WTG cloud modeling with spectral decomposition of heating

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Abstract A modified form of the relaxed weak temperature gradient (WTG) approximation is described and implemented in a cloud resolving model. In this scheme, the vertical heating profile is spectrally decomposed into normal modes. The relaxation time scale of each mode is set proportional to the corresponding vertical wave number of the mode, as would be expected if gravity waves were responsible for the relaxation. Buoyancy adjustment exhibits greater realism than in conventional relaxed WTG, with nonlocal adjustment occurring as in the weak pressure gradient model of Romps. Furthermore, no boundary layer taper is needed when using the improved scheme. Experiments from the literature are revisited; mass flux profiles are smoother and multiple equilibrium experiments are more robust.

1. Introduction

Single-column models (SCMs) and limited-domain cloud-resolving models (CRMs) are important tools used to investigate convection and its interaction with the large-scale tropical environment. While complex models can often be tuned to useful accuracy, these simplified models can provide insight into processes that are often parameterized based on theoretical assumption or minimal observations. Models with limited horizontal extent, however, lack interaction with an assumed surrounding environment. Compensating subsidence—normally a phenomenon occurring over a broad region around convective events—must occur within the limited model domain due to mass continuity. In addition, the temperature anomaly resulting from latent heat release cannot be dispersed and thus also must remain within the domain. These effects combine to alter the model state such that convection experiences an environment which is far from that typical of the real world. These issues can be addressed by incorporating extra dynamics.

For example, observed vertical velocities can be imposed in order to simulate convective mass fluxes of interest. However, the convective heat fluxes that result cannot feed back on the imposed velocities. Since tropical anomalies are a strong function of the delicate balance between adiabatic cooling and diabatic heating, this unidirectional mechanism can lead to large temperature errors in numerical simulations [Bergman and Sardeshmukh, 2004]. Furthermore, imposition of a vertical velocity profile largely fixes the precipitation rate.

A vertical velocity can also be parameterized and modulated based on diagnosed thermodynamics within the model. Prognostic variables can then be modified according to assumptions about the surrounding environment and its role in modulating the effects of convection. Several methods for parameterizing the effects of the large-scale environment have been proposed, including the weak temperature gradient (WTG) approximation [Sobel and Bretherton, 2000; Raymond and Zeng, 2005; Sobel et al., 2007; Sessions et al., 2010] and the damped gravity wave (DGW) (a. k. a. weak pressure gradient (WPG) approximation) [Bergman and Sardeshmukh, 2004; Kuang, 2008; Blossey et al., 2009; Kuang, 2011, 2012; Romps, 2012a, 2012b; Edman and Romps, 2014]. In addition, some authors have joined two single-column models in order to investigate the interaction between small-scale convection and the large-scale environment [Nilsson and Emanuel, 1999; Raymond and Zeng, 2000], while others have studied interactions between convection within two domains of similar size, as in the linked-CRM experiments of Daleu et al. [2012] and Daleu et al. (Transition from suppressed to active convection modulated by the weak–temperature–gradient–derived large–scale circulation, submitted to Journal of Atmospheric Sciences, 2014).

Each of these schemes is based on the observation that horizontal thermodynamic gradients are small in the tropics, but the schemes vary in how this status is maintained. In the strict WTG scheme presented by Sobel and Bretherton [2000], temperature is held constant and a vertical velocity is calculated based on the
convective heating term. The resulting vertical velocity can then be used to advect moisture vertically within the model domain, as well as horizontally into the domain from an assumed surrounding environment. The strict WTG approximation may be suitable for sufficiently long time scales, since the temperature tendency is negligible in that case.

On shorter time scales, however, the temperature should be free to vary in time, and the vertical velocity in a WTG simulation is then defined so as to relax the temperature anomaly toward zero at an assumed rate. The anomaly is typically the difference between the domain averaged virtual potential temperature profile and that of a reference profile, which is taken to represent the local environment surrounding the model domain. Such a mechanism was implemented in a CRM by Raymond and Zeng [2005] (hereafter RZ05) and is often termed relaxed WTG.

The strict WTG scheme is based on a scale analysis of the equations of motion in the tropics, which shows that temperature fluctuations must be small over large space and time scales. The relaxed WTG scheme of RZ05 assumes that gravity waves act to redistribute temperature anomalies away from the locus of diabatic heating associated with deep convection. In this way, the model domain is treated as an isolated region of convective clouds, which is assumed to emit gravity waves based on its domain averaged net heating, as in the study by Bretherton and Smolarkiewicz [1989]. The simple form of the relaxation used in RZ05 is equivalent to assuming that gravity waves of all vertical wavelengths are equally effective in redistributing temperature anomalies of corresponding vertical scales.

Some studies, however, suggest it is important to assign different relaxation times to each of a broad vertical spectrum of modes, particularly when the dominant simulated convection falls short of reaching the tropopause. For example, in a recent CRM study by Lane and Zhang [2011], the deepest convection only reached 11 km. Though this was an artifact of the truncated run time used, spectral analysis of the modeled troposphere and stratosphere each illustrated a dominant third mode response. Given a population of convective clouds with a diverse spectrum of depths, a suitably diverse spectrum of wave modes with proper relaxation times for each of the modes is needed to parameterize the approach to the mean state. This reasoning is consistent with the analysis of Romps [2012a, 2012b], who showed that a conventional relaxed WTG scheme tuned to the gravest mode adjusts the temperature profile too quickly compared to a 3-D Boussinesq model when confronted with transient temperature anomalies having vertical scales less than the depth of the troposphere.

To address the issues mentioned above, Bergman and Sardeshmukh [2004] improved the performance of an SCM by projecting heating rates onto Fourier modes and reconstructing the corresponding vertical advection of temperature by assigning appropriate relaxation times to each vertical mode. In a different study, Cohen and Craig [2004] predicted the timescale of convective adjustment to large-scale anomalous forcing up to a constant by projecting the mass flux profile in a CRM onto the nine deepest gravity wave modes and assuming a unique speed for each mode. Remarkably the authors found that, while the speed of the gravest mode varied considerably over a range of different environmental assumptions, a weighted-average gravity wave speed derived from a projection of the mass-flux profile remained relatively constant. Mapes [2004] parameterized large-scale dynamics in a CRM using a single vertical velocity wave structure and relaxation timescale. The author pointed out that conclusions based on the results were limited by the number of modes used and suggested expanding the CRM domain heating and parameterized large-scale vertical velocity in a Fourier series. If we assume the adjustment time scale of the environment is related to the time scale over which convective temperature anomalies relax back to the mean state, these studies all suggest that including more gravity wave modes in the WTG adjustment mechanism should improve the accuracy of the adjustment process.

In this paper, we seek to define and implement a more accurate form of WTG by using different relaxation rates for different Fourier components of the temperature anomaly profile. In section 2, we briefly describe the CRM used in this paper. Both conventional and spectral WTG schemes are then described in section 3. We present the results of a simple experiment as well as revisit the experiments of RZ05 and Raymond and Sessions (2007) using both the conventional and spectral WTG formulations in section 4. In section 5, we analyze spectral WTG in terms of the 3-D Boussinesq model used by Romps (2012a) and in terms of a simple numerical experiment presented by Romps (2012b). Implications when using WTG in multiple equilibrium experiments are discussed in section 6 and conclusions are given in section 7.
two important changes. A prognostic thermodynamic variable has been changed from an approximate equivalent potential temperature $\Theta_e$ to an approximate specific moist entropy $s$ in anticipation of implementing the complete treatment of moist entropy described in Raymond [2013]. Also, advection of intrinsic quantities via the WTG mechanism takes a different form. Less significant changes have also been implemented in the model over the period since RZ05. A detailed list of notable changes implemented since RZ05 is provided in Appendix B. Constants used throughout the paper are defined in Table 1.

### 3. Weak Temperature Gradient Vertical Velocity

During a weak temperature gradient experiment, sink terms enter the prognostic equations for specific moist entropy and total cloud water mixing ratio. Expressions for the entropy (A22) and mixing ratio (A21) sinks are given in Appendix A. The large-scale vertical velocity used in these expressions is the weak temperature gradient vertical velocity, obtained diagnostically from the model using two different schemes: conventional WTG and spectral WTG.

#### 3.1. Conventional WTG

The conventional WTG vertical velocity is defined in terms of the horizontally averaged potential temperature excess $\theta'$ in the model relative to the reference profile as

$$ w_{\text{wtg}}(z, t) = \frac{\theta'(z, t)}{\tau (d\theta'/dz)}, \quad (1) $$

where $\tau$ is the relaxation time, $\theta'(z, t) = \left[ \bar{\theta}(z, t) - \theta_0(z, t) \right] M(z)$ is the anomaly of the domain averaged potential temperature profile, $\theta_0$ is an externally defined reference profile of potential temperature. The masking function $M(z)$ is defined

$$ M(z) = \begin{cases} 
\sin \left( \frac{\pi z}{h} \right) & z < h \\
0 & z \geq h
\end{cases} \quad (2) $$

where $h$ is the height of the tropopause, and the domain-averaged vertical gradient of potential temperature $d\theta'/dz$ is computed using centered differencing. In the lowest layers of the numerical model, the conventional WTG vertical velocity is further tapered to zero at the surface using the simple linear function.
The above treatment has the problem that $\tau$ is only valid for potential temperature anomaly profiles with vertical scale equal to the vertical scale of gravity waves with horizontal phase speed $c$. Real profiles may be spectrally rich and produce gravity waves of many different vertical scales and corresponding phase speeds.

We solve this problem by decomposing the scaled potential temperature anomaly into a superposition of approximate gravity wave vertical eigenmodes, taken here to be sine functions bounded above by the tropopause and below by the surface. To simplify the expression, we rewrite the scaled potential temperature anomaly using

$$D_h(z, t) = \sum_j H_j(t) \sin(m_j z),$$

where $H_j(t)$ is a Fourier coefficient and the vertical wave number now is mode-dependent:

$$m_j = j \frac{\pi}{h}, \quad j = 1, 2, 3, \ldots$$

The sine functions constitute the vertical structure of hydrostatic gravity waves in the case in which the Brunt-Väisälä frequency of the troposphere is constant, there is zero mean horizontal wind, and a rigid lid exists at the tropopause. They are therefore an approximation to the actual situation. Inversion of this sine series yields the coefficients

$$H_j(t) = \frac{2}{N} \int_0^h D_h(z, t) \sin(m_j z) dz.$$

The spectral WTG vertical velocity is then written

$$W_{\text{swtg}}(z, t) = \sum_j \frac{\Theta_j(t)}{\tau_j} \sin(m_j z),$$

where the $\tau_j$ are the relaxation times for the various vertical scales. If $\tau_j = \tau$ independent of $j$, then this expression reduces to the conventional WTG vertical velocity $W_{\text{swtg}} = D_h / \tau$.

The relaxation times can now be written in terms of $L$ and the gravity wave speeds for each vertical mode,

$$\tau_j = \frac{L}{c_j} = \frac{Lmj}{N} = \frac{\pi Lj}{NhN},$$

where we have used the hydrostatic gravity wave speed for vertical wave number $m_j$. 

where the $z_{\text{pol}}$ is the assumed boundary layer top. This is to restrict gravity-wave adjustment of near-surface thermodynamics, which are considered a strong function of the adjacent SST value. The value $z_{\text{pol}} = 1$ km is used in all experiments described in this paper, except where noted. The relaxation time is specified in terms of a horizontal scale $L$, and the speed $c$ of the gravity waves that disperse the potential temperature anomaly:

$$\tau = L/c,$$

where a simple model of hydrostatic gravity waves yields

$$c = Nh / \pi,$$

and $N$ is the approximate constant Brunt-Väisälä frequency of the troposphere.
\[ c_j = N_c / m_j, \]

with \( N_c \) assumed constant in height. As with conventional WTG, a lower bound \( \gamma_{\text{swtg}} \) must be placed on \( d\theta/dz \) in (6) to maintain physically realistic values of \( w_{\text{swtg}} \). However, the mask \( M(z) \) and the special treatment of the boundary layer, (3), are not needed.

To calculate (9), we employ a simple numerical integration scheme. We verified that this leads to an accurate Fourier reconstruction of \( D_0(z,t) \) when the complete set of modes is used. To estimate the Brunt-Väisälä frequency, we calculate \( N^2 = g(d\theta/dz)/\theta \) at every time step and then average this quantity between the surface and tropopause.

The parameter \( \gamma_{\text{swtg}} \) deserves some discussion here, as it plays a role in both WTG schemes. The minimum value of the static stability in the upper troposphere over the intensive flux array from TOGA-COARE (western Pacific warm pool) and the GATE array (eastern Atlantic) was estimated to be \( (d\theta/dz)_{\text{min}} \approx 1.6 \text{ K km}^{-1} \) (Johnson et al. [1999], estimated using their Figures 5a and 5b). Time-averaged atmospheric profiles obtained during the East Pacific Investigation of Climate Processes in the Coupled Ocean-Atmosphere System (EPIC) East Pacific, lasting from 1 September to 10 October 2001 show a typical minimum static stability value of \( (d\theta/dz)_{\text{min}} \approx 2.0 \text{ K km}^{-1} \). In the implementations of WTG used here, we use a minimum value of \( \gamma_{\text{swtg}} = 0.3 \text{ K km}^{-1} \). In practice, the static stability in the model crosses this threshold about 2.5% of the time. Since this is considerably smaller than observed values and is not a common model state, we feel it is a limit that allows the model to evolve normally while avoiding the rare occurrence of nonphysical vertical velocities.

4. Results

We now compare the conventional WTG and spectral WTG (SWTG) schemes in a simple surface flux forcing experiment. This experiment is intended to model the case of an anomalous patch of SST amidst a broad tropical background wherein the SST is consistent with a certain radiative convective-equilibrium state. We first perform an RCE calculation from which we obtain time-averaged reference profiles of potential temperature and moisture. We then initialize the model using these reference profiles, but we increase the sea-surface temperature (SST) over the background value to provide additional surface flux forcing during the WTG experiments. In each simulation, we use a 20 km by 256 km 2-D domain with 250 m vertical resolution and 1 km horizontal resolution. The radiation is held fixed in time with an equivalent potential temperature sink of \( (d\theta_s/dt)_{\text{rad}} = 2 \text{ K d}^{-1} \) from the surface to \( z = 12 \text{ km} \). This cooling profile is tapered linearly to zero from \( z = 12 \text{ km} \) to the tropopause \( z = 15 \text{ km} \). The surface wind speed in each run is maintained near \( \nu_s = 5 \text{ m s}^{-1} \) throughout the simulation using a wind relaxation constant \( \tau_v \approx 6 \text{ h} \). The SST is set to a constant 298 K during the RCE simulation.

Having obtained the time-averaged reference profiles from the RCE simulation, we then apply a uniform SST anomaly of \( \Delta T = 4 \text{K} \) to the radiative convective-equilibrium value across the entire domain and run the model for 30 days under the WTG forcing imposed by both conventional and spectral WTG vertical velocities. Here, \( w_{\text{swtg}} \) is given by (1) and \( w_{\text{swtg}} \) is given by (10), each with horizontal relaxation scale \( L = 300 \text{ km} \). This value of \( L \) corresponds to a relaxation time constant \( \tau \approx 1.7 \text{h} \) for the first vertical mode in each scheme, since \( \tau = \tau_1 = L/c_1 = L/\nu_h \). The last 15 days are averaged to obtain profiles of the parameterized vertical velocities \( w_{\text{swtg}} \) and \( w_{\text{swtg}} \) shown in Figure 1. The time-averaged value of the gradient in potential temperature for the spectral simulation is also shown in the figure; the corresponding profile for the conventional simulation (not shown) is similar. Note that ~95% of the upper tropospheric gradient values exceed the imposed minimum gradient used in both simulations, \( d\theta/dz > \gamma_{\text{swtg}} = 0.3 \text{ K km}^{-1} \).

The spectral WTG velocity is considerably smoother than the original, yet the maximum value is near 11 km for both schemes. The most significant differences in the vertical velocity profile of spectral WTG compared to conventional WTG are: a large increase in magnitude above 2 km and a larger vertical gradient between the surface and 6 km. The vertical velocity profiles shown here are similar to other surface flux forcing experiments (Sobel and Bretherton [2000], Figures 5b and 7b; RZ05, Figure 9; Romps [2012b], Figures 3 and 5).

The precipitation rate is increased by ~40% (not shown) in spectral WTG compared to conventional WTG in this experiment, which is a direct result of the steeper mass flux profile and corresponding greater mass...
convergence implied by the velocity profile shown in Figure 1. This experiment corroborates the findings of
the above-mentioned surface forcing experiments in that the vertical velocity is positive throughout the tro-
posphere, with minor exception in the boundary layer.

The equilibrium domain-averaged potential temperature anomaly of the spectral WTG simulation is shown
in Figure 2; the corresponding plot for the conventional WTG simulation (not shown) is broadly similar.
Note the strong cool anomaly that persists in the boundary layer. Both schemes avoid converting this into a
strong parameterized downdraft as is evident in Figure 1. This is prevented using somewhat artificial means
in conventional WTG. However, it is suppressed in spectral WTG due to the longer relaxation times assigned
to shallow wave modes in the Fourier expansion. This is closer to the physics of the real atmosphere, where
gravity waves corresponding to shallow temperature anomalies have slower propagation speeds.

The equilibrium total cloud water mixing ratio also differs between the two schemes (see Figure 2). The
middle tropospheric mixing ratio is \( \frac{3}{4} \) greater in the spectral WTG simulation. This added moistening is
likely due to increased moisture convergence owing to the stronger WTG vertical velocity.

4.1. Earlier Studies Revisited

We now apply the spectral WTG scheme to a selection of earlier studies chosen for their simplicity. In doing
so, we illustrate how spectral WTG performs, and also test the robustness of each experiment to a modified
parameterized vertical velocity.

4.1.1. Anomalous Surface Wind Experiment

Raymond and Zeng [2005] introduced the conventional WTG scheme (1), which they implemented in an ear-
lier version of the CRM used here. Their interest was in examining the dependence of precipitation on
changes in surface heat flux in a tropical environment. In particular, their experiment modeled a region of
anomalous surface wind within a much broader area experiencing the RCE forcing conditions. The authors
performed a 2-D RCE calculation followed by a series of WTG experiments with reference profiles derived
from the RCE state. A range of surface wind speeds was used in the WTG experiments in order to mimic the
effects of different surface flux conditions upon the modeled convection.

We now revisit the RZ05 experiment, beginning with an RCE calculation using the latest version of the CRM.
We adopt similar initialization parameters used in that study, except we decrease the resolution (from
500 m) to \( \Delta x = \Delta y = 1000 \text{m} \) and increase the horizontal domain size from 50 to 192 km. We adopt the ver-
tical resolution of \( \Delta z = 250 \text{m} \) and vertical model domain of 20 km used in RZ05. The radiative cooling is fixed
at an equivalent potential temperature tendency of \(-2 \text{ K d}^{-1}\) up to 12 km, linearly decreasing to zero at
the tropopause height of 15 km. We maintain the same constant mean wind of \( v = 5 \text{ m s}^{-1}\) blowing across
the 2-D domain using a wind relaxation constant $\tau_v = 5.6h$. The sea surface temperature is held constant at 303 K.

We then perform the series of WTG experiments described in RZ05 using both spectral and conventional WTG schemes. The WTG reference profiles are time-averaged profiles of potential temperature and total water vapor mixing ratio, obtained from a statistically steady 70 day period of the RCE simulation. Each WTG simulation was run for 40 days. As is explained below, however, two simulations were extended to 80 days to check for instability. As in RZ05, we performed a series of simulations using various surface wind speeds: $v = 0, 5, 10, 15$, and $20$ m s$^{-1}$. As in RZ05, we employ a relaxation timescale of $\tau_v = 1.85h$ in the conventional WTG simulations. Since the Brunt-Väisälä frequency is continuously diagnosed from the model when the spectral WTG scheme is used, the resulting timescale varies by $\sim4\%$ over all the simulations, with a typical value of $\tau_v = 2h$ for the deepest gravity wave mode ($j = 1$). Time averages are computed over the last 20 days of each simulation for plotting.

**Figure 2.** Equilibrium warm anomaly for the spectral WTG simulation (left) and percent difference in equilibrium total cloud water mixing ratio between the two schemes (right), using $\% \text{ difference} = 200\% (r_{t_{\text{swtg}}} - r_{t_{\text{wtg}}}) / (r_{t_{\text{swtg}}} + r_{t_{\text{wtg}}})$. The equilibrium value is obtained by averaging the horizontal mean over the last 30 days of each simulation in each plot. Dotted lines indicate 95% confidence intervals around the mean profile, or approximately the mean value $\pm 2\sigma$.

**Figure 3.** The domain-averaged rain rate (left) and column precipitable water (right) for WTG simulations using the conventional WTG scheme. Results for the $v = 0$ m s$^{-1}$ (top) and $v = 5$ m s$^{-1}$ (bottom) runs are shown. Note the time axis differs between rows.
4.1.1.1. Conventional WTG - old versus new
We now investigate significant differences between WTG simulations. To account for changes made to the model since RZ05, we briefly compare RZ05 results to those using the conventional WTG scheme in the current version of the CRM. Time series of rain rate and precipitable water for the conventional WTG experiments using surface winds of $v = 0 \text{ m s}^{-1}$ and $v = 5 \text{ m s}^{-1}$ are shown in Figure 3 (compare with Figures 5 and 6 in RZ05).

The $v = 0 \text{ m s}^{-1}$ simulation (first row, Figure 3) is broadly similar to that of the original experiment. The $v = 5 \text{ m s}^{-1}$ (second row, Figure 3) simulation uses the same surface flux conditions as in the RCE, so that we expect it to evince similar diagnostics. However, a trend in the column-integrated precipitable water toward the end of the 40 day simulation suggested the model was not yet in equilibrium; we thus extended the run to 80 days. Ultimately, the precipitable water drops by 44% of its original value, evidence of a persistent instability in WTG experiments that employ surface forcing conditions similar to those used in the RCE simulation. In RZ05, the precipitable water dropped twice as quickly, though the corresponding simulation only ran for 20 days in that study. The authors speculated that the instability might have been due to lack of equilibrium in the RCE profile.

In the course of the current study, we found that this reduction in moisture occurs earlier in the simulation when the model domain is small. This may indicate that smaller domains cannot usefully accommodate both convection and subsidence events such that the resulting RCE profiles are too stable. Longer RCE simulations and time averages may give more robust reference profiles, but this instability seems to be a robust behavior when the RCE surface fluxes and radiation are used.

The instability found here is similar to that obtained by Daleu et al. [2012] in a study using the Met Office Large Eddy Model. In that study, as well as in RZ05, the corresponding rainfall rate fell by $\sim 80\%$. An even greater reduction is likely in the Met Office model for relaxation timescales less than 2 h (see Figure 5, Daleu et al. [2012]). This is also consistent with our study: we use a smaller timescale (1.85 h) and our rainfall ceases completely. It is likely that the simulation in RZ05 would have ceased precipitating as well had the run been extended further. Precipitation rates and precipitable water values for the $v = 10 \text{ m s}^{-1}$ and $v = 20 \text{ m s}^{-1}$ simulations resemble those of the original RZ05 experiment and are not shown.

Time-averaged vertical mass flux profiles using conventional WTG are shown at left in Figure 4 (compare to Figure 9 in RZ05). These are similar to those found in RZ05 except the upper tropospheric vertical mass flux magnitudes in our study are $\sim 40\%$ smaller for the $v = 10 \text{ m s}^{-1}$ and $v = 20 \text{ m s}^{-1}$ simulations. The magnitude difference in upper tropospheric mass flux profiles between RZ05 and our conventional WTG experiment is due to a change in the implementation of the WTG advection scheme (see Appendix B).

4.1.1.2. Conventional WTG versus spectral WTG
We now compare conventional and spectral WTG schemes using the diagnostics of the RZ05 experiment. The rainfall rate and precipitable water time series for the $v = 0 \text{ m s}^{-1}$ and $v = 5 \text{ m s}^{-1}$ spectral WTG simulations are shown in Figure 5. As with conventional WTG, when the surface wind is zero there is an initial small rain event followed by negligible rain thereafter. A rain event occurs near day 20, but the trend illustrated by the precipitable water is clear: the model environment descends to a steady state of negligible precipitation.

The rate of drying is slower than in the conventional WTG scheme—an effect likely related to slightly positive vertical velocity in the boundary layer, not evident at the scale shown in Figure 4. This feature allows moist parcels to be advected above the surface and is consistent with the few precipitation events that occur, acting to slow the drying trend above the boundary layer.

The rain rate and precipitable water for the $v = 5 \text{ m s}^{-1}$ case are initially similar to the corresponding conventional WTG case; however, both quantities appear stable throughout the run. To check if drying occurs over the same period, it takes the conventional WTG simulation to reach the nonprecipitating equilibrium state, we extended this simulation to 80 days. The corresponding equilibrium mass flux profiles (right, Figure 4) lend some insight here. While the mass flux profile for conventional WTG is descending through the depth of the free troposphere, that of the spectral WTG is slightly ascending there. The spectral WTG simulation thus avoids the descent seen in conventional WTG when RCE surface conditions are used.

Vertical mass flux profiles for the greater surface wind speed cases are smoother and of greater magnitude in the spectral scheme than in the corresponding conventional WTG scheme simulations (compare dashed and solid lines between left and right plots of Figure 4). The smoothness of the spectral scheme results...
from the dominance of faster wave modes, which have deeper wave structures. The high wave number modes needed to describe, for example, the sharp peak at 250 hPa in the mass flux profile of the $v = 20 \text{ m s}^{-1}$ conventional WTG simulation, correspond to very slow adjustment times and are thus negligible contributors to the mass flux profile. This smoothing effect also helps to minimize WTG adjustments in the modeled boundary layer, even when there is significant heating due to shallow convection. Since the boundary layer is shallow, a temperature anomaly there projects most strongly onto wave modes that have small adjustment timescales. The steady state vertical mass flux near 250 hPa is greater than the corresponding conventional WTG profile by $\sim 25\%$ for the $v = 20 \text{ m s}^{-1}$ case. We verified that the increase in the mass flux profile aloft is not due to the sign or magnitude of the WTG vertical velocity in the boundary layer. Indeed, if the boundary layer mass flux were positive in these simulations, the tropospheric mass flux profiles for spectral WTG would be even larger (see section 6). The greater mass flux values aloft in these cases are likely due to the greater mass flux values in the lower free troposphere, which act to advect more moisture aloft, leading to greater upper tropospheric warming via latent heat release. At 800 hPa, the vertical mass flux profile is $\sim 60\%$ greater compared to the corresponding conventional WTG value for the $v = 20 \text{ m s}^{-1}$ case. While this

![Figure 4](#)

*Figure 4.* Equilibrium mass flux profiles for wind forcing experiments averaged over the last 20 days of each run for conventional WTG (left) and spectral WTG (right).
proximate cause is likely, the ultimate cause of the shape of the profile is difficult to determine as it represents a balance of many interactive processes.

The difference in lower tropospheric vertical mass flux profiles between the two WTG schemes leads to a significant change in the surface rainfall rate. Figure 6 shows equilibrium precipitation rates for both WTG schemes and for each surface wind speed used. Results from the original RZ05 experiment are also shown for reference. The increase in rainfall rate from conventional WTG to spectral WTG for the $v = 20 \text{ m s}^{-1}$ case is $\sim 50\%$. Other wind speed cases also show increases in rainfall rate from those of the corresponding conventional WTG simulations. This is consistent with differences in mass flux profiles in each case.

### 4.1.2. Anomalous Reference Profile Experiment

While the RZ05 comparison illustrates distinguishing features between conventional and spectral WTG, only modified surface fluxes are used to alter the state of convection in that experiment. The smoothing effect of the spectral scheme is more significant when perturbations are applied to the reference profile, as in the experiment of Raymond and Sessions [2007] (hereafter RS07).

The authors used conventional WTG to test the hypothesis that increased stability in the tropical environment leads to a lowering of the vertical mass flux maximum such that the resulting inflow driven by mass convergence occurs over a shallower layer. The CRM used was an earlier version of that described in section 2, and changes in the model since RS07 have been minimal. The experiment involved increasing the stability of the assumed environment outside the model domain by applying perturbations to the reference potential temperature profile (see Figure 2a and text in RS07 for more details). This experiment models changes to the thermodynamic profiles of the RCE state which could occur, e.g., during the passing of a tropical wave through a broad region of uniform forcing conditions. However, any stimulus that modifies the temperature and/or moisture within the tropical atmosphere can be considered a target of this simplified experiment.

Here, we perform the WTG experiment again with the latest version of the CRM using both conventional and spectral WTG, except we use a larger 2-D domain size (200 km instead of 50 km) with slightly coarser horizontal resolution ($\Delta x = 1000\text{ m}$ instead of $\Delta x = 500\text{ m}$). The resulting mass flux profiles using both WTG schemes are shown in Figure 7. The profile obtained using the conventional WTG scheme is broadly similar to the results of RS07, except the magnitudes are smaller. This seems to be due to the difference in domain...
sizes used, as a smaller domain simulation (not shown) gives similar magnitudes as in RS07. Less significant differentiating factors are due to several minor changes made to the CRM code, described in Appendix B.

More significant differences are evident between the conventional and spectral WTG schemes. The spectral scheme is much smoother, again owing to the fact that the effects of higher wave number modes are attenuated. This again prevents discontinuities such as that seen at the freezing level near $z = 5$ km. In addition, although the boundary layer taper (3) is not used, the spectral scheme nevertheless produces negligible vertical mass flux there. As in our reconstruction of the RZ05 experiment, the spectral WTG mass flux magnitudes are larger than those using the conventional scheme.

5. Response to Transient Temperature Anomalies

We now examine the difference between conventional and spectral WTG in the context of recent analyses presented in Romps [2012a, 2012b]. The author performed several novel analytical and numerical comparisons of WTG and WPG (a. k. a. DGW). Although we feel the term damped gravity wave (DGW) is a more illustrative name for this scheme as it highlights the used of damping in the adjustment mechanism, we use the term WPG throughout this section to facilitate comparison with Romps’s work. Here, we revisit Romps’s analysis of WTG in the presence of transient, localized buoyancy anomalies. This experiment considers the events following the spontaneous appearance of a localized temperature anomaly, as due to convective heating or adiabatic lifting. While such an onset of buoyancy anomaly is not physical, we use Romps’s simplification for comparison purposes. The question of how these schemes behave during a more gradually formed buoyancy anomaly will be explored in a later publication.

As in Romps [2012a], we obtain an expression for the time-dependent buoyancy when spectral WTG is used. The linearized, hydrostatic Boussinesq equations form the basis for the derivation. We compare the evolution of a given buoyancy anomaly for the full 3-D Boussinesq system to that predicted by the conventional and spectral WTG approximations. To begin, we derive the spectral WTG buoyancy and compare it to the solutions derived in the previous work.

The linear, hydrostatic, Boussinesq buoyancy anomaly evolves according to

$$\frac{\partial b}{\partial t} = -N^2 w + Q.$$  \hspace{1cm} (13)

where $b = \frac{g h'}{\theta}$ is the buoyancy, $N$ is the Brunt-Väisälä frequency, $w$ is the vertical velocity, and $Q$ is the buoyancy source. In the transient case where no heating or frictional dissipation occurs, $Q = 0$, and (13) becomes

$$\frac{\partial}{\partial t} \sum_j b_j \sin(m_jz) = -N^2 \left( \frac{NH}{\pi L} \sum_j \Theta_j \sin(m_jz) \right).$$ \hspace{1cm} (14)

where we have replaced the vertical velocity by the spectral WTG expression (10) and represented the buoyancy by a Fourier series with coefficients $b_j$. Since $b = N^2 D \theta$ from (6), $b_j = N^2 \Theta_j$ for constant $N$, and (14) can be rewritten

$$\sum_j \frac{\partial b_j}{\partial t} \sin(m_jz) = -N^2 \left( \frac{NH}{\pi L} \sum_j b_j \sin(m_jz) \right).$$ \hspace{1cm} (15)

Since the sine functions are linearly independent, (15) must be satisfied on a term-by-term basis:

$$\frac{\partial b_j}{\partial t} = -\frac{NH}{\pi j L} b_j.$$ \hspace{1cm} (16)

This equation has the solution $b_j(t) = b_j(0) \exp(-Nht/\pi j L)$ and we find that

$$b_{\text{hyp}}(t) = b_{\text{hyp}}(0) \exp(-Nht/\pi j L).$$ \hspace{1cm} (17)

The unknown free parameter in (17) is the horizontal scale $L$. It is thought to be related to the distance over which gravity waves must travel in order to suppress local buoyancy anomalies, though its precise meaning is unclear, i.e., the relevant distance in a limited domain CRM or SCM is not obvious. However, we can
choose a particular value for \( L \) in this comparison so that the parameterization schemes respond as closely as possible to the 3-D system of equations when confronted with a buoyancy anomaly of a given depth, or vertical wave number \( \nu \) and radius \( r \). Given that the vertical mode \( j \) of such a buoyancy anomaly evolves according to

\[
b_{3D-j}(t) = b_0 \exp \left( -\frac{Nht}{\pi r} \right),
\]

comparison with (17) suggests that we set \( L = r \), i.e., \( L \) becomes the assumed radius of the buoyancy anomaly being neutralized by the outgoing gravity waves in SWTG. The corresponding expression for conventional WTG takes the form:

\[
b_{\text{wtg-j}}(t) = b_0 \exp \left( -\frac{Nht}{\pi L} \right).
\]

Similar expressions for WPG are derived in Romps [2012a] and are not copied here.

Note that the effective decay timescale for the full 3-D equations has a \( j^{-1} \) dependence. The same is true for spectral WTG. In contrast, conventional WTG has no \( j \) dependence and that of WPG is \( j^{-2} \). These are summarized in Table 2.

The evolution of each buoyancy expression is illustrated in Figure 8, as in Romps [2012a]. Cases for two initial buoyancy anomalies at \( \pi = 0 \) are shown: \( j = 1 \) and \( j = 5 \), corresponding to vertical depths of 15 km and 3 km, respectively. For a buoyancy anomaly spanning the full 15 km depth of the troposphere (left side of Figure 8), all parameterized schemes are tuned to approximate the results of the full 3-D system and are thus superimposed. However, for an imposed depth of 3 km (right side of Figure 8) WPG adjusts too slowly and conventional WTG adjusts too quickly. In contrast, spectral WTG continues to agree approximately with the 3-D model results. This is because spectral WTG is the only scheme that exhibits the same dependence on \( j \) as the 3-D model (see Table 2).

To examine the actual behavior of the two WTG schemes in a CRM when confronted with a strictly transient temperature anomaly, we recreated the conditions described in Romps [2012b], section 5. As in the latter experiment, we shut off surface and radiative fluxes in the CRM, set the mixing ratio to zero everywhere in the column, and impose an initial potential temperature lapse rate of 3.6 K km\(^{-1}\). We then initialize a WTG experiment with a \( +1 \) K potential temperature anomaly across the horizontal domain between 2.5 km and 7.5 km, using both conventional and spectral WTG schemes with \( L = 86 \) km (\( t \approx 30 \) min for the deepest wave mode). The reference profile used also has zero moisture, but the potential temperature is that of the original imposed lapse rate without the anomaly.

Each WTG scheme then acts to eliminate the temperature anomaly \( \theta^0 = 1 \) at \( t = 0 \). As noted in Romps [2012b], the depth over which parcels must be lifted \( D_0 \) is given by a simple expression. The

![Time series of the scaled buoyancy for each of the schemes defined in equations (17) through (19) for buoyancy anomalies of vertical wave number \( j = 1 \) (left) and \( j = 5 \) (right).](image-url)
A thermodynamic equation in pressure coordinates in the absence of diabatic heating and horizontal temperature gradients is \( \frac{\partial T}{\partial t} = -\omega S \), where the static stability \( S = -\frac{L}{\Gamma_d - \Gamma} / \rho g \), and \( \Gamma \) and \( \Gamma_d \) are the lapse rate and dry adiabatic lapse rate, respectively. Approximating \( \frac{\partial T}{\partial t} = \frac{\Delta T}{\Delta t} \) and \( \frac{\partial \theta}{\partial z} = \frac{\Delta \theta}{\Delta z} \), we find that \( D_\theta = \Delta z \approx \Delta T / (\Gamma_d - \Gamma) \approx 300 \text{m} \). We can diagnose this from the CRM by integrating the respective parameterized vertical velocity over time: \( D_\theta(z) = \sum \Delta t \theta_{wgt}(z) \).

The results are shown in Figure 9 and are easily compared with corresponding plots in Figure 8 of Romps [2012b]. As predicted, both conventional and spectral WTG schemes lift parcels through a depth of \( D_\theta \approx 300 \text{m} \) over a long-time integration (1 day). This is shown in the top row of plots of the figure. However, only spectral WTG produces transient nonlocal parcel adjustment, as seen in the bottom row of plots of the figure. Over the first and second hours of the simulation, conventional WTG undergoes strong vertical adjustment, but only within the limits of the original temperature anomaly; this was also found in Romps [2012b]. In contrast, however, the spectral WTG vertical velocity lifts parcels above and below the initial anomaly during the first hour and lowers parcels during the second hour. As asserted in Romps [2012b], the conventional WTG scheme lacks the ability to produce nonlocal adjustment. However, the simulation shows that this is indeed possible with spectral WTG.

6. Multiple Equilibria and the PBL Treatment

We now explore consequences of the boundary layer treatment used in calculating the conventional WTG vertical velocity. In particular, we examine the results of a simple multiple equilibrium experiment, modeled after that described in Sessions et al. [2010] (hereafter S10). Multiple equilibrium experiments are generally...
performed to investigate the respective characteristics of adjacent moist and dry regions of the tropical atmosphere, as in the case of dry patches near organized convection. We use the same domain, resolution, relaxation timescale, surface wind values, and RCE reference profile described in subsection 4.1 for the RZ05 experiment. Surface fluxes used to calculate the RCE state in S10 were based on the same parameters as used in our study: \( \text{SST} = 303\text{K} \) and \( \nu = 5\text{m s}^{-1} \). Fixed radiation was used unlike the experiment performed by S10, who used interactive radiation.

We run the WTG simulations for 40 days in order to give enough time for an equilibrium state to develop. Runs are extended beyond 40 days when a trend exists in the column integrated precipitable water at the end of the 40 day run. Each simulation is performed twice using both the WTG and SWTG schemes: once with the RCE reference profile value of relative humidity (moist) and once with zero relative humidity (dry). The run times used seem appropriate here: for initially dry runs that do not precipitate during the simulation, none show a trend in column-integrated precipitable water. Thus, we do not expect these runs to precipitate, even in a longer simulation. The resulting rain rates are compared between the moist (taken from the RZ05 experiment) and the corresponding dry simulations. In each of the simulations, the boundary layer depth parameter is set to \( z_{\text{b0}} = 1\text{ km} \).

Equilibrium surface precipitation rates and times to reach equilibrium for each simulation are shown in Figure 10. Precipitation rates are calculated as the time average of surface precipitation from the approximate time when equilibrium occurs to the end of the run. The time to reach an equilibrium state is estimated as the time when the slope of the column integrated precipitable water time series becomes approximately flat. This is trivial to estimate to a precision of \( 1\text{ day} \) in all cases, except for that of the \( \nu = 5\text{m s}^{-1} \) simulation using spectral WTG, which takes much longer to equilibrate than the other simulations for reasons discussed below.

The moist-initialized simulations using conventional WTG exhibit nonzero precipitation for surface wind speeds greater than \( \nu = 8\text{ m s}^{-1} \). For spectral WTG, however, precipitation also occurs for the \( \nu = 5\text{ m s}^{-1} \) case. The conventional WTG scheme produces multiple equilibria for two of the cases we tested: \( \nu = 8\text{ m s}^{-1} \) and \( \nu = 10\text{ m s}^{-1} \). There may be additional cases of multiple equilibria spanning the range of \( \nu = (5, 15)\text{ms}^{-1} \).

In contrast, spectral WTG does not show any evidence of multiple equilibria in our experiment. Although the spectral WTG \( \nu = 5\text{ m s}^{-1} \) dry simulation took a very long time to reach equilibrium (110 days), it regains a precipitating state during this time. It is possible that multiple equilibria might exist for a slightly lesser wind speed. However, a search for that wind speed may be computationally expensive and was not done. Nevertheless, the range of possible multiple equilibria in the case of spectral WTG is limited to the interval

![Figure 10](image-url)
In a broader study of multiple equilibria using fixed radiation and a range of domain sizes (results not shown), the spectral WTG scheme shows evidence of multiple equilibria, but only for surface flux values similar to that of the RCE reference profile; in each of these experiments conventional WTG produces multiple equilibria over a broader range of surface flux values.

Given the cases of multiple equilibria found for the conventional WTG scheme, we want to know if these results are sensitive to the $z_{pbl}$ parameter that defines the assumed top of the boundary layer in (1). To test this, we recalculate two simulations over a range of $z_{pbl}$ values and again examine the model output as described above. The surface wind speeds we choose to test are those at the upper end of the range of cases exhibiting multiple equilibria. We therefore ask the following questions: (1) Can we induce precipitation in the greatest wind speed simulation that remains dry ($v = 10$ m s$^{-1}$; open squares) and the first precipitating simulation following the dry run ($v = 15$ m s$^{-1}$; solid squares) are shown.

\[
v = (3, 5) \text{ m s}^{-1}\]

In this paper, we present a form of the relaxed weak temperature-gradient approximation in which each gravity wave vertical mode is assigned a unique speed. This is an improvement over the conventional WTG scheme, which uses the same wave speed for each vertical mode. Several distinct benefits arise from this innovation:

1. The relaxation of domain-averaged temperature anomalies occurs primarily via the superposition of low-wave number (smooth) parameterized gravity wave modes. This is beneficial for numerical experiments that may be sensitive to grid-scale discontinuities.
2. The response of spectral WTG to transient temperature anomalies occurs at a rate similar to that of the linearized, hydrostatic Boussinesq equations for a complete set of vertical wave modes. This is not true of other large-scale parameterization schemes.

3. The adjustment due to spectral WTG is vertically non-local due to the natural dominance of low-wave number vertical modes. Such nonlocality can drive vertical motion beyond model levels exhibiting temperature anomalies.

4. No restriction on the WTG mechanism is needed in the boundary layer when spectral WTG is used and thus the $z_{pol}$ parameter is not needed. This is another effect of the scheme’s preference for low-wave number vertical modes, by which temperature anomalies occurring over the boundary layer depth are largely suppressed.

In addition to these favorable qualities, spectral WTG has somewhat different stability characteristics than does conventional WTG. In our study, no spectral WTG simulation using RCE surface wind speeds and SST evolves to a nonprecipitating state. This is in contrast to conventional WTG, which ceases to precipitate in this case.

It is likewise noteworthy that multiple equilibria occur in our study when conventional WTG is used with fixed radiation. The theoretical study of Emanuel et al. [2014] and the numerical study by Wing and Emanuel [2014] both suggest that the radiative-convective instability, a mechanism thought to herald the onset of convective self-aggregation, can only occur when radiative processes depend upon atmospheric moisture. Since multiple-equilibria are often likened to a manifestation of the self-aggregation process, it should abide by similar forcing constraints. Is the existence of multiple equilibria under fixed radiation physically meaningful, or is it simply an aberrant state of the conventional WTG parameterization scheme? Results of the brief multiple-equilibrium experiment presented here and other simulations (not shown) suggest that our refinement of the WTG mechanism also refines the behavior of the instability associated with a multiplicity of states. In light of the noted improvements of spectral WTG over conventional WTG, we thus speculate that using spectral WTG in multiple-equilibrium experiments will lead to a more robust exploration of this process.

Appendix A: Description of Cloud Resolving Model

The conservation of mass is expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (A1)$$

where $\rho$ is the density, $\mathbf{v}$ is the velocity, and $\nabla$ is the 3-D gradient operator. The momentum then evolves according to

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} - K \mathbf{D}) + \nabla \rho + g \rho \mathbf{k} = \rho (\mathbf{F}_s - \mathbf{F}) - \rho (\mathbf{v} - \mathbf{v}_0), \quad (A2)$$

where $\mathbf{v}_0(z)$ is a target horizontal velocity profile, $\mathbf{F}_s(z) = \mathbf{F}_s(z - h)/(z_{top} - h)$ is a damping profile turned on only in the stratosphere ($z > h$) for the purpose of absorbing upward-propagating gravity waves with $z_{top}$ and $h$ being heights of the model top and tropopause, $K$ is the eddy-mixing coefficient, $\rho$ is the pressure, $g$ is the acceleration of gravity, $\mathbf{k}$ is the vertical unit vector, $\mathbf{F}_s$ is the force due to surface stresses, $\mathbf{F}$ is an external momentum sink, used generally to constrain the horizontally averaged wind profile to a desired form, and $\mathbf{D}$ is the strain rate tensor

$$D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (A3)$$

The prognostic thermodynamic variables are specific moist entropy, total cloud water mixing ratio (vapor plus advected condensate), and precipitation mixing ratio. The specific moist entropy $s$ evolves according to

$$\frac{\partial s}{\partial t} + \nabla \cdot (\rho s \mathbf{v} - K \nabla s) = \rho (S_{ss} - S_{sr} - S_{se}), \quad (A4)$$

where $S_{ss}$ is the source of entropy from surface fluxes, $S_{sr}$ is the source from radiation, and $S_{se}$ is the external entropy sink, to be discussed later. The total cloud water mixing ratio $r_t$ evolves according to
\[
\frac{\partial \rho r_t}{\partial t} + \nabla \cdot (\rho v r_t - K \nabla r_t) = \rho (S_{r_t} + S_{r_p} - S_{r_t}),
\]

where \( r_t = r_v + r_c, r_c \) is the water vapor mixing ratio, \( r_t \) is the mixing ratio of advected condensate, \( S_{r_p} \) is minus the conversion rate of cloud water to precipitation, \( S_{r_t} \) is the source of total cloud water mixing ratio from surface evaporation, and \( S_{r_p} \) is the external sink of total cloud water mixing ratio. The equation for precipitation mixing ratio \( r_p \) is

\[
\frac{\partial \rho r_p}{\partial t} + \nabla \cdot \left[ \rho (v - w_t \hat{\kappa}) r_p - K \nabla r_p - K_n \frac{\partial r_p}{\partial z} \right] = -\rho S_{r_p}.
\]

where \( w_t \) is the terminal velocity of precipitating moisture—taken as positive, \( K_n = \hat{\lambda} \rho (\Delta z^2 / \Delta t) \) represents numerical viscosity, \( \hat{\lambda} \) is the numerical eddy viscosity factor, \( \Delta z \) is the vertical grid size, and \( \Delta t \) is the time step. The latter acts to suppress grid-scale oscillations that only occur in the precipitation mixing ratio.

The radiative source of specific moist entropy \( S_{r_t} \) is calculated either from a fixed profile or from an interactive radiative transfer calculation. The fixed profile is either a specified equivalent potential temperature or is a time-varying function of height taken from an initialization profile. The interactive radiation is computed by a slightly modified version of the toy radiation package of Raymond (2001); this is not called at every time step in order to save computation.

The eddy mixing coefficient is nonzero when the square of the deformation rate minus twice the Brunt frequency is positive:

\[
K = \rho C \left[ \sum_{ij} D_i D_j - \frac{2q_d}{q} \nabla \cdot D \right]^{1/2} \Delta z^2
\]

where \( C \) is a constant scaling factor. We set \( K = 0 \) when the quantity inside the square brackets is negative. This corresponds to having the Richardson number \( Ri > 1/4 \), indicating quasi-laminar flow. The effective stability is

\[
\Gamma_e = A \frac{\partial q_e}{\partial z} + (1 - A) \frac{\partial \theta}{\partial z}
\]

where \( \theta \) is the potential temperature, \( \theta_e \) is the equivalent potential temperature, \( A = [(H-1)/\partial H + 1]/2 \), and where \( H = r_t / r_c, r_c \) being the saturation mixing ratio. The parameter \( \partial H \) determines the rate of transition from an unsaturated to a saturated atmosphere in the eddy mixing calculation. We limit \( A \) to the range \( 0 \leq A \leq 1 \).

The above-mentioned sinks of momentum, moist entropy, and moisture just balance the sources of these quantities due to convective and radiative processes in the steady state, and thus on the average are equivalent to the large scale sources for these quantities. They therefore approximate the source terms for the large-scale dynamical equations. The large-scale sink term is omitted from the rainwater equation since the time scale for this equation is in general much shorter than that for large-scale vertical advection. The forms of the sink terms are discussed below in section 3.

We use an approximate formula for the specific moist entropy \( s \):

\[
s = C_p \ln \left( \frac{T}{T_s} \right) - R_v \ln \left( \frac{p}{p_s} \right) + (L_v + L_f) r_c / T_s.
\]

where \( C_p \) is the specific heat of dry air at constant pressure, \( T \) is the temperature, \( T_s \) is a reference temperature, \( R_v \) is the gas constant for dry air, \( p_s \) is a reference pressure, and \( L_v \) and \( L_f \) are the latent heats of condensation and freezing. The equivalent potential temperature is related to the entropy by

\[
\theta_e = T_s \exp \left( \frac{s}{C_p} \right)
\]

The saturated moist entropy \( s_s \) is obtained by replacing the mixing ratio with the saturated mixing ratio in the entropy formula. The saturation mixing ratio \( r_t = R_v e_s(T) / R_v p \) with \( R_v \) being the gas constant for water vapor and \( e_s(T) \) is the saturation vapor pressure in Pascals at temperature \( T \) (in Kelvin), approximated by Teton's formula:

\[
e_s(T) = 611.2 \exp \left[ 17.67(T - 273.15) / (T - 29.65) \right].
\]

The ideal gas law
The particle terminal velocity is externally specified with different values above $(w_t)$ and below $(w_{s0})$ the freezing level. The total water source term due to the formation and evaporation of precipitation is given by

$$S_{ep} = -\lambda_p (r_t - r_s), \quad r_t > r_s$$  \hspace{1cm} (A15)

in the condensing case and

$$S_{ep} = -\lambda_w (r_t - r_s), \quad r_t < r_s$$  \hspace{1cm} (A16)

in the evaporating case. The constants $\lambda_p$ and $\lambda_w$ are externally specified rates and $\lambda_p$ uses different values above $(\lambda_{p0})$ and below $(\lambda_{pww})$ the freezing level.

The surface flux sources are concentrated in the lowest model layer and are derived from the bulk flux formula:

$$S_{st} = C_d U_c [s_{ss} - s(0)] / (\Delta z/2)$$ \hspace{1cm} (A17)

where $C_d$ is the transfer coefficient, $s_{ss}$ is the saturated sea surface entropy, $s(0)$ is the moist entropy of air at the lowest model level, where

$$U_c = [v_x(0)^2 + v_y(0)^2 + W^2]^{1/2}$$ \hspace{1cm} (A18)

is the effective surface wind with $W$ being an externally set gustiness parameter;

$$S_{nt} = C_d U_c [r_{nt} - r_t(0)] / (\Delta z/2)$$ \hspace{1cm} (A19)

where $r_{ss}$ is the saturation mixing ratio at the sea surface temperature and pressure;

$$F_s = -C_d U_c v(0) / (\Delta z/2).$$ \hspace{1cm} (A20)

The surface fluxes are deposited in a layer of thickness $\Delta z/2$ because this is the part of the lowest grid cell which is above the surface.

The conversion of surface thermodynamic fluxes from the difference in equivalent potential temperatures in earlier versions of the model to the difference in moist entropy values as in (A17) leads to a small change in the magnitude of the surface entropy forcing. This and other negligible artifacts of the numerical integration scheme contribute to a slightly cooler ($\Delta T \approx -1K$ at mid troposphere) temperature profile in the radiative convective-equilibrium state when the moist entropy implementation is used.

The external sink of total water mixing ratio in (A5) is a combination of entrainment from the surrounding environment, as represented by the reference profiles of moist entropy and total cloud water $s_0(z)$ and $r_0(z)$, and a large-scale vertical advection by the mean vertical velocity $w_{avg}(z)$ (or $w_{swag}$), described in section 3:

$$S_{re} = \lambda_{em} (r_t - r_{t-ent}) \frac{1}{\rho_0} \frac{\partial \rho_0 w_{avg}}{\partial z} + w_{avg} \frac{\partial r_t}{\partial z} + \lambda_{sm} (r_t - r_{o}).$$ \hspace{1cm} (A21)

where $r_t$ is the domain averaged total water mixing ratio profile. When lateral entrainment is used $\lambda_{em} = 1$, otherwise $\lambda_{em} = 0$. Similarly, $\lambda_{sm}$ controls relaxation of thermodynamic profiles to environmental values, as in Sobel et al. [2007]. We set $r_{t-ent}$ to $r_t$ at detraining levels, i.e., where $\partial \rho_0 w_{avg} / \partial z < 0$, and to the reference profile $r_{o}$ where $\partial \rho_0 w_{avg} / \partial z > 0$. A similar equation is written for the external entropy sink in (A4):

$$S_{se} = \lambda_{em} (s - s_{ent}) \frac{1}{\rho_0} \frac{\partial \rho_0 w_{avg}}{\partial z} + w_{avg} \frac{\partial s}{\partial z} + \lambda_{sm} (s - s_{o}).$$ \hspace{1cm} (A22)

The external sink of momentum in (A2) is a relaxation toward a reference profile of horizontal velocity $v_0$ at a rate $\lambda_{v0}$.
\[ F = \dot{\lambda}_{rad} (v - v_0). \]  

We ignore vertical advection of momentum by \( w_{wtg} \) because in general we make the effect of relaxation strong enough to force the mean wind to the specified wind profile.

A notable change made to the large-scale advection concerns the external entropy sink. For the experiments performed in RZ05, the total cloud water sink was calculated using \( \langle A21 \rangle \) but the entropy sink was calculated differently. The entropy-like prognostic variable used in RZ05 was the equivalent potential temperature and the external sink \( S_e \) in that model was calculated using the chain rule. Note the symbol for this quantity differs from \( S_{se} \) used in \( \langle A4 \rangle \) because the prognostic variable is different:

\[
S_e = \left( \frac{\partial \theta_g}{\partial t} \right)_{ext} = \left( \frac{\partial \theta_g}{\partial t} \right)_{ext} + \frac{\partial \theta_g}{\partial t} \left( \frac{\partial v}{\partial t} \right)_{ext}
\]

\[
= \frac{\theta_0}{\theta_0} \left( w_{wtg} \frac{\partial \theta_g}{\partial z} + (L_c + L_f) \rho_c \frac{\partial r_e}{\partial \theta_e} / T_R \right), \tag{A24}
\]

where \( \langle \rangle_{ext} \) indicates external processes, \( \theta_0 \) and \( \theta_0 \) are reference values of equivalent potential temperature and potential temperature, \( w_{wtg} \) comes from \( \langle 1 \rangle \), and \( S_{se} \) was given by \( \langle A21 \rangle \). Thus, while the current version of the model computes the entropy sink directly, the RZ05 version inferred it from the cloud water mixing ratio and potential temperature sinks. This change led to modest reductions in the amplitude of the vertical mass flux profile and of the precipitation rate during surface flux forcing experiments. See Appendix B for details.

**Appendix B: List of Significant Model Changes Since RZ05**

The CRM used in this study has evolved since RZ05, leading to model behavior that differs from earlier results in a quantitative, though not necessarily qualitative way. We examine these changes in terms of various diagnostics, chiefly the column-averaged percent differences in steady state WTG vertical mass flux and surface precipitation between conventional WTG experiments using the original RZ05 model and later model versions. An imposed wind of \( v = 20 \text{ m s}^{-1} \) with reference profiles from an RCE calculation using \( v = 5 \text{ m s}^{-1} \) defined the WTG simulation in each case.

In the model used for Raymond and Sessions [2007] and also for Sessions et al. [2010], minor changes were made to the parallelization scheme. The diabatic warm anomaly \( \theta' = \theta - \theta_0 \) was smoothed using a 1-2-1 filter and conservative smoothing was added to the precipitation mixing ratio, both to suppress oscillations in the vertical dimension. Entrainment due to the WTG adjustment scheme, previously limited to cells above the lowest grid cell, was allowed to occur at the surface. A minor change to a Newton’s method scheme used to obtain the potential temperature was made; the lower troposphere was slightly warmer and the mass flux profile was slightly stronger due to this change. The most significant change at that time was that the sink term entering the prognostic variable representing equivalent potential temperature, previously a relaxation to the potential temperature reference profile calculated using (A24), then represented explicit entrainment and vertical advection of equivalent potential temperature via the WTG vertical velocity according to an expression analogous to (A22). All the changes listed for this model version accounted for a reduction in the vertical mass flux profile of \(-23\%\) mostly in the upper troposphere. The equilibrium precipitation rate was \(-15\%\) less than in RZ05.

The current version of the model used in this paper incorporates all the changes listed above, plus several additional changes. The thermodynamic prognostic variable has been changed from equivalent potential temperature to moist entropy and WTG entrainment is done via equation (A22). Defining the bulk thermodynamic flux in terms of moist entropy alters the magnitude of surface sensible heat and moisture fluxes such that the resulting RCE state is cooler and drier than when equivalent potential temperature was used. Also, changes to the numerical scheme affecting boundary conditions on prognostic fluxes were implemented and the surface bulk flux scheme now deposits fluxes into a shallower layer. In addition, \( \frac{\partial \theta}{\partial z} \) is computed with centered-difference, not upstream differencing and the interactive radiation scheme is shifted from cell edges to cell centers. Last, the minimum limiting value of the vertical gradient of potential temperature is reduced from \( \gamma = 1.0 \text{ K km}^{-1} \) to \( \gamma = 0.3 \text{ K km}^{-1} \). Together, these changes account for a slight increase in the column-averaged vertical mass flux profile over that of the previously described model.
version. This makes the mass flux profile ~20% less than that of RZ05. The precipitation rate is unchanged from the previous model, however, so the difference is again ~15% less than in RZ05.

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