which we can use equation (7.26) to derive the relationship between Montgomery potential and temperature:

$$T = \frac{\theta}{C_p} \frac{\partial M}{\partial \theta}.$$
(7.29)

Figure 7.2 shows a typical tropical sounding in isentropic coordinates. This particular sounding is a three-week average taken at 95° W, 10° N during September 2001 (Raymond et al., 2004). Note how η diminishes above 350 K. This decrease represents the tropopause, with the troposphere below and the stratosphere above. The isentropic density η increases dramatically at the lowest levels due to the fact that the atmosphere tends to be near-neutrally stable there, which means that $\partial\theta/\partial z = \rho/\eta$ is small. The atmosphere has small static stability just below the tropopause as well, i. e., near $\theta = 345$ K in the tropics.

7.3 Geostrophic wind

Geostrophic balance comes from setting the acceleration term and the external force to zero in the momentum equation, as in the shallow water case, resulting in

$$(v_{gx}, v_{gy}) = \left(-\frac{1}{f}\frac{\partial M}{\partial y}, \frac{1}{f}\frac{\partial M}{\partial x}\right)$$
(7.30)

which is similar to the shallow water results except that gh is replaced by the Montgomery potential M.

For the atmosphere we get the so called thermal wind equations by taking the θ derivative of the geostrophic balance equations and using equation (7.26):

$$\left(\frac{\partial v_{gx}}{\partial \theta}, \frac{\partial v_{gy}}{\partial \theta}\right) = \left(-\frac{1}{f}\frac{\partial \Pi}{\partial y}, \frac{1}{f}\frac{\partial \Pi}{\partial x}\right).$$
(7.31)

Thus, the westerly wind increases with height in the northern hemisphere if surfaces of constant Exner function (or pressure) tilt down toward the north in isentropic coordinates, such that $\partial \Pi / \partial y < 0$. Similar relations exist in the ocean.

7.4 Potential vorticity

The circulation theorem leads us to the potential vorticity in both the oceanic and atmospheric cases. In the case of no friction or heating, the circulation theorem in three dimensions



Figure 7.2: Typical tropical oceanic sounding in isentropic coordinates. Isentropic density η , Exner function Π , geopotential Φ , and Montgomery potential M as a function of potential temperature θ .



Figure 7.3: Cylindrical pillbox with ends of area A separated by distance Δz . The ends coincide with constant potential temperature surfaces, so that the circulation loop Γ is confined to a constant potential temperature surface as well.

gives us

$$\frac{d\Gamma}{dt} = -\oint \frac{dp}{\rho} = -\oint \theta d\Pi \tag{7.32}$$

where we recall that Π is the Exner function. The first form applies to the ocean, the second to the atmosphere. If the circulation loop is confined to a layer of constant density (ocean) or potential temperature (atmosphere) then we have closed line integrals of perfect differentials and we are left in both cases with

$$\frac{d\Gamma}{dt} = 0. \tag{7.33}$$

Using arguments similar to those made for shallow water flow, this equation reduces to

$$\frac{dq}{dt} = 0 \tag{7.34}$$

where the potential vorticity is defined

$$q = \frac{\zeta_a}{\eta} \tag{7.35}$$

for both isopycnal and isentropic coordinates, where

$$\zeta_a = f + \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{7.36}$$

is the absolute vorticity.

7.4.1 Potential vorticity evolution

We now derive the potential vorticity evolution equation in isentropic coordinates for the case with frictional forces and heating. We begin with the identity

$$\zeta_z \hat{\mathbf{z}} \times \mathbf{v} = \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} - \boldsymbol{\nabla} (v^2/2), \qquad (7.37)$$

where ζ_z is the vertical component of relative vorticity, from which we rewrite the momentum equation (7.21) as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{\nabla} (M + v^2/2) + S_\theta \frac{\partial \mathbf{v}}{\partial \theta} + \zeta_a \hat{\mathbf{z}} \times \mathbf{v} = \mathbf{F}.$$
(7.38)

Taking the vertical component of the curl of this equation results in the flux form of the vorticity equation

$$\frac{\partial \zeta_a}{\partial t} + \boldsymbol{\nabla} \cdot (\zeta_a \mathbf{v} + \mathbf{Z}) = 0 \tag{7.39}$$

where

$$\mathbf{Z} = \left(S_{\theta} \frac{\partial v_y}{\partial \theta} - F_y, -S_{\theta} \frac{\partial v_x}{\partial \theta} + F_x\right).$$
(7.40)

This equation shows that the vorticity tendency equals minus the horizontal divergence of the advective flux of vorticity $\zeta_z \mathbf{v}$ and a non-advective flux \mathbf{Z} , which only occurs when there is friction or heating in vertical shear of the horizontal wind.

Substitution of $\zeta_a = \eta q$ (see equation (7.35)) results in the flux form of the potential vorticity equation

$$\frac{\partial \eta q}{\partial t} + \boldsymbol{\nabla} \cdot (\eta q \mathbf{v} + \mathbf{Z}) = 0, \qquad (7.41)$$

which in combination with the mass continuity equation (7.24) can be reduced to an advective form:

$$\frac{dq}{dt} = \zeta_a \frac{\partial(\eta S_\theta)}{\partial \theta} + \frac{\hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times \mathbf{Z}}{\eta}.$$
(7.42)

Haynes and McIntyre (1990) make an analogy between the potential vorticity q and the mixing ratio of some substance such as water vapor. In this analogy $\eta q = \zeta_a$ is the density of *potential vorticity substance* just as ηr is the density of water substance (r is the mixing ratio of water vapor). Integration of equation (7.41) over a volume in isentropic space bounded by two isentropic surfaces shows that

$$\int \eta q dV = 0, \tag{7.43}$$

i. e., the amount of potential vorticity substance between these two layers never changes. (The integral is assumed to be carried out horizontally to a point where the horizontal flux of potential vorticity is zero.) This is called the *impermeability theorem*. In reality, it is just a different way of looking at the circulation theorem.

7.4.2 Potential vorticity inversion

Approximating velocities in the definition of potential vorticity given by equation (7.35) by geostrophic velocities and eliminating η with equation (7.27) results in the potential vorticity inversion equation

$$f + \frac{\partial}{\partial x} \left(\frac{1}{f} \frac{\partial M}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{f} \frac{\partial M}{\partial y} \right) + q \frac{\partial}{\partial \theta} \left[\frac{p_R}{g} \left(\frac{1}{C_p} \frac{\partial M}{\partial \theta} \right)^{1/\kappa} \right] = 0.$$
(7.44)

Given the potential vorticity distribution $q(x, y, \theta, t)$, this equation may in principle be solved for the Montgomery potential, from which the isentropic density (7.27), Exner function (7.26), geopotential (7.28), and geostrophic wind (7.30) may be derived. A similar equation exists for isopycnal coordinates. However, note that whereas the isopycnal coordinate inversion equation is linear, the isentropic equivalent is nonlinear, and thus more difficult to solve exactly.

7.4.3 Ageostrophic flow

Once the isentropic density η and the geostrophic flow \mathbf{v}_g are diagnosed from the Montgomery potential, the mass continuity equation can be used to determine the ageostrophic flow. Since the geostrophic flow is primarily rotational, we assume that the ageostrophic flow is irrotational and divergent, and thus derivable from a velocity potential χ , $\mathbf{v}_a = -\nabla \chi$. Substituting into the mass continuity equation (7.24) and realizing that $\nabla \cdot \mathbf{v}_g = -\beta v_{gy}/f$ where $\beta = df/dy$ results in a diagnostic equation for χ :

$$\eta \nabla^2 \chi + \boldsymbol{\nabla} \eta \cdot \boldsymbol{\nabla} \chi = \frac{\partial \eta}{\partial t} + \mathbf{v}_g \cdot \nabla \eta + \frac{\partial \eta S_\theta}{\partial \theta} - \frac{\eta \beta v_{gy}}{f}.$$
 (7.45)

This is in the form of a modified Poisson equation, with everything on the right side known. The time derivative can be approximated by backward differencing in this case.

7.5 Boundary conditions

As long as the Coriolis parameter f and the potential vorticity q are of the same sign, equation (7.44) is elliptic, with solutions in the interior of the domain governed by conditions on the boundary. Of the boundary conditions, the most important are those at the upper and lower surfaces. At the lower surface, g times the surface elevation d(x, y) must equal the geopotential at the potential temperature of the air in contact with the surface $\theta_B(x, y, t)$:

$$gd(x,y) = \Phi(\theta_B). \tag{7.46}$$

The potential temperature at the lower surface θ_B is simply advected along the surface unless there is a surface potential temperature source $S_{\theta B}$:

$$\frac{\partial \theta_B}{\partial t} + \mathbf{v}_B \cdot \boldsymbol{\nabla} \theta_B = S_{\theta B}, \tag{7.47}$$

where \mathbf{v}_B is the flow velocity at the lower boundary.

We assume that the pressure, and hence the Exner function, are zero at the upper boundary. If the potential temperature θ_T is known there, then

$$\Pi(\theta_T) = \left(\frac{\partial M}{\partial \theta}\right)_T = 0. \tag{7.48}$$

Often it is sufficient to assume that θ_T is a constant independent of space and time at the upper boundary. Otherwise θ_T obeys an equation analogous to equation (7.47).

7.6 Linearization in isentropic coordinates

Equations (7.44), (7.45), and (7.46) are nonlinear, and therefore difficult to solve. Linearization allows small amplitude solutions to be obtained. Let us assume that $M = M_0(\theta) + M'$ where $M_0(\theta)$ is a reference profile of Montgomery potential, and define similar splits in other dependent variables. Reference profiles of geopotential $\Phi_0(\theta)$, Exner function $\Pi_0(\theta)$, temperature $T_0(\theta)$, and isentropic density $\eta_0(\theta)$ consistent with $M_0(\theta)$ are obtained by using equations (7.28), (7.26), (7.29), and (7.27) respectively. We also split the potential vorticity into a reference profile plus a perturbation, $q = q_0(\theta) + q'$ where $q_0 = f/\eta_0$.

The linearizations used here leave the momentum equation (7.21) unchanged. However, the mass continuity equation (7.24) becomes

$$\frac{\partial \eta'}{\partial t} + \mathbf{v} \cdot \nabla \eta' + \eta_0 \nabla \cdot \mathbf{v} + \frac{\partial \eta_0 S_\theta}{\partial \theta} = 0$$
(7.49)

on the assumption that the isentropic vertical velocity S_{θ} is also a small perturbation quantity.

The split of isentropic density into reference and linear perturbation parts is particularly important to the linearized potential vorticity inversion equation. Linearization of equation (7.27) results in a relation between the perturbation density and Montgomery potential

$$\eta' = -\frac{\partial}{\partial\theta} \left(a_0 \frac{\partial M'}{\partial\theta} \right) \tag{7.50}$$

where

$$a_0(\theta) = \frac{p_R}{gR} \left(\frac{\Pi_0(\theta)}{C_p}\right)^{1/\kappa - 1} = \frac{p_R}{gR} \left(\frac{T_0(\theta)}{\theta}\right)^{1/\kappa - 1} = \frac{p_R}{gR} \left(\frac{p}{p_R}\right)^{1-\kappa}.$$
 (7.51)

Analogous perturbation relations exist for geopotential

$$\Phi' = \theta^2 \frac{\partial}{\partial \theta} \left(\frac{M'}{\theta} \right) \tag{7.52}$$

and Exner function

$$\Pi' = \frac{\partial M'}{\partial \theta}.\tag{7.53}$$

The linearized form of the potential vorticity inversion equation (7.44) is thus

$$\frac{\partial}{\partial x} \left(\frac{1}{f} \frac{\partial M'}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{f} \frac{\partial M'}{\partial y} \right) + q_0 \frac{\partial}{\partial \theta} \left(a_0 \frac{\partial M'}{\partial \theta} \right) = \eta_0 q' \tag{7.54}$$

This is a generalized three-dimensional Poisson equation for M' in terms of q' with potentially variable coefficients. The velocity potential equation derives from equation (7.49) and simplifies to a two-dimensional Poisson equation

$$\eta_0 \nabla^2 \chi = \frac{\partial \eta'}{\partial t} + \frac{\partial \eta_0 S_\theta}{\partial \theta} - \frac{\eta_0 \beta v_{gy}}{f}.$$
(7.55)

The lower boundary condition (7.46) is also nonlinear. In linearizing this equation we define θ_R as a constant reference potential temperature obeying the implicit equation

$$\Phi_0(\theta_R) = 0. \tag{7.56}$$

We split the lower surface temperature into reference and perturbation parts $\theta_B(x, y, t) = \theta_R + \theta'_B(x, y, t)$, substitute this into equation (7.46), and use a Taylor series expansion to approximate quantities at $\theta = \theta_B$ in terms of quantities at $\theta = \theta_R$, e. g.,

$$X(\theta_B) \approx X(\theta_R) + \left(\frac{\partial X}{\partial \theta}\right)_R \theta'_B,$$
 (7.57)

to get

$$gd = \left(\frac{d\Phi_0}{d\theta}\right)_R \theta'_B + \Phi'(\theta_R).$$
(7.58)

Two terms are omitted in this linearization. The term $\Phi_0(\theta_R)$ is zero by virtue of equation (7.56) and the term involving Φ' and θ'_B is nonlinear and therefore dropped. This lower boundary condition is valid for small amplitude unbalanced solutions as well as for potential vorticity inversions.

7.7 Boussinesq approximation

Under the so-called Boussinesq approximation the oceanic and atmospheric equations are simplified so as to make them tractable for analytic work. The degree of simplification is much more severe in the atmospheric case.

7.7.1 Atmosphere

The Boussinesq approximation in the atmospheric case is a simplification of the linearized isentropic equations which assumes that potential temperature and pressure variations small compared to the actual potential temperature and pressure values. As a result we can assume that $a_0 = a_R$ is constant, resulting in

$$\eta' = -a_R \frac{\partial^2 M'}{\partial \theta^2}.\tag{7.59}$$

In addition, we simplify the geopotential perturbation equation to

$$\Phi' = -\theta_R \frac{\partial M'}{\partial \theta},\tag{7.60}$$

which means that the Exner function perturbation differs from the geopotential perturbation only by a constant factor:

$$\Pi' = \frac{\partial M'}{\partial \theta} = -\frac{\Phi'}{\theta_R}.$$
(7.61)

The perturbation isentropic density can also be written in terms of the geopotential and Exner function perturbations as

$$\eta' = -a_R \frac{\partial \Pi'}{\partial \theta} = \frac{a_R}{\theta_R} \frac{\partial \Phi'}{\partial \theta}.$$

The potential vorticity inversion equation also simplifies to

$$\frac{\partial}{\partial x} \left(\frac{1}{f} \frac{\partial M'}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{f} \frac{\partial M'}{\partial y} \right) + q_0 a_R \frac{\partial^2 M'}{\partial \theta^2} = \eta_0 q'.$$
(7.62)

The momentum equation (7.21) and the linearized mass continuity equation (7.49) are unchanged. The only reference profile which survives this approximation is that of the isentropic density $\eta_0(\theta)$. The Boussinesq approximation is only quantitatively valid for small vertical scales in the atmosphere, but the qualitative character of the solutions is in many cases representative of the exact solutions in deep atmospheric layers.

7.7.2 Ocean

In the ocean we simply set the ordinary density to a constant reference density ρ_R in perturbation equations where this amounts to a very minor approximation. Thus the isopycnal density equation (7.12) becomes

$$\eta' = -\frac{\rho_R}{g} \frac{\partial \Phi'}{\partial \rho},\tag{7.63}$$

and the geopotential equation (7.16) simplifies to

$$\Phi' = \rho_R \frac{\partial M'}{\partial \rho},\tag{7.64}$$

which means that the relationship between isopycnal density and Montgomery potential becomes

$$\eta' = -\frac{\rho_R^2}{g} \frac{\partial^2 M'}{\partial \rho^2}.$$
(7.65)

7.8 References

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7.9 Laboratory

1. Convert a zonally averaged GFS run into isentropic coordinates and plot contours of the zonal wind and Montgomery potential. Then plot contours of zonal wind shear $(\partial v_x/\partial \theta)$ and Exner function. Plot also contours of isentropic density and potential vorticity. Finally, plot contours of geopotential.

7.10 Problems

- 1. Suppose that the density in the ocean goes as $\rho(z) = \rho_0 \Delta \rho \exp(z/d)$, assuming that the ocean surface occurs at z = 0, with negative z downward.
 - (a) Determine the geopotential as a function of density.
 - (b) Determine the Montgomery potential as a function of density, assuming that the pressure (and hence the Montgomery potential) is zero at the surface.
 - (c) Determine the pressure as a function of density.
 - (d) If $\rho_0 = 1037$ kg m⁻³, $\Delta \rho = 8$ kg m⁻³, d = 100 m, and the ocean is 4000 m deep, determine the geometrical depth which corresponds to a depth equal to half the depth of the ocean in isopycnal coordinates.
- 2. Consider an atmosphere with an isothermal reference profile $T_0(\theta) = 300$ K.
 - (a) From this compute the reference profile of Exner function and Montgomery potential. Assume that the potential temperature equals the temperature at the surface where the geopotential is also zero.
 - (b) Compute the reference profile of isentropic density.
 - (c) From the profile of Montgomery potential compute the reference profile of geopotential.
 - (d) Compare your profiles with those in the tropical profiles shown in figure 7.2 and explain the differences.
- 3. Gravity waves in isentropic coordinates:
 - (a) Linearize the Boussinesq isentropic momentum and mass continuity equations about a state of rest with a constant value of η_0 .
 - (b) Assume that v_x and η' are proportional to $\exp[i(kx + m\theta \omega t)]$ and obtain the dispersion relation $\omega(k, m, t)$.
 - (c) Compute the horizontal trace velocity $u_{tx} = \omega/k$ and determine how this depends on the vertical wavenumber m.
 - (d) Compute the vertical group velocity of the resulting waves, $u_{g\theta} = \partial \omega / \partial m$, and compare with the vertical trace velocity $u_{t\theta} = \omega / m$.