Chapter 1

Harmonic Oscillator

Figure 1.1 illustrates the prototypical harmonic oscillator, the mass-spring system. A mass M is attached to one end of a spring. The other end of the spring is attached to something rigid such as a wall. The spring exerts a restoring force $F = -kx$ on the mass when it is stretched by an amount x, i. e., it acts to return the mass to its initial position. This is called *Hooke's law* and k is called the *spring constant*.

1.1 Energy Analysis

The potential energy of the mass-spring system is

$$U(x) = kx^2/2 \tag{1.1}$$

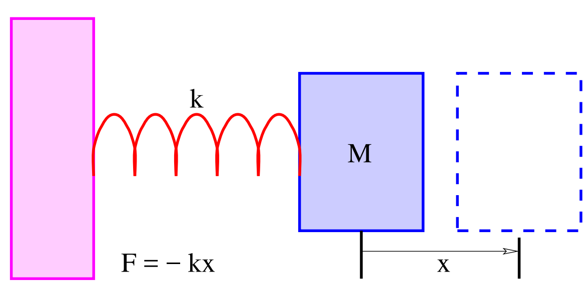

Figure 1.1: Illustration of a mass-spring system.

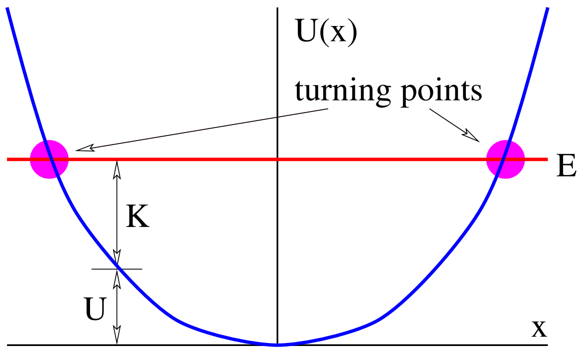

Figure 1.2: Potential, kinetic, and total energy of a harmonic oscillator plotted as a function of spring displacement x.

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which may be verified by noting that the Hooke's law force is derived from this potential energy: $F = -d(kx^2/2)/dx = -kx$. This is shown in figure 1.2. Since a potential energy exists, the total energy $E = K + U$ is conserved, i. e., is constant in time. If the total energy is known, this provides a useful tool for determining how the kinetic energy varies with the position x of the mass M: $K(x) = E - U(x)$. Since the kinetic energy is expressed (non-relativistically) in terms of the velocity u as $K = Mu^2/2$, the velocity at any point on the graph in figure 1.2 is

$$u = \pm \left(\frac{2(E - U)}{M} \right)^{1/2} . \tag{1.2}$$

Given all this, it is fairly evident how the mass moves. From Hooke's law, the mass is always *accelerating* toward the *equilibrium position*, $x = 0$. However, at any point the *velocity* can be either to the left or the right. At the points where $U(x) = E$, the kinetic energy is zero. This occurs at the turning points

$$x_{TP} = \pm \left(\frac{2E}{k} \right)^{1/2} . \tag{1.3}$$

If the mass is moving to the left, it slows down as it approaches the left turning point. It stops when it reaches this point and begins to move to the right. It accelerates until it passes the equilibrium position and then begins to decelerate, stopping at the right turning point, accelerating toward the left, etc. The mass thus oscillates between the left and right turning points. (Note that equations (1.2) and (1.3) are only true for the harmonic oscillator.)

How does the period of the oscillation depend on the total energy of the system? Notice that from equation (1.2) the maximum speed of the mass (i. e., the speed at $x = 0$) is equal to $u_{max} = (2E/M)^{1/2}$. The average speed must be some fraction of this maximum value. Let us guess here that it is half the maximum speed:

$$u_{average} \approx \frac{u_{max}}{2} = \left(\frac{E}{2M}\right)^{1/2} \quad \text{(approximate)}. \tag{1.4}$$

However, the distance d the mass has to travel for one full oscillation is twice the distance between turning points, or $d = 4(2E/k)^{1/2}$. Therefore, the period of oscillation must be approximately

$$T = \frac{d}{u_{average}} \approx 4\left(\frac{2E}{k}\right)^{1/2}\left(\frac{2M}{E}\right)^{1/2} = 8\left(\frac{M}{k}\right)^{1/2} \quad \text{(approximate)}. \tag{1.5}$$

1.2 Analysis Using Newton's Laws

The acceleration of the mass at any time is given by Newton's second law:

$$a = \frac{d^2x}{dt^2} = \frac{F}{M} = -\frac{kx}{M}. \tag{1.6}$$

An equation of this type is known as a *differential equation* since it involves a derivative of the dependent variable x. Equations of this type are generally more difficult to solve than algebraic equations, as there are no universal techniques for solving all forms of such equations. In fact, it is fair to say that the solutions of most differential equations were originally obtained by *guessing*!

We already have the basis on which to make an intelligent guess for the solution to equation (1.6) since we know that the mass oscillates back and forth with a period that is independent of the amplitude of the oscillation. A function which might fill the bill is the cosine function. Let us try substituting $x = \cos(\omega t)$, where ω is a constant, into this equation. The second derivative of x with respect to t is $-\omega^2 \cos(\omega t)$, so performing this substitution results in

$$-\omega^2 \cos(\omega t) = -\frac{k}{M}\cos(\omega t). \tag{1.7}$$

Notice that the cosine function cancels out, leaving us with $-\omega^2 = -k/M$. The guess thus works if we set

$$\omega = \left(\frac{k}{M}\right)^{1/2}. \tag{1.8}$$

The constant ω is the angular oscillation frequency for the oscillator, from which we infer the period of oscillation to be $T = 2\pi(M/k)^{1/2}$. This agrees with the earlier approximate result of equation (1.5), except that the approximation has a numerical factor of 8 rather than $2\pi \approx 6$. Thus, the earlier guess is only off by about 30%!

It is easy to show that $x = B\cos(\omega t)$ is also a solution of equation (1.6), where B is any constant and $\omega = (k/M)^{1/2}$. This confirms that the oscillation frequency and period are independent of amplitude. Furthermore, the sine function is equally valid as a solution: $x = A\sin(\omega t)$, where A is another constant. In fact, the most general possible solution is just a combination of these two, i. e.,

$$x = A\sin(\omega t) + B\cos(\omega t) = C\cos(\omega t - \phi). \tag{1.9}$$

The values of A and B depend on the position and velocity of the mass at time $t = 0$. The right side of equation (1.9) shows an alternate way of writing the general harmonic oscillator solution that uses a cosine function with a phase factor ϕ.

1.3 Forced Oscillator

If we wiggle the left end of the spring by the amount $d = d_0\cos(\omega_F t)$, as in figure 1.3, rather than rigidly fixing it as in figure 1.1, we have a *forced* harmonic oscillator. The constant d_0 is the amplitude of the imposed wiggling motion. The *forcing frequency* ω_F is not necessarily equal to the natural or *resonant* frequency $\omega = (k/M)^{1/2}$ of the mass-spring system. Very different behavior occurs depending on whether ω_F is less than, equal to, or greater than ω.

Given the above wiggling, the force of the spring on the mass becomes $F = -k(x - d) = -k[x - d_0\cos(\omega_F t)]$ since the length of the spring is the difference between the positions of the left and right ends. Proceeding as for

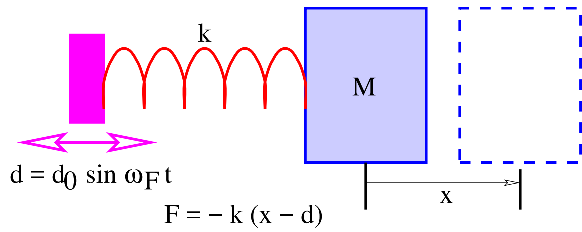

Figure 1.3: Illustration of a forced mass-spring oscillator. The left end of the spring is wiggled back and forth with an angular frequency ω_F and a maximum amplitude d_0.

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the unforced mass-spring system, we arrive at the differential equation

$$\frac{d^2x}{dt^2} + \frac{kx}{M} = \frac{kd_0}{M}\cos(\omega_F t). \tag{1.10}$$

The solution to this equation turns out to be the sum of a forced part in which x is proportional to $\cos(\omega_F t)$ and a free part which is the same as the solution to the unforced equation (1.9). We are primarily interested in the forced part of the solution, so let us set $x = x_0 \cos(\omega_F t)$ and substitute this into equation (1.10):

$$-\omega_F^2 x_0 \cos(\omega_F t) + \frac{kx_0}{M}\cos(\omega_F t) = \frac{kd_0}{M}\cos(\omega_F t). \tag{1.11}$$

Again the cosine factor cancels and we are left with an algebraic equation for x_0, the amplitude of the oscillatory motion of the mass.

Solving for the ratio of the oscillation amplitude of the mass to the amplitude of the wiggling motion, x_0/d_0, we find

$$\frac{x_0}{d_0} = \frac{1}{1 - \omega_F^2/\omega^2}, \tag{1.12}$$

where we have recognized that $k/M = \omega^2$, the square of the frequency of the *free* oscillation. This function is plotted in figure 1.4.

Notice that if $\omega_F < \omega$, the motion of the mass is in phase with the wiggling motion and the amplitude of the mass oscillation is greater than the amplitude of the wiggling. As the forcing frequency approaches the natural frequency of the oscillator, the response of the mass grows in amplitude.

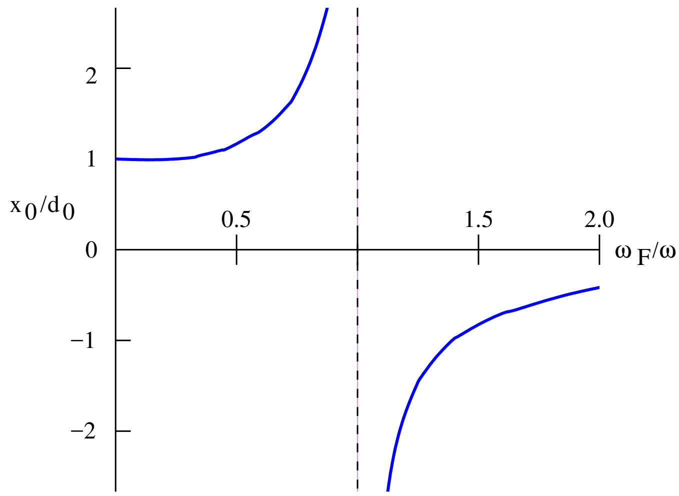

Figure 1.4: Plot of the ratio of response to forcing vs. the ratio of forced to free oscillator frequency for the mass-spring system.

When the forcing is at the resonant frequency, the response is technically infinite, though practical limits on the amplitude of the oscillation will intervene in this case — for instance, the spring cannot stretch or shrink an infinite amount. In many cases friction will act to limit the response of the mass to forcing near the resonant frequency. When the forcing frequency is greater than the natural frequency, the mass actually moves in the opposite direction of the wiggling motion — i. e., the response is out of phase with the forcing. The amplitude of the response decreases as the forcing frequency increases above the resonant frequency.

Forced and free harmonic oscillators form an important part of many physical systems. For instance, any elastic material body such as a bridge or an airplane wing has harmonic oscillatory modes. A common engineering problem is to ensure that such modes are damped by friction or some other physical mechanism when there is a possibility of exitation of these modes by naturally occurring processes. A number of disasters can be traced to a failure to properly account for oscillatory forcing in engineered structures.

1.4 Complex Exponential Solutions

Complex exponential functions of the form $x = \exp(\pm i\omega t)$ also constitute solutions to the free harmonic oscillator governed by equation (1.6). This makes sense, as the complex exponential is the sum of sines and cosines. However, for the frictionless harmonic oscillator, the exponential solutions provide no particular advantage over sines and cosines. Furthermore, oscillator displacements are real, not complex quantities.

The superposition principle solves the problem of complex versus real solutions. For an equation like (1.6) which has real coefficients, if $\exp(i\omega t)$ is a solution, then so is $\exp(-i\omega t)$, so the superposition of these two solutions is also a solution. Furthermore

$$\exp(i\omega t) + \exp(-i\omega t) = 2\cos(\omega t) = 2\mathrm{real}[\exp(i\omega t)]. \tag{1.13}$$

This shows a shortcut for getting the physical part of a complex exponential solution to equations like the harmonic oscillator equation; simply take the real part.

Complex exponential solutions come into their own for more complicated equations. For instance, suppose the force on the mass in the mass-spring system takes the form

$$F = -kx - b\frac{dx}{dt}. \tag{1.14}$$

The term containing b represents a frictional damping effect on the harmonic oscillator and the governing differential equation becomes

$$\frac{d^2x}{dt^2} + \frac{b}{M}\frac{dx}{dt} + \frac{k}{M}x = 0. \tag{1.15}$$

Trying the exponential function $\exp(\sigma t)$ in this equation results in

$$\sigma = \frac{1}{2}\left[-\frac{b}{M} \pm \left(\frac{b^2}{M^2} - \frac{4k}{M}\right)^{1/2}\right] = -\beta \pm i(\omega_0^2 - \beta^2)^{1/2} \tag{1.16}$$

where we have set

$$\beta = \frac{b}{2M} \qquad \omega_0 = \left(\frac{k}{M}\right)^{1/2}. \tag{1.17}$$

The quantity $\omega \equiv (\omega_0^2 - \beta^2)^{1/2}$ is the actual frequency of oscillation of the damped oscillator, which one can see is less than the oscillation frequency

ω_0 that occurs with the damping turned off. The physical solution to the damped oscillator is thus

$$x(t) = \mathrm{real}[\exp(\sigma t)] = \mathrm{real}[\exp(i\omega t)\exp(-\beta t)] = \cos(\omega t)\exp(-\beta t) \quad (1.18)$$

as long as $\omega_0^2 > \beta^2$. Notice that this solution is in the form of an oscillation $\cos(\omega t)$ multiplied by a decaying exponential $\exp(-\beta t)$. This confirms that the b term decreases the amplitude of the oscillation with time.

1.5 Quantum Mechanical Harmonic Oscillator

The quantum mechanical harmonic oscillator shares the characteristic of other quantum mechanical bound state problems in that the total energy can take on only discrete values. Calculation of these values is too difficult for this book, but the problem is sufficiently important to warrant reporting the results here. The energies accessible to a quantum mechanical mass-spring system are given by the formula

$$E_n = (n + 1/2)\hbar(k/M)^{1/2}, \qquad n = 0, 1, 2, \ldots . \quad (1.19)$$

In other words, the energy *difference* between successive quantum mechanical energy levels in this case is constant and equals the classical resonant frequency for the oscillator, $\omega = (k/M)^{1/2}$, times $\hbar$.

1.6 Problems

1. An oscillator (non-harmonic) has the potential energy function $U(x) = Cx^4$, where C is a constant. How does the oscillation frequency depend on energy? Explain your reasoning.

2. Show that $C\cos(\omega t - \phi)$ is an alternate way of writing $A\sin(\omega t) + B\cos(\omega t)$ by finding the values of A and B in terms of the constants C and ϕ. Hint: Expand $\cos(\omega t - \phi)$ by using the trigonometric identity for the cosine of the sum of two angles.

3. If a mass-spring harmonic oscillator has displacement $x = 0$ and velocity $dx/dt = V$ at time $t = 0$, determine the values of A and B as well as those of C and ϕ in equation (1.9).

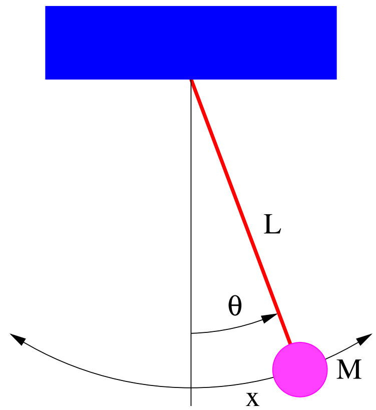

Figure 1.5: The pendulum as a harmonic oscillator.

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4. A mass M is suspended against gravity by a spring of spring constant k. The unstretched length of the spring is x_0 and under the influence of gravity the spring is stretched to a resting length $x_1 > x_0$.
   (a) Compute the length of the spring x_1 in the steady, resting case.
   (b) Set up the equation of motion for the mass moving under the influence of the two forces, gravity and spring. Solve the equation for the frequency of the oscillation and the position of the spring as a function of time $x(t)$. Does the oscillation frequency change from the case without gravity?
5. Determine the two real solutions to the damped harmonic oscillator problem in the case in which $\omega_0^2 < \beta^2$.
6. Consider the pendulum in figure 1.5. The mass M moves along an arc with x denoting the distance along the arc from the equilibrium point.
   (a) Find the component of the gravitational force tangent to the arc (and thus in the direction of motion of the mass) as a function of the angle θ. Use the small angle approximation on $\sin(\theta)$ to simplify this answer.
   (b) Get the force in terms of x rather than θ. (Recall that $\theta = x/L$.)

(c) Use Newton's second law for motion in the x direction (i. e., along the arc followed by the mass) to get the equation of motion for the mass.

(d) Solve the equation of motion using the solution to the mass-spring problem as a guide.

7. Forced damped oscillator:

   (a) Add a damping term to the forced harmonic oscillator equation (1.10) and solve for the forced part of the solution using complex exponential methods. Hint: Change the cosine on the right side of this equation to $\exp(i\omega_F t)$ to convert the equation to complex form and then try the solution $x = x_0 \exp(i\omega_F t)$ where x_0 will depend on ω_F. Also, write the equation in terms of $\beta = b/(2M)$ and $\omega^2 = k/M$.

   (b) Find the physical part of this solution by taking the real part of $x(t)$. Hint: While taking the real part of x, it may be helpful to recall that the inverse of any complex number can be written $1/(a + ib) = (a - ib)/(a^2 + b^2)$.

   (c) Determine how x_0 differs from that in the undamped case when ω_F is near the resonant frequency of the unforced oscillator. In particular, show how the phase and amplitude of the oscillation change as the forcing frequency changes from less than to greater than the resonant frequency.

8. A *massless* particle is confined to a box of length a. (Think of a photon between two mirrors.) Treating the particle classically, compute the period of one round trip from one end of the box to the other and back again. From this compute an angular frequency for the oscillation of this particle in the box. Does this frequency depend on the particle's energy?

9. Compute the ground state energy E_{ground} of a massless particle in a box of length a using quantum mechanics. Compare $E_{ground}/\hbar$ with the angular frequency computed in the previous problem.