Frictional Convergence in a Decaying Weak Vortex

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ABSTRACT

Cross-isobaric flow and Ekman pumping are investigated in the frictional spindown of an initially barotropic vortex in a stratified atmosphere. Consistent with early work by Holton and others, it is found that the stratification limits the vertical penetration of the secondary circulation driven by friction, resulting in more rapid spindown than in the unstratified case. As a result, the cross-isobaric flow and Ekman pumping are weaker and shallower than classical calculations ignoring the stratification would lead one to believe. The effect of stability becomes stronger as the vortex becomes smaller for fixed boundary layer depth. For weak geostrophic vortices with horizontal scales of several hundred kilometers or less, such as tropical easterly waves, the reduction is particularly pronounced, which raises questions about the efficacy of Ekman pumping in forcing convection in such vortices. These results suggest a revised conceptual model for the role of Ekman pumping in the atmosphere. The theory as it stands is limited to weak, linear vortices in which geostrophic balance holds approximately, corresponding to small Rossby number, though extensions of the analytical theory to stronger vortices may be possible.

1. Introduction

The idea that deep atmospheric convection may be aided by frictionally induced convergence and lifting in the atmospheric boundary layer (Ekman pumping) is a common assumption in meteorological theory. Its application in idealized models appears to have originated in Charney and Eliassen (1949) and was used in the context of tropical meteorology by Charney and Eliassen (1964), Ooyama (1969), Holton et al. (1971), Charney (1971, 1973), Holton (1975), Wang (1988), Wang and Rui (1990), etc.

The simple, most widely used version of the model was challenged by Raymond and Herman (2012, hereafter RH12). In this version, Charney and Eliassen (1964) assumed that the secondary circulation arising from surface friction would extend through most or all of the troposphere, which means that the time constant for global spindown would be large compared to the time required to bring the boundary layer by itself to a halt by friction. Holton (1965) pointed out a potential flaw in this argument, noting that the upward penetration depth of the secondary circulation is limited by the stratification of the atmosphere. In particular, for a horizontal scale $L$ of the boundary layer flow, and hence of the secondary circulation, the vertical penetration depth of this circulation scales as $Z = fL/N$, where $f$ is the Coriolis parameter and $N$ is the Brunt–Väisälä frequency. Thus, for small to mesoscale disturbances in the tropics where $Nf \approx 300$ and $L \approx 300$ km, $Z \leq 1$ km, or much less than the depth of the troposphere, contrary to the assumption of Charney and Eliassen (1964). As a consequence, the time for spindown is small enough that the steady-state idealization behind the usual Ekman pumping formula may be invalid. RH12 showed in a linearized, slab-symmetric model that this effect has major consequences for Ekman pumping in weak (i.e., low Rossby number) disturbances in the tropics with horizontal scales less than a few hundred kilometers.
Aside from possible nonlinearity, two situations could invalidate the analysis of RH12. First, if the boundary layer flow is driven directly by some external mechanism, then it could be maintained as a steady flow in the face of the rapid spindown tendency produced by surface friction. An example of this is the case in which surface temperature gradients drive the boundary layer, as envisioned by Lindzen and Nigam (1987). Spatial variations in these gradients could then result in quasi-steady regions of convergence and divergence. Though friction plays an important role in determining the structure of such convergence patterns, it is not correct to ascribe the convergence to the friction per se, as the prime mover in this case is the pattern of surface temperature gradient.

The second possibility is that deep convection operates in a manner that can be idealized by an effective reduction in the Brunt–Väisälä frequency of the atmosphere. This would result in an increase in the penetration depth of the secondary circulation and a corresponding reduction in the Brunt–Väisälä frequency in its spindown time. Yano and Emanuel (1991), Emanuel et al. (1994), and Neelin and Yu (1994), among others, have postulated such a model, with a typical reduction in the Brunt–Väisälä frequency to approximately 30% of its dry value.

In their idealized model, Emanuel et al. (1994) assumed that convective inhibition is negligible over the tropical oceans. Actual measurements of convective inhibition over the ocean (e.g., Raymond et al. 2003) show convective inhibition values that are undoubtedly much less than they are over, say, the American high plains in the spring, but are nevertheless large enough to play a significant role in tropical convective dynamics. Furthermore, Raymond (1995) showed that surface heat and moisture fluxes were often more effective in reducing convective inhibition than lifting by the weak mesoscale vertical motions typical of oceanic regions. In such a situation, the initiation of deep convection by Ekman pumping may not occur at all, especially if the Ekman pumping is being weakened by the low-level spindown of the parent disturbance.

One might argue that the low-level convergence produced by a preexisting region of convection is a result of Ekman pumping: if the convection is in a statistically steady state, then the steady relationship between friction, pressure gradient, and Coriolis force characteristic of Ekman pumping must exist. However, this diagnostic relationship does not prove that the Ekman pumping “caused” the convection. If Ekman pumping in the absence of convection is insufficient to get the convection started in the first place, then the cause of the convection must be sought elsewhere.

In modern numerical models of the atmosphere, no assumptions are made about the nature of frictionally induced convergence, so the considerations raised here do not apply directly to the model results. However, they potentially apply to the interpretation of these results. Such interpretations may not matter to the producers of model forecasts, but they are important for understanding the physics of such models and attempts to make the models better. Thus, we believe that it is of practical as well as of theoretical importance to sort out issues of causality in the frictional atmospheric boundary layer.

As noted above, RH12 confined themselves to slab symmetry and to the frictional convergence in single spectral modes. The present paper extends this work to boundary layer flows in the decay of initially barotropic vortices. Axial symmetry is important because tropical cyclones are often idealized as being axisymmetric and frequently have Ekman pumping, that is, cloud-base mass fluxes matched to frictionally converged mass, invoked as a forcing mechanism for convection (e.g., Charney and Eliassen 1964; Ooyama 1964, 1969). Zhu et al. (2001) and Zehnder (2001) admit more complex convective closures in their minimal cyclone models, comparing pure Ekman pumping closures with schemes based on surface heat and moisture fluxes. Not surprisingly, significant differences in tropical cyclone development exist among their various alternatives.

Mathematical tractability currently limits us to the linear case, which imposes significant restrictions on the direct comparison with tropical cyclone observations. Nevertheless, the results are interesting in their own right and can form the basis for future calculations not limited by the constraints of linearization.

2. Model for spindown in axisymmetry

The scaled, hydrostatic, rotating Boussinesq equations in cylindrical coordinates \((r, \theta, z)\) for a stably stratified atmosphere in axisymmetry \((\partial / \partial \theta = 0)\) are

\[
\frac{\partial u}{\partial t} + \mathcal{R} v \cdot \nabla u - \mathcal{V} u = - \frac{\partial p}{\partial r} + F_r, \tag{1}
\]

\[
\frac{\partial v}{\partial t} + \mathcal{R} v \cdot \nabla v + u = F_\theta, \tag{2}
\]

\[
\frac{\partial p}{\partial z} = b = 0, \tag{3}
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial w}{\partial z} = 0, \tag{4}
\]

\[
\frac{\partial b}{\partial t} + \mathcal{R} v \cdot \nabla b + w = 0. \tag{5}
\]

The velocity vector with components in the radial, azimuthal, and vertical directions is \(\mathbf{v} = (u, v, w)\), the
buoyancy perturbation is $b$, the kinematic pressure perturbation (the mean potential temperature times the Exner function) is $\pi$, and $r = (x^2 + y^2)^{1/2}$.

The scaling is accomplished by replacing dimensioned variables with the product of scaling constants, represented in uppercase, and dimensionless variables:

$$(x,y) \rightarrow (x,y)L,$$

$$(u,v) \rightarrow (u,v)U,$$

$$w \rightarrow wW,$$

$$b \rightarrow bB,$$

$$\pi \rightarrow \pi \Pi.$$  \hspace{1cm} (6)

The values of $L$ and $U$ are imposed and the rest of the scaling parameters take on the values

$$Z = L(f/N),$$

$$T = 1/f,$$

$$W = U(f/N),$$

$$B = UN,$$

$$\Pi = LUf,$$  \hspace{1cm} (7)

where $f$ is the Coriolis parameter and $N$ is the Brunt–Väisälä frequency, assumed to be constant. The dimensionless quantity $\mathcal{R} = U/(Lf)$ is the Rossby number.

Surface friction enters in the radial and azimuthal directions ($F_r, F_\theta$) as a frictional force per unit mass, approximated to be in linear in velocity and decreasing exponentially with height. In dimensionless form, these are

$$F_r = -\lambda u_s \exp(-\mu z),$$  \hspace{1cm} (8)

$$F_\theta = -\lambda u_s \exp(-\mu z).$$  \hspace{1cm} (9)

The exponential vertical structure is convenient for our analysis but may not reflect the real vertical structure of momentum flux divergence. The importance of this issue remains uncertain.

The vector $(u_s, v_s)$ is the dimensionless surface wind, and $\mu = Z/h_\mu = Lf/(Nh_\mu)$ is the inverse of the non-dimensionalized depth $h_\mu$ over which surface friction acts. The quantity $\lambda$ is the inverse of the dimensionless spindown time scale and is approximated by $\lambda = C_D U/(Nh_\mu)$, where $C_D = 10^{-3}$ is the drag coefficient. There are thus three nondimensional constants governing the behavior of this system: $\mathcal{R}, \mu$, and $\lambda$.

Assuming that the Rossby number is small $\mathcal{R} \ll 1$ allows the nonlinear advection terms to be ignored. This linearizes the system of (1)–(5), which can then be combined into a single differential equation for the time tendency of the kinematic pressure perturbation. Incorporating (8) and (9) to

$$\frac{\partial^2 \pi_t}{\partial t^2} + \frac{1}{\sigma^2 + 1} \frac{\partial}{\partial r} \left[ \frac{\partial \pi_t}{\partial r} \right]$$

$$= -\frac{1}{\sigma^2 + 1} \frac{\partial}{\partial r} \left[ \frac{\partial \pi_t}{\partial r} \right] \exp(-\mu z),$$  \hspace{1cm} (10)

where $\pi_t = \partial \pi/\partial t$. To solve (10), we use the barotropic initial condition that $\pi(r,z) = \pi_C(r)$ at all levels and assume that the surface wind $(u_s, v_s)$, surface pressure perturbation $\pi_s$, and time tendency of kinematic pressure perturbation $\pi_t$, decay exponentially to zero with time according to $u_s, v_s, \pi_s, \pi_t = \exp(-\sigma t)$ with $\sigma$ to be determined later, as in RH12. Substituting the aforementioned time dependence into (10) results in

$$\frac{\partial^2 \pi_t}{\partial t^2} + \frac{1}{\sigma^2 + 1} \frac{\partial}{\partial r} \left[ \frac{\partial \pi_t}{\partial r} \right]$$

$$= -\frac{1}{\sigma^2 + 1} \frac{\partial}{\partial r} \left[ \frac{\partial \pi_t}{\partial r} \right] \exp(-\mu z).$$  \hspace{1cm} (11)

Evaluating (1) and (2) at the surface and solving them for surface winds $u_s$ and $v_s$ in terms of the surface pressure perturbation, we obtain

$$u_s = -\frac{\lambda - \sigma}{1 + (\lambda - \sigma)^2} \frac{\partial \pi_s}{\partial r},$$  \hspace{1cm} (12)

$$v_s = \frac{1}{1 + (\lambda - \sigma)^2} \frac{\partial \pi_s}{\partial r}.$$  \hspace{1cm} (13)

Substituting these into (11), we find

$$\frac{\partial^2 \pi_t}{\partial t^2} + \frac{1}{\sigma^2 + 1} \frac{\partial}{\partial r} \left[ \frac{\partial \pi_t}{\partial r} \right]$$

$$= -\frac{\lambda G(\sigma)}{\sigma^2 + 1} \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r \frac{\partial \pi_t}{\partial r} \right],$$  \hspace{1cm} (14)

where

$$G(\sigma) = \frac{1 + \sigma(\lambda - \sigma)}{1 + (\lambda - \sigma)^2}.$$  \hspace{1cm} (15)

The solution to this equation can be written as the sum of homogeneous $\pi_{ht}$ and inhomogeneous $\pi_h$ parts. For the homogeneous part we assume $z$ dependence of the form $\pi_{ht} \approx \exp(-nz)$, which allows the equation for $\pi_{ht}$ to be written in the form

$$\frac{\partial^2 \pi_{ht}}{\partial r^2} + \frac{1}{r} \frac{\partial \pi_{ht}}{\partial r} + \left( \sigma^2 + 1 \right) m^2 \pi_{ht} = 0.$$  \hspace{1cm} (16)
This is an unscaled form of Bessel’s equation for the Bessel function of order zero. It has the solution

$$\pi_{rt} \propto J_0(kr), \quad (17)$$

where $J_0$ is the zeroth-order Bessel function and

$$k = (\sigma^2 + 1)^{1/2} m \quad (18)$$

is the nondimensional radial wavenumber $k$. We set the horizontal length scale equal to the inverse of a typical dimensional value of $k$. Note that $\sigma$ will be determined later.

We write the full surface pressure distribution as a superposition of Bessel functions with different radial wavenumbers $k$—that is, in the form of an inverse Hankel transform

$$\pi_s = \int_0^\infty D(k) \exp(-\sigma t) J_0(kr) k \, dk, \quad (19)$$

where $D(k)$ is the forward Hankel transform of the radial surface pressure distribution at the initial time. The assumed time dependence $\exp(-\sigma t)$ is included explicitly. Note that $\sigma$ in general is a function of $k$. The inhomogeneous part of the solution for $\pi_t$ can therefore be written as an inverse Hankel transform as well:

$$\pi_{ht} = \exp(-\mu z) \int_0^\infty C(k) \exp(-\sigma t) J_0(kr) k \, dk, \quad (20)$$

where $C$ is proportional to $D$, but remains to be determined. The homogeneous part of the solution can then be written

$$\pi_{ht} = \int_0^\infty A(k) \exp(-mz - \sigma t) J_0(kr) k \, dk, \quad (21)$$

where the inverse of the dimensionless vertical penetration depth of the secondary circulation $m$ is obtained from (18):

$$m(k) = \frac{k}{(\sigma^2 + 1)^{1/2}}. \quad (22)$$

Putting these together results in

$$\pi_t(r, z) = \int_0^\infty \left[ C(k) \exp(-\mu z - \sigma t) + A(k) \exp(-mz - \sigma t) \right] J_0(kr) k \, dk. \quad (23)$$

We determine $C(k)$ by noting that the time derivative of the surface pressure $\pi_s(r, 0)$ at the initial time is given by

$$-\sigma \pi_s(r) = \pi_t(r, 0) \quad (24)$$

since this quantity is proportional to $\exp(-\sigma t)$. Using this relation at time $t = 0$ in conjunction with (19) and (23) implies that $-\sigma D(k) = C(k) + A(k)$, or

$$C = -A - \sigma D. \quad (25)$$

Hence,

$$\pi_t = \int_0^\infty \left\{ A(k) \exp(-mz) - [A(k) + \sigma D(k)] \exp(-\mu z) \right\} \exp(-\sigma t) J_0(kr) k \, dk. \quad (26)$$

Requiring zero buoyancy perturbation at the surface implies via the hydrostatic equation (3) that

$$\left( \frac{\partial \pi_t}{\partial z} \right)_{z=0} = 0, \quad (27)$$

which yields $A(k) = -\sigma \mu D(k)/(\mu - m)$ and, hence,

$$\pi_t = \int_0^\infty \sigma D(k) \left[ \frac{m \exp(-\mu z) - \mu \exp(-mz)}{\mu - m} \right] \exp(-\sigma t) J_0(kr) k \, dk. \quad (28)$$

Substituting the solution (28) into the differential equation (14) leads to a dispersion relation that is identical to the one found using slab symmetry in RH12:

$$\sigma(k) = \frac{\lambda m G(\sigma)}{m + \mu}. \quad (29)$$

Finally, our solution for the kinematic pressure perturbation is obtained by integrating (28) in time and requiring that $\pi = \pi_s(r, 0) = \pi_{C}(r)$ at $t = 0$:

$$\pi = \int_0^\infty D(k) \left\{ 1 - \frac{m \exp(-\mu z) - \mu \exp(-mz)}{m - \mu} \right\} \left[ 1 - \exp(-\sigma t) \right] J_0(kr) k \, dk. \quad (30)$$
The initial state of the flow is therefore barotropic (unsheared in the vertical) with arbitrary radial structure. Setting \( z = 0 \) reduces this to the assumed surface pressure perturbation (19).

\[
b = \int_0^\infty D(k) \frac{m \mu [\exp(-\mu z) - \exp(-m z)]}{m - \mu} [1 - \exp(-\sigma t)] J_0(kr) k \, dk,
\]

(31)

\[
w = -\int_0^\infty D(k) \frac{\sigma m \mu [\exp(-\mu z) - \exp(-m z)]}{m - \mu} \exp(-\sigma t) J_0(kr) k \, dk.
\]

(32)

Using the identity

\[
\frac{1}{r} \frac{\partial r J_1(kr)}{\partial r} = k J_0(kr),
\]

(33)

where \( J_1 \) is the Bessel function of order 1, plus the mass continuity equation (4), we get an equation for the radial velocity component:

\[
u = -\int_0^\infty D(k) \frac{\sigma m \mu [\exp(-\mu z) - \exp(-m z)]}{m - \mu} \exp(-\sigma t) J_1(kr) k \, dk.
\]

(34)

The azimuthal velocity component is obtained from (1):

\[
v = \lambda \nu_s \exp(-\mu z) + \frac{\partial \nu}{\partial t} + \frac{\partial \pi}{\partial r},
\]

(35)

where

\[
\frac{\partial \pi}{\partial r} = -\int_0^\infty D(k) \left\{1 - \frac{m \exp(-\mu z) - \mu \exp(-m z)}{m - \mu} [1 - \exp(-\sigma t)] \right\} J_1(kr) k^2 \, dk
\]

(36)

and

\[
\frac{\partial \nu}{\partial t} = \int_0^\infty D(k) \frac{\sigma^2 m \mu [\exp(-\mu z) - \exp(-m z)]}{N^2(m - \mu)} \exp(-\sigma t) J_1(kr) k \, dk.
\]

(37)

Equation (35) can be written explicitly with help of (34), (36), and (37), but it is not shown here owing to its complexity.

3. Results and interpretation

We first analyze the surface winds derived in the previous section by invoking a simplified form of the dispersion relation. We then present computations of the radial structure of the radial and vertical winds for a specified radial distribution of surface pressure, given this simplified analysis. Results in this section are converted to dimensional form except where noted for ease of interpretation.

a. Surface wind analysis

Our primary circulation is a rotating disturbance on which frictionally induced cross-isobaric flow acts, creating a secondary circulation. As noted above, friction is assumed to have an exponential decay in \( z \): \( F_{r,0} \propto \exp(-z/h_\mu) \) where we call \( h_\mu = Z/\mu \) the surface friction depth. The characteristics of a secondary circulation of interest in this analysis are the penetration depth of the secondary circulation \( h_m = Z/m \) and the cross-isobaric wind \( u \).

We simplify our system of equations by assuming a typical tropical boundary layer in a weakly disturbed region where winds are not too strong, that is, with \( \lambda = C_D U/(h_\mu) \ll 1 \). We note that \( G(\sigma) \approx 1 \) for spin-down times \( \sigma^{-1} \) greater than the rotational time scale (unity in nondimensional form), which constrains \( \sigma \approx \lambda \) for all positive real values of \( m \). We further examine a particular radial mode with a dimensional wavenumber equal to the inverse of the characteristic scale of the disturbance, leading to a nondimensional wavenumber equal to \( k = 1 \). Since \( \lambda \ll 1 \), \( G(\sigma) \approx 1 \), resulting in \( m(k) \approx k \approx 1 \) and therefore \( \sigma(k) \approx \lambda k/(k + \mu) = \lambda/(1 + \mu) \). The penetration depth of our secondary circulation is then \( h_m = Z/m \approx Z/k = Z = fL/N \). Thus, the dimensional secondary circulation depth increases in proportion to the horizontal scale of the disturbance. In this case \( \mu = Z/h_\mu = h_m/h_\mu \).
Using these approximations, we now compare the surface radial wind $u_s$ predicted by our model for the specified radial mode with that for the steady-state surface wind $u_{ss}$ arising from classical Ekman pumping analysis. From (12) and (13), we infer that

$$u_s = -(\lambda - \sigma)u_s = -\left(\frac{\lambda u}{1 + \mu}\right)u_s.$$  

(38)

Neglecting the time derivative in (2) corresponds to setting $\sigma = 0$ in (38), so the steady state radial velocity is just

$$u_{ss} = -\lambda u_{ss}.$$  

(39)

Taking the ratio of $u_s$ to $u_{ss}$ and using $\mu = h_m/h_\mu$ results in

$$u_s = \frac{\mu}{1 + \mu}u_{ss} = \frac{h_m}{h_m + h_\mu}u_{ss}.$$  

(40)

Thus, the actual cross-isobaric flow at the surface is less than the steady-state cross-isobaric flow by the factor $h_m/(h_m + h_\mu)$. Since Ekman pumping is related to the cross-isobaric flow $u$, we conclude that

$$h_\mu \ll h_m \Rightarrow u_s \approx u_{ss} \Rightarrow \text{normal Ekman pumping},$$

(41)

$$h_\mu \gg h_m \Rightarrow u_s \ll u_{ss} \Rightarrow \text{suppressed Ekman pumping}.$$  

(42)

Given that $h_m = fL/N$ under these conditions, with $f = 3 \times 10^{-5}\text{s}^{-1}$ and $N = 10^{-2}\text{s}^{-1}$, and taking a plausible tropical boundary layer value of $h_\mu = 600\text{m}$, we find that $h_m \approx 300\text{m}$ for $L = 100\text{km}$ while $h_m \approx 3000\text{m}$ for $L = 1000\text{km}$. Thus, at the smaller scale, $u_s \approx (1/3)u_{ss}$, while $u_s \approx u_{ss}$ for the larger scale. The steady-state assumption for the cross-isobaric flow is therefore poor for scales smaller than several hundred kilometers, at least for low Rossby number, $\mathcal{R} = U/(Lf) \ll 1$.

### b. Radial and vertical structure

The nondimensional steady state solutions aloft for $u$ and $w$ are relatively simple to compute from (2) and (4):

$$u_{steady} = u_{ss} \exp(-\mu z),$$  

(43)

$$w_{steady} = -\frac{1}{\mu r} \frac{\partial u_{ss}}{\partial r} \left[1 - \exp(-\mu z)\right]$$

(44)

in which $u_{ss}(r)$ is the steady surface radial wind defined above. However, the full solutions, as represented by (32) and (34), are nonlocal and require the specification of the surface pressure as a function of radius at the initial time $t = 0$. As an example, we assume the surface pressure to take the dimensionless form:

<table>
<thead>
<tr>
<th>$L$ (km)</th>
<th>$r_{max}$ (km)</th>
<th>$\mathcal{R}$</th>
<th>$\mu$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>58</td>
<td>1.67</td>
<td>0.5</td>
<td>0.28</td>
</tr>
<tr>
<td>300</td>
<td>173</td>
<td>0.56</td>
<td>1.5</td>
<td>0.28</td>
</tr>
<tr>
<td>1000</td>
<td>577</td>
<td>0.17</td>
<td>5.0</td>
<td>0.28</td>
</tr>
<tr>
<td>3000</td>
<td>1732</td>
<td>0.06</td>
<td>15.0</td>
<td>0.28</td>
</tr>
</tbody>
</table>

(45)

To calculate the dispersion relation (29), Radius of maximum tangential wind $r_{max}$ and dimensionless parameters resulting for various scaling radii $L$, assuming $U = 5\text{m s}^{-1}, f = 3 \times 10^{-5}\text{s}^{-1}$, and $N = 10^{-2}\text{s}^{-1}$. More details are provided in Table 1.
Figure 1 shows the dimensional radial surface winds for the decaying vortex and for the steady-state approximation to these winds at time $t = 0$ for the cases listed in Table 1. The actual radial winds are always less than the steady-state approximation, with a significant deficit for the $L = 300$ km and $L = 100$ km cases. The radius of maximum radial inflow is also slightly greater for the actual winds in comparison to the steady state.

Figure 2 shows the vertical velocity profile for the secondary circulation at a radius of $r = L/10$, which is well within the radius of maximum winds for the cases of Table 1. A broad peak in vertical velocity exists within this radius, with a maximum value at $r = 0$. The vertical velocity for the decaying vortex is significantly less than that arising from the steady-state assumption, particularly at levels above the penetration depth of the secondary circulation in the decaying case. Furthermore, the elevation of maximum upward motion decreases as $L$ decreases, reflecting the decrease in the penetration depth $h_m$. The difference in the vertical velocity between the steady and actual cases is particularly pronounced above the $z = h_m$ level, with significant differences persisting out to the largest scales. In particular the steady vertical velocity asymptotes to a constant positive value for large $z$, while the actual vertical velocity in the decaying case decreases rapidly toward zero above $z = h_m$. The peak vertical velocity in the decaying case is actually within the boundary layer for $L \leq 300$ km.

4. Conclusions

The idea of Ekman pumping as a forcing mechanism for convection in the tropics has a long history in tropical
meteorology. The classical formulation of Ekman pumping assumes that the free-tropospheric secondary circulation induced by surface friction has a time scale for spindown that is long compared to the spindown time of an isolated boundary layer. As a consequence of this, the time derivatives in the horizontal components of the momentum equation are ignored in the boundary layer, resulting in the classical expression for the cross-isobaric flow there and the associated vertical motion (Ekman pumping). This vertical motion has been assumed to promote convection by a number of possible mechanisms: from providing a direct mass source for convective updrafts to simply destabilizing and moistening the lower troposphere.

Holton (1965) showed that the vertical scale of the secondary circulation resulting from surface friction is limited by the stable stratification of the free troposphere, resulting in a free-tropospheric spindown time much smaller than is normally assumed. This result implies that the momentum equation time derivatives cannot be ignored for a decaying vortex in the derivation of the cross-isobaric flow in the boundary layer. In this case, the classical Ekman pumping equation is incorrect, and the vertical motion can actually be much weaker and shallower than estimated by classical theory.

RH12 put numbers to this result for a zonal jet structure periodic in the meridional direction and found that the actual cross-isobaric flow and secondary circulation are much shallower and weaker than the classical result for lateral jet scales of order several hundred kilometers or less. We extend this analysis to axisymmetric weak
vortices with arbitrary radial structure, with similar consequences. A particularly interesting result illustrated here is that the vertical velocity is not only weaker than the classical Ekman pumping result but also exhibits a vertical scale that decreases with vortex size. This reflects the penetration depth of the secondary circulation.

The current results would be modified in the direction of classical Ekman pumping if the disturbances of interest were coupled to moist convection in a manner that results in a reduction of the effective Brunt–Väisälä frequency. However, we argue in the introduction of this paper that such a model for the interaction of moist convection with the boundary layer flow is oversimplified. Our experience in the tropics argues for a much looser relationship between weak, large-scale vertical motion and moist convection, a relationship in which it is easier to separate cause from effect. If convection exists in association with frictionally modulated convergence in the boundary layer, then the convergence cannot be thought of as being caused by this convergence unless the convergence would have existed initially in the absence of the convection. Unless this is so, the convergence is more likely to be a consequence of the convection rather than vice versa, and what is naively perceived as convection being forced by Ekman pumping in fact often is not.

This result is important since it gives us a completely different conceptual picture of the forcing of moist convection in the tropics than classical Ekman pumping theory would imply. A flawed conceptual picture can lead us to causally incorrect choices in such things as the construction of cumulus parameterizations in large-scale atmospheric models.

These results are applicable, at least approximately, to tropical waves with wind perturbations of order 5 m s\(^{-1}\) and scales of several hundred kilometers. Since these waves have no external source of energy aside from convection once they leave regions of strong barotropic or baroclinic forcing, the causality analysis with respect to convection should apply to them. For tropical cyclones, the scales are too small and the winds are too large for these results to be directly applicable, as the Rossby number in this case is large.

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