

Figure 1: Charge of mass  $M$  and charge  $q$  moving in a circle of radius  $R$  at speed  $v$ .

## 1 Angular momentum and magnetic moment

The Rutherford model of the atom has electrons moving in orbit around a small, positively charged nucleus. Thus, in general electrons have both angular momentum and magnetic moments. These magnetic moments are important for understanding the magnetic properties of matter. Even though the Rutherford model has been superceded by quantum mechanics, the classically derived relationship between angular momentum and magnetic moment remains useful.

Imagine a particle with mass  $M$  and charge  $q$  moving in a circle of radius  $R$  with speed  $v$ . Treating the motion of this particle classically, with speed much less than the speed of light, the orbital angular momentum of the particle is  $L = MvR$ . The time for the particle to make one revolution is  $T = 2\pi R/v$ , so the circular current represented by this moving charge is  $i = q/T = qv/(2\pi R)$ . The magnetic moment generated by this circular current is the current times the area of the circle, or  $m = iA = i\pi R^2$ . The ratio of the magnetic moment to the angular momentum of the particle

$$\gamma = \frac{m}{L} = \frac{q}{2M} \quad (1)$$

is called the *gyromagnetic ratio*.

This equation turns out to be generally true, even if particles are not moving in circular orbits. However, elementary particles have an intrinsic angular momentum that has nothing to do with orbital motion. For the electron in particular, the gyromagnetic ratio is (to within a tenth of a percent)

$$\gamma_e = \frac{e}{M_e}, \quad (2)$$

where  $e$  is the charge on the electron and  $M_e$  is its mass, or twice the gyromagnetic ratio of the electron in orbital motion. (Technically, the gyromagnetic ratio of particles with negative charge is negative, though this sign is often ignored.) Since the spin angular momentum of the electron is  $\hbar/2$ , the intrinsic magnetic moment associated with the electron spin is

$$m_B = \frac{\hbar\gamma_e}{2} = \frac{\hbar e}{2M_e}. \quad (3)$$

This is sometimes called the *Bohr magneton*, hence the subscripted  $B$ . The factor of 2 missing from equation (2) in comparison to equation (1) results from relativistic factors, but the derivation is rather complex, so we will not present it here.

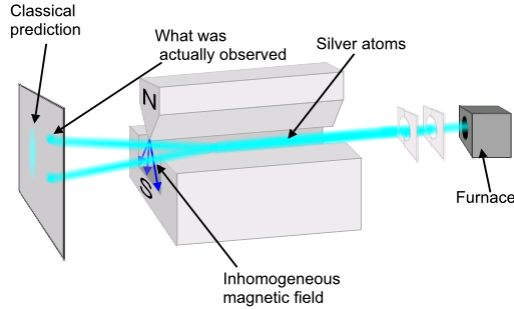


Figure 2: Stern-Gerlach experiment. Licensed from Wikimedia Commons under Creative Commons BY-SA 3.0.

Magnetic dipoles from orbital and spin angular momenta have important practical consequences; they are central to the phenomena of *paramagnetism* and *ferromagnetism*. Paramagnetism occurs when atoms have an odd number of electrons, which means that the angular momentum of these electrons does not add up to zero (see next chapter). In this case, there is a net magnetic dipole moment for each atom. If a magnetic field is imposed, this dipole tends to align with the magnetic field, which in turn reinforces the imposed field, resulting in a stronger total field. If the imposed magnetic field is taken away, the atoms reorient themselves randomly, resulting in zero field.

A few materials, such as iron, exhibit a more complex phenomenon called ferromagnetism. In this case the imposition of an external magnetic field causes very strong dipole alignment effects to the point where interactions between the newly aligned dipoles prevent the reversion to random orientation when the imposed field is removed. “Permanent” magnets may be created this way. As the temperature increases, the thermal jiggling of atoms increases in proportion. If this jiggling becomes strong enough, the tendency of neighboring dipoles to maintain a common orientation is overcome and the material becomes paramagnetic. The temperature at which this occurs for a particular material is called the *Curie temperature*.

If there are no unpaired electrons in a material, then there are no pre-existing dipoles to align. However, the magnetic field still distorts the motions of electrons via the Lorentz force. The result is the production of induced magnetic dipoles that anti-align with the field, as happens with the electrons in a plasma. The resulting induced magnetic field opposes the original field, resulting in a total field that is less than originally imposed. Such a material is called *diamagnetic*.

## 2 Problems

1. Consider the hydrogen atom, which has one electron circling the nucleus.
  - (a) Suppose that the electron moves in a circle of radius  $R$  with velocity  $v$ , as shown in figure 1. (This can only occur in an excited state of the atom.) Use Ampère’s law to estimate the magnetic field produced at the nucleus. Hint: Derive an

equation for the average magnetic field over a magnetic circulation loop of radius  $R$  that circles the trajectory of the electron. Assume that the field at the nucleus is somewhat greater than this average value. (Pick a number.)

- (b) The radius of the hydrogen atom in an excited state is of order  $10^{-10}$  m. Compute the orbital velocity of the electron for a circular orbit, given the electron mass and the charges on the electron and the proton nucleus.
  - (c) From the above information compute an approximate value for the magnetic field at the nucleus caused by the electron's orbital motion.
2. The Stern-Gerlach experiment demonstrated the quantization of angular momentum by passing a beam of neutral silver atoms (approximately 108 times mass of proton) through a magnetic field with a strong gradient. (See figure 2.) Since silver has one unpaired electron, the magnetic dipole moment is non-zero. Assuming a vertical gradient in the vertical component of the magnetic field, the silver atoms will be deflected either up or down, depending on the orientation of the dipole moment.
- (a) For a given value of  $\partial B_z/\partial z$ , derive an expression for the  $z$  component of the force on a silver atom, assuming that the dipole moment equals one Bohr magneton aligned (1) in the  $+z$  and (2) in the  $-z$  directions.
  - (b) If the silver atom has mass  $M$  and initial speed  $v$ , compute the angular deflection up or down of the trajectory of the silver atom as it passes through the field of the magnet, assumed to have length  $L$ .
  - (c) Assuming  $\partial B_z/\partial z = 10^2$  T/m,  $v = 300$  m/s, and  $L = 10$  cm, compute the above deflection angle.
  - (d) Explain why the classical result is different from the quantum result as illustrated in figure 2.