Chapter 7

Atmospheric Models

An accurate model of the atmosphere requires the representation of continuous vertical profiles, leading to a fully three-dimensional treatment. However, many aspects of atmospheric flow can be represented qualitatively by a small number of layers. In this chapter we consider single and two layer models of the atmosphere.

7.1 Atmospheric structure

Figure 7.1 shows a highly schematic profile of the potential temperature in the earth's atmosphere. Typically, a boundary layer exists next to the earth's surface which has nearly constant potential temperature through its depth. Above the tropopause in the stratosphere the potential temperature increases strongly with height. The free troposphere between the top of the boundary layer and the tropopause exhibits a less strong increase in potential temperature with height than the stratosphere. The boundary layer thickness ranges typically from 500 m over the ocean to $2-3$ km over land, while the tropopause ranges from 8 km above sea level in polar regions to 16 km in the tropics.

7.2 Single layer model of atmosphere

Recall that the hydrostatic equation in terms of potential temperature and Exner function is

$$\theta\frac{\partial\Pi}{\partial z} = -g \tag{7.1}$$

where g is the acceleration of gravity. This is useful for layer models of the atmosphere in which the potential temperature is constant in each layer, since the layer thickness h is proportional to the change in Exner function across the layer:

$$h = -\theta\Delta\Pi/g. \tag{7.2}$$

Though of limited applicability, we could in principle define a single layer model of the earth's atmosphere with a uniformly constant potential temperature equal to the average

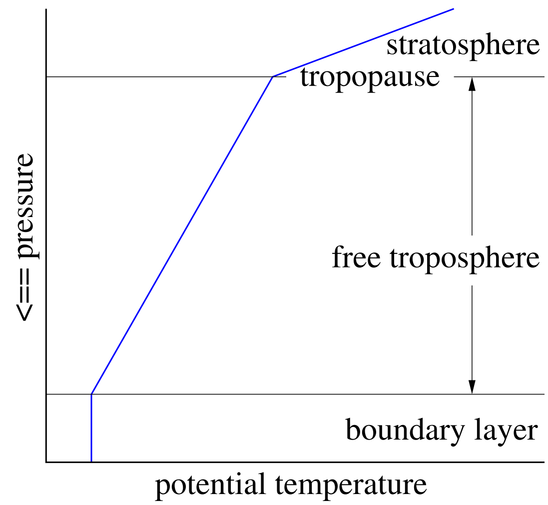

Figure 7.1: Schematic profile of potential temperature as a function of height in the earth's atmosphere.

potential temperature θ of the atmosphere. From the hydrostatic equation (7.1) the Exner function as a function of height would be

$$\Pi = \frac{g(h + d - z)}{\theta} \tag{7.3}$$

where $d(x, y)$ is the terrain height. In the atmospheric momentum equation we replace $\rho^{-1}\boldsymbol{\nabla} p$ with $\theta\boldsymbol{\nabla}\Pi$. The horizontal pressure gradient term in this case is $\theta\boldsymbol{\nabla}\Pi = g\nabla h$, and the momentum equation is therefore

$$\frac{d\mathbf{v}}{dt} + g\boldsymbol{\nabla}(h + d) + f\hat{\mathbf{z}} \times \mathbf{v} = \mathbf{F} \tag{7.4}$$

where $\mathbf{F}$ is an externally applied force, typically surface friction. Notice that this equation is identical to the momentum equation for the shallow water flow of an incompressible fluid.

The mass per unit area in a layer of fluid of thickness h is $\overline{\rho}h$ where $\overline{\rho}$ is the vertical average of the density over the layer. The mass continuity equation thus becomes

$$\frac{\partial\overline{\rho}h}{\partial t} + \boldsymbol{\nabla} \cdot (\overline{\rho}h\mathbf{v}) = \overline{\rho}M \tag{7.5}$$

where a mass source term $\overline{\rho}M$ has been added to this equation.

For a nearly incompressible fluid of almost uniform density such as ocean water, the average density $\overline{\rho}$ can be accurately approximated by a constant value, which as we saw earlier can then be extracted from the space and time derivatives. A similar approximation is sometimes used for the atmosphere. This has the effect of making the atmospheric governing equations identical to those for the ocean, but is less justified in the case of the atmosphere than it is in the ocean. In this approximation we equate $\overline{\rho}$ to the mean density of the atmospheric layer in its unperturbed state, ρ_m. This could be obtained by dividing the mass per unit area in the layer $\Delta p_0/g$ by the layer thickness:

$$\rho_m = \frac{\Delta p_0}{gh}. \tag{7.6}$$

The quantity Δp_0 is the constant pressure thickness of the layer in the reference state. Since the density is now taken to be constant, the mass continuity equation can then be approximated by

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot (h\mathbf{v}) = M, \tag{7.7}$$

which is identical to the mass continuity equation for an incompressible fluid.

7.2.1 Geostrophic wind

As in the oceanic case, one can define a geostrophic wind, which results from a balance between the pressure gradient and Coriolis forces in the case of no terrain:

$$v_{gx} = -\frac{g}{f}\frac{\partial h}{\partial y} \qquad v_{gy} = \frac{g}{f}\frac{\partial h}{\partial x}. \tag{7.8}$$

7.2.2 Surface friction

Recall that the force per unit area of the atmosphere on the ocean is given by the so-called bulk flux formula. By Newton's third law, the force of the ocean on the atmosphere is equal and opposite to the force of the atmosphere on the ocean resulting in a frictional force per unit area on the atmosphere of

$$\mathbf{T} = -\rho C_D |\mathbf{v}|\mathbf{v} \tag{7.9}$$

where ρ is the atmospheric density at the surface, $\mathbf{v}$ is the atmospheric surface wind (actually the wind minus the surface ocean current), and $C_D \approx 1 - 2 \times 10^{-3}$ is the drag coefficient.

The atmosphere generally has a turbulent, neutrally stratified layer next to the surface known as the *boundary layer* in which surface friction is thought to be distributed more or less uniformly. The force per unit mass acting on the air in the boundary layer is thus $\mathbf{F} = \mathbf{T}/(\rho h)$ where h is the thickness of the boundary layer. As noted above, in the free atmosphere above the boundary layer we often approximate the flow by the geostrophic wind, which is a result of geostrophic balance, i. e., a balance between the pressure gradient force and the Coriolis force. In the boundary layer a better approximation is a three-way balance between the pressure gradient force, the Coriolis force, and surface friction. This balance is called *Ekman balance*, and as with geostrophic balance, it is obtained by ignoring parcel accelerations.

In a single layer model we can write the two components of the momentum equation absent the acceleration terms as

$$g\frac{\partial h}{\partial x} - f v_y + (C_D v/h) v_x = 0 \tag{7.10}$$

$$g\frac{\partial h}{\partial y} + f v_x + (C_D v/h) v_y = 0 \tag{7.11}$$

where $v = (v_x^2 + v_y^2)^{1/2}$.

Let us specialize to the case in which $\partial h/\partial x = 0$, which constitutes no loss of generality since we can orient the coordinate axes any way we like. We divide equations (7.10) and (7.11) by f and recognize $-(g/f)(\partial h/\partial y)$ as the geostrophic wind in the x direction, v_{gx}.

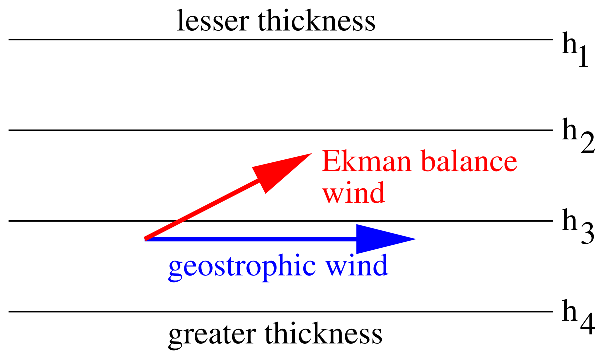

Figure 7.2: Illustration of geostrophic wind and Ekman balance wind in the atmospheric boundary layer.

Further defining $\epsilon = C_D/(hf)$ as a measure of the strength of friction, (7.10) and (7.11) simplify to

$$\epsilon v v_x - v_y = 0 \tag{7.12}$$

$$v_x + \epsilon v v_y = v_{gx} \tag{7.13}$$

with the resulting solutions

$$v_x = \frac{v_{gx}}{1 + \epsilon^2 v^2} \qquad v_y = \frac{\epsilon v v_{gx}}{1 + \epsilon^2 v^2}. \tag{7.14}$$

These solutions are not completely explicit, because v remains undetermined. However, squaring and adding the equations for v_x and v_y results in a quadratic equation for v^2 which has the solution

$$v^2 = \frac{(1 + 4\epsilon^2 v_{gx}^2)^{1/2} - 1}{2\epsilon^2}. \tag{7.15}$$

A not very accurate approximation to equation (7.14) is to assume that v is constant, presumably taking on a value determined by the mean geostrophic wind in equation (7.15). This *linear Ekman balance* approximation is used when a linear relationship between the boundary layer wind and the geostrophic wind is needed to simplify computations.

Figure 7.2 provides a schematic illustration of the boundary layer wind resulting from Ekman balance. In this figure the thickness decreases to the north, resulting in the illustrated geostrophic wind (assuming $f > 0$). The Ekman balance wind is smaller in magnitude and is rotated in direction down the thickness or pressure gradient.

The single layer model of the boundary layer ignores the effect of the overlying atmosphere, which is a major approximation. If the free troposphere is approximated as the upper layer in a two-layer model, the flow in the boundary layer responds to the thickness gradient in this layer as well as in the boundary layer.

7.2.3 Potential vorticity in single layer model

The treatment of potential vorticity q in our constant layer density atmospheric model is identical to that in the single layer shallow water model studied earlier with the exception that the mass source M must be accounted for in the potential vorticity evolution and ageostrophic velocity equations. We earlier derived the expression for potential vorticity evolution

$$\frac{dq}{dt} = \frac{1}{V}\frac{d\Gamma}{dt} - \frac{\Gamma}{V^2}\frac{dV}{dt} \tag{7.16}$$

where $\Gamma = A\zeta_a$ is the circulation around a test volume $V = Ah$ and the potential vorticity is $q = \zeta_a/h$, where ζ_a is the absolute vorticity. The first term on the right side of this equation equals $\boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{z}}/h$ by the circulation theorem. Previously we set dV/dt to zero. However, with a mass source M in the continuity equation this is no longer true; more properly, M is a volume source per unit area, which means that $dV/dt = MA$. Thus, equation (7.16) becomes

$$\frac{dq}{dt} = \frac{\boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{z}}}{h} - \frac{qM}{h}. \tag{7.17}$$

The linearized potential vorticity inversion equation takes the form

$$\frac{c^2}{f}\boldsymbol{\nabla} \cdot \left[\frac{1}{f}\boldsymbol{\nabla}\left(\eta + d/h_0\right)\right] - \eta = q'/q_0 \tag{7.18}$$

where we have retained the possibility of a Coriolis parameter $f(y)$ which varies with latitude. The thickness has been written $h = h_0(1+\eta)$ as usual and the reference potential vorticity $q_0 = f/h_0$ also varies with latitude. The Rossby radius as previously defined is not a constant, so we do not the above equation in terms of this quantity. Instead, we write $gh_0 = c^2$, the square of the gravity wave speed.

The addition of a mass source term to the mass continuity equation (7.7) results in an additional term in the linearized ageostrophic wind equation as well:

$$\nabla^2\chi = \frac{\partial\eta}{\partial t} - \frac{M}{h_0}, \tag{7.19}$$

where we recall that the velocity potential χ provides the ageostrophic wind $\mathbf{v}_a = -\boldsymbol{\nabla}\chi$.

7.3 Two-layer model

The single layer model of the atmosphere is of limited validity, and as in the ocean, a two layer model describes a much wider range of observed phenomena. Figure 7.3 shows a model of the atmosphere containing two homogeneous layers with constant potential temperature θ. The upper layer has potential temperature θ_1 and thickness h_1, while θ_2 and h_2 represent the corresponding variables for the lower layer.

We compute the Exner function in layer 1 to be

$$\Pi_1 = \frac{g}{\theta_1}(h_1 + h_2 + d - z), \tag{7.20}$$

where we have assumed that $\Pi = 0$ at the top of layer 1. The Exner function at the interface between the layers is

$$\Pi_I = \frac{g}{\theta_1}h_1 \tag{7.21}$$

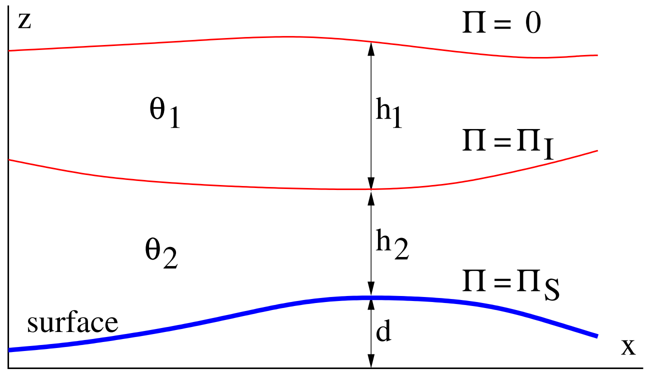

Figure 7.3: Two layer model of the atmosphere. The effect of terrain is represented by the terrain height $d(x, y)$.

and in layer 2 is

$$\Pi_2 = \Pi_I + \frac{g}{\theta_2}(h_2 + d - z) = g[h_1/\theta_1 + (h_2 + d - z)/\theta_2]. \tag{7.22}$$

The surface Exner function ($z = d$) is thus

$$\Pi_S = g(h_1/\theta_1 + h_2/\theta_2). \tag{7.23}$$

Proceeding as in the single layer model, the momentum equations for the two layers are therefore

$$\frac{d\mathbf{v}_1}{dt} + g\nabla(h_1 + h_2 + d) + f\hat{\mathbf{z}} \times \mathbf{v}_1 = \mathbf{F}_1, \tag{7.24}$$

$$\frac{d\mathbf{v}_2}{dt} + g\nabla(\nu^2 h_1 + h_2 + d) + f\hat{\mathbf{z}} \times \mathbf{v}_2 = \mathbf{F}_2 \tag{7.25}$$

where $\nu^2 = \theta_2/\theta_1$. These look a lot like the corresponding momentum equations for the two layer ocean, the only difference being the replacement of ρ_1/ρ_2 by $\nu^2 = \theta_2/\theta_1$. For generality an arbitrary external force per unit mass is included for each level.

The mass continuity equations for the constant layer density approximation are derived as in the single layer case, resulting in

$$\frac{\partial h_1}{\partial t} + \nabla \cdot (h_1 \mathbf{v}_1) = M_1 \tag{7.26}$$

$$\frac{\partial h_2}{\partial t} + \nabla \cdot (h_2 \mathbf{v}_2) = M_2, \tag{7.27}$$

where as in the single layer case we have added source terms M_1 and M_2. The quantities $\rho_{m1} M_1$ and $\rho_{m2} M_2$ represent the mass of air per unit area added to each layer as a result of heating or cooling associated with convection or radiation. The quantities ρ_{m1} and ρ_{m2} are the (constant) mean densities in each layer in analogy with ρ_m defined above for the single layer model. Conservation of mass implies that mass lost in one layer reappears in the other layer, i. e.,

$$\rho_{m1} M_1 + \rho_{m2} M_2 = 0. \tag{7.28}$$

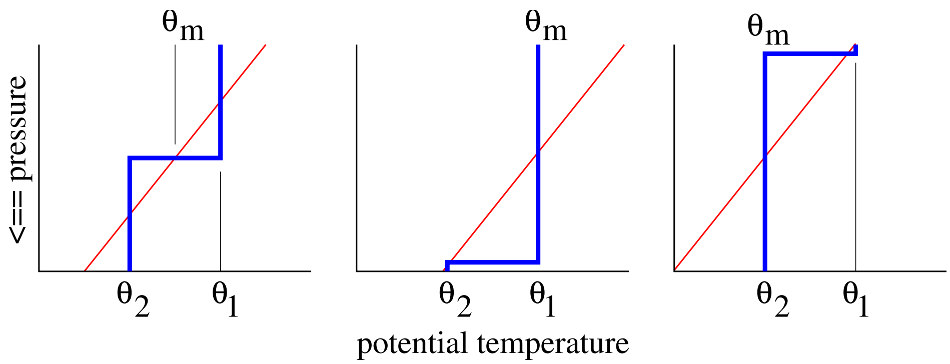

Figure 7.4: Sketch of the potential temperature as a function of pressure in the two layer model (thick lines) and the constant gradient profile it is assumed to approximate (slanted thin lines).

We think of the two layer model as approximating an atmosphere with a constant gradient in potential temperature with respect to pressure, as illustrated in figure 7.4. The level separating the upper and lower layers is adjusted so that the average potential temperature in each layer of the actual atmosphere is the same as the potential temperature of the layer. In this way an atmosphere with horizontal variability in mass-weighted average potential temperature (but no variation in vertical structure) can be represented approximately by the two-layer model. The mean potential temperature of the atmosphere in this model is given by

$$\theta_m = \frac{\rho_{m1} h_1 \theta_1 + \rho_{m2} h_2 \theta_2}{\rho_{m1} h_1 + \rho_{m2} h_2}. \tag{7.29}$$

The surface potential temperature is sometimes needed for calculating surface heat fluxes. Examination of figure 7.4 shows that the actual surface potential temperature, as opposed to θ_2 the potential temperature of the lower layer, is given by

$$\theta_S = \theta_m - (\theta_1 - \theta_2). \tag{7.30}$$

A similar equation gives us the temperature at the tropopause:

$$\theta_T = \theta_m + (\theta_1 - \theta_2). \tag{7.31}$$

We sometimes need the surface and tropopause winds as well. Assuming constant shear through the troposphere, similar reasoning yields

$$\mathbf{v}_S = \mathbf{v}_m - (\mathbf{v}_1 - \mathbf{v}_2) \qquad \mathbf{v}_T = \mathbf{v}_m + (\mathbf{v}_1 - \mathbf{v}_2) \tag{7.32}$$

where $\mathbf{v}_m$ is the mass-weighted mean of the upper and lower layer winds.

7.3.1 External and internal modes

As with the two-layer ocean model, external and internal modes exist in this model. Here we develop independent governing equations for these two modes in the special case in which the mean thicknesses of the two layers are the same, h. This case is particularly easy to analyze.

Assuming that $h_1 = h(1+\eta_1)$ and $h_2 = h(1+\eta_2)$, we first investigate free, non-rotating gravity waves on a horizontally homogeneous environment at rest. The linearized governing equations are for waves moving in the x direction are

$$\frac{\partial v_1}{\partial t} + gh\frac{\partial}{\partial x}(\eta_1+\eta_2) = 0 \tag{7.33}$$

$$\frac{\partial v_2}{\partial t} + gh\frac{\partial}{\partial x}\left(\nu^2\eta_1+\eta_2\right) = 0 \tag{7.34}$$

$$\frac{\partial \eta_1}{\partial t} + \frac{\partial v_1}{\partial x} = 0 \tag{7.35}$$

$$\frac{\partial \eta_2}{\partial t} + \frac{\partial v_2}{\partial x} = 0 \tag{7.36}$$

where v_1 and v_2 are the x components of the velocity. Assuming waves of the form $\exp[i(kx-\omega t)]$ and defining the phase velocity as $c = \omega/k$, we note that $v_1 = c\eta_1$ and $v_2 = c\eta_2$. The result is the set of homogeneous equations represented in matrix form as

$$\begin{pmatrix} -c^2+gh & gh \\ \nu^2 gh & -c^2+gh \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \tag{7.37}$$

which has the secular equation

$$c^4 - 2c^2gh + (1-\nu^2)g^2h^2 = 0. \tag{7.38}$$

This equation has two solutions, the external mode with $c^2 = c_X^2 = gh(1+\nu)$ and polarization relation $\eta_2 = \nu\eta_1$, and the internal mode with $c^2 = c_I^2 = gh(1-\nu)$ and $\eta_2 = -\nu\eta_1$. Given these polarization relations, we can define external and internal fractional thickness variations

$$\eta_X = \frac{\nu\eta_1+\eta_2}{2} \qquad \eta_I = \frac{\nu\eta_1-\eta_2}{2} \tag{7.39}$$

and corresponding external and internal velocity variations

$$v_X = \frac{\nu v_1+v_2}{2} = c_X\eta_X \qquad v_I = \frac{\nu v_1 - v_2}{2} = c_I\eta_I. \tag{7.40}$$

Inverting these, we see that

$$\eta_1 = (\eta_X+\eta_I)/\nu \qquad \eta_2 = \eta_X - \eta_I, \tag{7.41}$$

etc. For a pure external mode, the internal mode dependent variables are zero, which means that velocities and thickness perturbations in the two layers are related by $\eta_2 = \nu\eta_1$ and $v_2 = \nu v_1$. Similarly a purely internal mode has $\eta_2 = -\nu\eta_1$ and $v_2 = -\nu v_1$. In other words, in an approximate sense, the two layers vary in phase for the external mode and out of phase for the internal mode.

The non-rotating gravity wave case we analyzed above is a very special case. However, Linearizing the full governing equations (7.24)-(7.27) about a state of rest and combining them in a manner to be described results in explicit equations for the external

$$\frac{\partial \mathbf{v}_X}{\partial t} + c_X^2\boldsymbol{\nabla}(\eta_X + d/2h) + f\hat{\mathbf{z}}\times\mathbf{v}_X = \mathbf{F}_X \tag{7.42}$$

$$\frac{\partial \eta_X}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v}_X = \frac{M_X}{h} \tag{7.43}$$

and internal

$$\frac{\partial \mathbf{v}_I}{\partial t} + c_I^2 \boldsymbol{\nabla}(\eta_I + d/2h) + f\hat{\mathbf{z}} \times \mathbf{v}_I = \mathbf{F}_I \tag{7.44}$$

$$\frac{\partial \eta_I}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v}_I = \frac{M_I}{h} \tag{7.45}$$

modes. These equations are obtained respectively for the external and internal modes by adding and subtracting the layer 2 equations from ν times the corresponding layer 1 equations and dividing by 2. Thus, $\mathbf{F}_X = (\nu \mathbf{F}_1 + \mathbf{F}_2)/2$, $\mathbf{F}_I = (\nu \mathbf{F}_1 - \mathbf{F}_2)/2$, $M_X = (\nu M_1 + M_2)/2$, and $M_I = (\nu M_1 - M_2)/2$. Inverting to get the layer velocities and fractional thickness perturbations results in

$$\mathbf{v_1} = (\mathbf{v}_X + \mathbf{v}_I)/\nu \qquad \mathbf{v}_2 = \mathbf{v}_X - \mathbf{v}_I \tag{7.46}$$

and

$$\eta_1 = (\eta_X + \eta_I)/\nu \qquad \eta_2 = \eta_X - \eta_I. \tag{7.47}$$

This superposition process only works for the linearized equations. If the advection or source terms in the momentum equations are important, the superposition process does not provide a clean separation between the internal and external modes. However, the decomposition of two-layer motions into these modes can be useful even in the presence of inhomogeneous terms.

We now make the Boussinesq approximation, which sets the densities equal in the two layers:

$$\rho_{m1} = \rho_{m2} \equiv \rho_m. \tag{7.48}$$

This choice in conjunction with equation (7.28) simplifies the the relationship between mean potential temperature and fractional thickness perturbations. In linearized form we have

$$\begin{aligned} \theta_m &= \theta_{m0} + (\eta_1 - \eta_2)(\theta_1 - \theta_2)/4 \\ &= \theta_{m0} + \theta_{mX}\eta_x + \theta_{mI}\eta_I. \end{aligned} \tag{7.49}$$

where $\theta_{m0} = (\theta_1 + \theta_2)/2$ is the average of the potential temperature in the two layers under unperturbed conditions, $\theta_{mX} = (1 - \nu)(\theta_1 - \theta_2)/(4\nu)$, and $\theta_{mI} = (1 + \nu)(\theta_1 - \theta_2)/(4\nu)$. The internal mode contributes much more strongly to mean temperature changes than does the external mode.

Starting from equation (7.32), we approximate the mean wind as the external mode wind and the difference between the upper and lower layer winds as twice the internal mode wind in order to make estimates of the surface and tropopause winds under our internal-external mode model:

$$\mathbf{v}_S \approx \mathbf{v}_X - 2\mathbf{v}_I \qquad \mathbf{v}_T \approx \mathbf{v}_X + 2\mathbf{v}_I. \tag{7.50}$$

7.3.2 Geostrophic balance

As with the single layer model, we can define geostrophic winds in the two layer model. The most fruitful way to represent these winds is in the context of the external and internal modes. For the two modes we have

$$\mathbf{v}_{Xg} = \frac{c_X^2}{f}\left(-\frac{\partial \eta_X}{\partial y}, \frac{\partial \eta_X}{\partial x}\right) \qquad \mathbf{v}_{Ig} = \frac{c_I^2}{f}\left(-\frac{\partial \eta_I}{\partial y}, \frac{\partial \eta_I}{\partial x}\right). \tag{7.51}$$

Given that $\theta_{mX} \ll \theta_{mI}$, $\boldsymbol{\nabla}\theta_m \approx \theta_{mI}\boldsymbol{\nabla}\eta_I$

$$\boldsymbol{\nabla}\theta_m \approx \theta_{mI}\boldsymbol{\nabla}\eta_I = -(\theta_{mI} f/c_I^2)\hat{\mathbf{z}} \times \mathbf{v}_{Ig}. \tag{7.52}$$

This is called the thermal wind, since the magnitude of the internal mode wind is proportional to the horizontal temperature gradient. This "wind" actually represents nearly opposing winds in the upper and lower layers.

7.3.3 Effect of heating

In our two-layer model, heating increases the mean temperature of the atmospheric column, not by increasing θ_1 or θ_2 , but by transferring mass from the lower layer to the upper layer. This is accomplished in the model by assigning a positive value of M_1 and a negative value of M_2, with the ratio of the two source terms adjusted to satisfy equation (7.28). We continue to assume the Boussinesq approximation here, so that $\rho_{m1} = \rho_{m2} = \rho_m$, which means that $M_2 = -M_1$.

If ΔQ is the heat added to the atmosphere per unit area in time interval Δt, we can relate ΔQ to M_1:

$$\Delta Q = \rho_m C_p \Delta T \Delta h_1 = \rho_m \Pi_m \Delta\theta M_1 \Delta t \tag{7.53}$$

where $\Delta T = T_1 - T_2$ is the temperature difference between the layers at the layer interface and $\Delta\theta = \theta_2 - \theta_1$. Assuming that the Exner function at the interface is Π_m, we have used $C_p\Delta T = \Pi_m\Delta\theta$ and further assume that Π_m is constant.

$$\frac{dQ}{dt} = \rho_m \Pi_m \Delta\theta M_1, \tag{7.54}$$

from which we can infer M_1 and $M_2 = -M_1$.

7.4 Global atmospheric circulation

In this section we apply what we have learned to the global circulation of the atmosphere.

7.4.1 Equatorial tropospheric structure

The potential temperature profile in the deep tropics is determined by moist convective processes, which maintain it near a moist adiabat. The equivalent potential temperature of this profile is approximately that of 80% relative humidity air (typical value over warm tropical oceans) at the sea surface temperature and pressure. Taking these as 300 K and 1000 hPa respectively, the equivalent potential temperature is roughly 350 K. At the tropopause the mixing ratio is so small that the potential temperature is essentially the same as the equivalent potential temperature there.

This result suggests that the two-layer model developed above should have $\theta_1 = 337.5$ K and $\theta_2 = 312.5$ K(mid-points of the upper and lower halves of the troposphere respectively; we neglect the stratosphere). Thus $\nu = (\theta_2/\theta_1)^{1/2} = 0.962$. A reasonable value for $h =$ 8000 m, or half the tropopause height and a plausible approximation for the density is $\rho_{m2} = 0.57$ kg m^{-3}, which is the average density of the air from the surface to 100 hPa. The value of $\Pi_m \approx 700$ J kg^{-1} K^{-1}. Internal and external gravity wave speeds are $c_I = [(1-\nu)gh]^{1/2} \approx 55$ m s^{-1} and $c_X = [(1+\nu)gh]^{1/2} \approx 390$ m s^{-1}.

7.4.2 Symmetric zonal flow

As was indicated earlier, the earth's atmosphere transports about 1 PW of energy per unit time northward out of the tropics to middle latitudes, and about 2 PW from middle latitudes to higher northern latitudes. Similar transports exist in the southern hemisphere. These transports depend on the existence of meridional (north-south) flows of air, whose motions actually carry the heat.

The simplest possible model of the atmosphere has no meridional flows at all, only zonal (east-west) flows. In this axisymmetric flow the zonal wind is in geostrophic balance with meridional pressure gradients. We now explore this model and show why it is not tenable. The arguments presented here follow in part those made by Lindzen (1990).

Our two-layer model of the atmosphere is sufficient to get across the essence of the argument, which assumes that each latitude belt exhibits local radiative-convective equilibrium independent of other latitudes. During the equinoxes, when the sun is directly over the equator, the local radiative temperature at latitude ϕ was shown earlier to be

$$T_{loc} = \left[\frac{F_s(1-A)\cos\phi}{\pi\sigma}\right]^{1/4} \tag{7.55}$$

in contrast to the global mean radiative equilibrium temperature

$$T_{rad} = \left[\frac{F_s(1-A)}{4\sigma}\right]^{1/4}. \tag{7.56}$$

In these equations F_s is the solar energy flux, A is the earth's albedo, and σ is the Stefan-Boltzmann constant.

In local radiative equilibrium, the actual temperature of the atmosphere at the effective level at which outward-going thermal radiation originates equals the radiative equilibrium temperature. One might imagine that the vertically averaged potential temperature of the atmosphere scales with this equilibrium radiative temperature under these conditions. In our simplified treatment we set the mean potential temperature of the atmosphere given by equation (7.49) equal to the equilibrium potential temperature at the effective radiative level, indicated by the Exner function at this level Π_{rad}, $\theta_m = C_p T_{loc}/\Pi_{rad}$.

In this analysis we are interested primarily in behavior in the equatorial regions where $|\phi| \ll 1$ is a reasonable approximation. In this case we can use $\cos\phi \approx 1 - \phi^2/2$. Defining $\theta_{eq} = C_p T_{loc}(0)/\Pi_{rad}$ and applying the small angle approximation for cosine, we find that

$$\theta_m = \theta_{eq}\left(1 - \frac{\phi^2}{8}\right) = \theta_{eq}\left(1 - \frac{y^2}{8a^2}\right) \tag{7.57}$$

where a is the radius of the earth. From the geostrophic wind analysis of the two-layer model we note that

$$\frac{\partial \theta_m}{\partial y} \approx -\frac{\theta_{mI} f v_{Igx}}{c_I^2}. \tag{7.58}$$

We now apply the equatorial beta plane approximation, $f = \beta y$. As shown previously, $\beta = 2\Omega/a$, where Ω is the angular rotation rate of the earth. Putting in everything we know, we find that

$$v_{Igx} = \frac{\nu(1+\nu^2)gh}{4(1+\nu)^2\Omega a} \approx 20 \text{ m s}^{-1}. \tag{7.59}$$

In steady zonal flow, the surface wind is zero due to the effects of surface friction. Equation (7.50) shows us that $\mathbf{v}_X = 2\mathbf{v}_I$ under these conditions, so that the tropopause wind $v_{Tgx} = (v_{Xgx} + 2v_{Igx}) = 4v_{Igx} = 80 \text{ m s}^{-1}$. Thus, the meridional temperature gradient results in zonal winds toward the east independent of latitude, at least in tropical regions where $|\phi| \ll 1$. On and some distance off the equator the upper troposphere is in *super rotation*, which means that it has specific angular momentum of rotation greater than that of any part of the solid earth, or more particularly, the earth's surface at the equator.

Hide's theorem states that a planetary atmosphere in zonally symmetric motion cannot exhibit super rotation. Zonal symmetry means that angular momentum cannot be transmitted by parcel motions to a latitudinal ring of air. In the absence of frictional forces, the angular momentum of such a ring is therefore conserved. A frictionless atmosphere starting from rest and maintaining zonal symmetry will never be able to produce super rotation. Furthermore, if there is friction between the atmosphere and the air, friction itself will not produce super rotation of the air since it will always act to drive the specific angular momentum of the air toward that of the underlying surface. Even if super rotation existed initially, this decay toward a state without super rotation would occur.

The specific angular momentum of a ring of air moving toward the east with speed v_x at latitude ϕ is

$$m = \Omega a^2 \cos^2\phi + v_x a \cos\phi, \tag{7.60}$$

whereas the maximum specific zonal angular momentum of the earth's surface at the equator is $m_e = \Omega a^2$. Hide's theorem is thus violated if $m > m_e$, i. e., if

$$v_x > \Omega a \phi^2 \tag{7.61}$$

to first order in ϕ^2. Given a zonal wind value of $v_x = 42$ m s, the maximum latitude for which this occurs is

$$\phi_{hide} \approx \left(\frac{v_x}{\Omega a}\right)^{1/2} \approx 24°. \tag{7.62}$$

We conclude that a zonally symmetric zonal flow is not an allowable solution for the general circulation of the atmosphere, at least in the tropics.

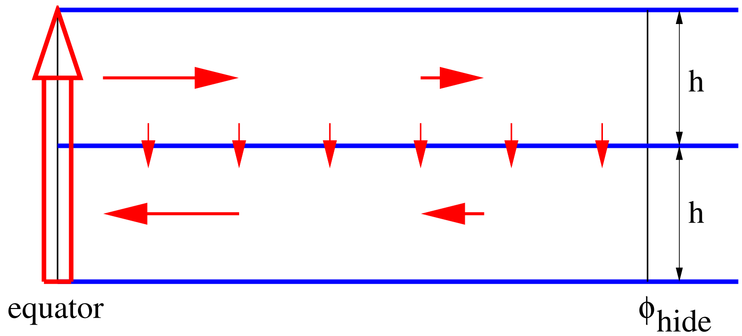

Figure 7.5: Hadley circulation in two-layer model.

7.4.3 Hadley circulation

If the wind shear between the upper and lower layers is reduced or eliminated from the value required to balance the meridional temperature gradient, then equations (7.44) and (7.52) tell us that there is a tendency to generate a meridional circulation, i. e.,

$$\frac{\partial v_{Iy}}{\partial t} = -gh\frac{\partial \eta_I}{\partial y} = -\frac{gh}{\theta_{mI}}\frac{\partial \theta_m}{\partial y} \tag{7.63}$$

which is positive in the northern hemisphere and negative in the summer hemisphere, i. e., away from the equator. Recall that the flow in the upper layer is in the same direction as $\mathbf{v}_I$ while the lower layer flow is in the opposite direction. Thus, a circulation tends to develop as illustrated in figure 7.5, with rising motion over the equator where the temperature is the greatest. This rising motion occurs exclusively in moist atmospheric convection and is therefore accompanied by latent heat release and precipitation.

This circulation, which is called the Hadley circulation, continues poleward as far as is needed to compensate for the lack of meridional balance implied by Hide's theorem. Though figure 7.5 shows the circulation cutting off abruptly at latitude ϕ_{hide}, the cutoff is more gradual than this. The intensity of the circulation is governed by the strength of the subsidence from layer 1 to layer 2, which in turn is governed by the radiative cooling rate of the atmosphere. Since solar radiation is deposited primarily in the underlying ocean, this vertically integrated radiative cooling results primarily from the emission of thermal radiation. We approximate this as the black body radiation at the global radiative temperature $T_{rad} = 255$ K, i. e., $\sigma T_{rad}^4 \approx 240$ W m^{-2}. This heat source (actually a sink) results in a negative mass source in the upper layer

$$M_1 = -\frac{\sigma T_{rad}^4}{\rho_m \Pi_m \Delta\theta} \approx 0.024 \text{ m s}^{-1} \tag{7.64}$$

and a corresponding source in the lower layer $M_2 = -M_1$.

The linearized, zonally symmetric, time independent mass continuity equation in the lower layer is

$$h\frac{\partial v_{2y}}{\partial y} = M_2, \tag{7.65}$$

which integrates to

$$v_{2y}(\phi) = \frac{aM_2(\phi - \phi_{hide})}{h}. \tag{7.66}$$

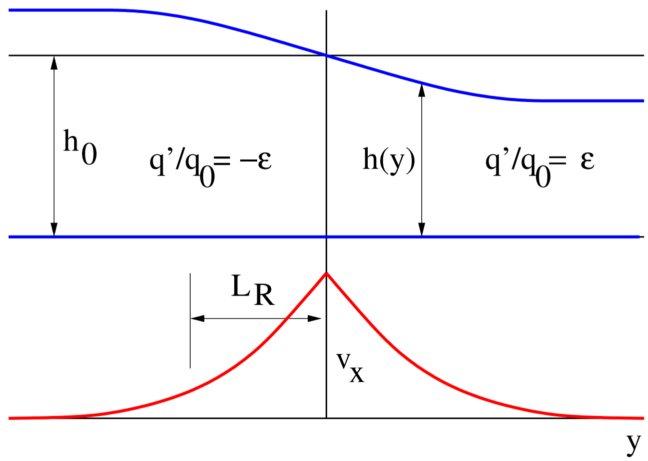

Figure 7.6: Layer thickness $h(y)$ and zonal wind $v_x(y)$ resulting from a step increase in potential vorticity equal to $\Delta q = 2\epsilon q_0$ at $y = 0$.

At the equator we get $v_{2y}(0) \approx -8$ m s^{-1}. A balancing poleward flow exists in the upper layer, as indicated in figure 7.5.

7.4.4 Origin of jet stream

The Coriolis force on the equatorward flux of air in the lower layer of the Hadley circulation accelerates this air toward the west. However, surface friction counters this westward acceleration, resulting in a moderate westward velocity component in the lower layer. This flow in the northern hemisphere is called the northeasterly trade winds, whereas in the southern hemisphere it is called the southeasterly trades.

There is much less frictional resistence to the flow in the upper layer, which means that the eastward acceleration due to the Coriolis force in this layer is essentially unrestrained. In the limit of zero friction, the angular momentum of this poleward-moving air is conserved. If this air starts out with the specific angular momentum of the earth's surface at the equator, then its zonal velocity will vary with latitude according to equation (7.61).

Equation (7.17) shows that a mass source in a layer tends to reduce the potential vorticity of the layer. Since our model of the Hadley circulation has the mass source on or near the equator, the planetary vorticity is small or zero. Furthermore, north-south symmetry indicates that the initial relative vorticity should also be small in this region. Thus, air exiting from equatorial deep convection will have very small potential vorticity. This small potential vorticity will be maintained as the air moves poleward in the upper layer.

At the termination of the poleward flow at upper levels, the Hadley circulation air impinges on mid-latitude air with much larger potential vorticity. The result is a sharp meridional potential vorticity gradient in the upper troposphere.

In order to understand the implications of this potential vorticity gradient, we now examine such a meridional gradient in a single layer f-plane model illustrated in figure 7.6. The potential vorticity perturbation for $y < 0$ is $q' = -\epsilon q_0$ where ϵ is a constant and $q_0 = f/h_0$ is the planetary potential vorticity. For $y > 0$, $q' = +\epsilon q_0$. Solving the potential vorticity inversion equation

$$L_R^2 \frac{\partial^2 \eta}{\partial y^2} - \eta = \frac{q'}{q_0} \tag{7.67}$$

results in

$$\eta(y) = -\epsilon \left[1 - \exp\left(-|y|/L_R\right)\right] \operatorname{sgn}(y) \tag{7.68}$$

for the fractional thickness perturbation, where $L_R = (gh_0)^{1/2}/f$ is the Rossby radius. The zonal wind takes the form

$$v_x(y) = \epsilon c \exp\left(-|y|/L_R\right) \tag{7.69}$$

where $c = (gh_0)^{1/2}$ is the speed of gravity waves. Thus, the layer thickness decreases from south to north crossing the potential vorticity discontinuity. The distance over which the layer thickness changes, scales with the Rossby radius. The geostrophic wind associated with this thickness gradient maximizes at the latitude of the potential vorticity discontinuity and scales in magnitude with ϵc.

Though the actual situation in the atmosphere is more complex, with a strong potential vorticity gradient in the upper troposphere but not in the lower part, as well as latitudinal variation of the Coriolis parameter, this simple model gets at the fundamental mechanism of upper tropospheric jet streams in the atmosphere. The jet originating on the poleward limit of the Hadley circulation is called the subtropical jet. Sometimes there are other jets closer to the poles as well. The decreasing layer thickness across the jet corresponds to a decrease in tropopause height across real atmospheric jet streams.

7.4.5 Mid-latitude eddies

Under construction.

7.5 References

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7.6 Laboratory

Under construction.

7.7 Problems

1. Consider the unbalanced internal mode response of a two-layer model with rotation but no friction to a mass source periodic in space and time:

   (a) Linearize and solve for the fractional thickness perturbation and fluid velocity in response to the internal mode mass source

   $$M_I = M_0 \cos(kx)\sin(\omega t)$$

   where M_0, k, and ω are externally specified constants. Hint: Try solutions of the form $v_{Ix} \propto \sin(kx)\sin(\omega t)$, $v_{Iy} \propto \sin(kx)\cos(\omega t)$, and $\eta_I \propto \cos(kx)\cos(\omega t)$.

   (b) For fixed k, determine how the layer thickness and the wind components respond to the mass forcing as a function of ω. Note particularly the value of ω for which the solution blows up. Give a physical interpretation of this blowup.

2. Repeat the above problem except consider the balanced response to the mass forcing. In particular:

   (a) Use the linearized potential vorticity advection equation to obtain q_I^* from M_I.

   (b) Invert the linearized potential vorticity perturbation q_I^* equation to obtain the fractional thickness perturbation η_I.

   (c) From η_I obtain the geostrophic wind.

   (d) Also from η_I, obtain the ageostrophic wind. Combine with the geostrophic wind to obtain the total wind.

   (e) Determine the range of ω values for which the linearized balanced response is in reasonable agreement with the linearized full response.

3. Given the Hadley circulation model presented above, compute the global poleward flux of heat as a function of latitude in the Hadley circulation. Compare with the observed poleward flux of heat at low latitudes.

4. Show that air in a zonally symmetric flow with zero zonal wind at the equator and constant specific angular momentum as a function of latitude has zero potential vorticity.