

VECTOR IDENTITIES⁴

Notation: f, g , are scalars; \mathbf{A}, \mathbf{B} , etc., are vectors; \mathbf{T} is a tensor; \mathbf{l} is the unit dyad.

- (1) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$
- (2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
- (3) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$
- (4) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$
- (5) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D}$
- (6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$
- (7) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
- (8) $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$
- (9) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$
- (10) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
- (11) $\mathbf{A} \times (\nabla \times \mathbf{B}) = (\nabla \mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
- (12) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
- (13) $\nabla^2 f = \nabla \cdot \nabla f$
- (14) $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$
- (15) $\nabla \times \nabla f = 0$
- (16) $\nabla \cdot \nabla \times \mathbf{A} = 0$

If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthonormal unit vectors, a second-order tensor \mathbf{T} can be written in the dyadic form

$$(17) \quad \mathbf{T} = \sum_{i,j} T_{ij} \mathbf{e}_i \mathbf{e}_j$$

In cartesian coordinates the divergence of a tensor is a vector with components

$$(18) \quad (\nabla \cdot \mathbf{T})_i = \sum_j (\partial T_{ji} / \partial x_j)$$

[This definition is required for consistency with Eq. (29)]. In general

$$(19) \quad \nabla \cdot (\mathbf{AB}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$(20) \quad \nabla \cdot (f\mathbf{T}) = \nabla f \cdot \mathbf{T} + f\nabla \cdot \mathbf{T}$$

Let $\mathbf{r} = ix + jy + kz$ be the radius vector of magnitude r , from the origin to the point x, y, z . Then

$$(21) \quad \nabla \cdot \mathbf{r} = 3$$

$$(22) \quad \nabla \times \mathbf{r} = 0$$

$$(23) \quad \nabla r = \mathbf{r}/r$$

$$(24) \quad \nabla(1/r) = -\mathbf{r}/r^3$$

$$(25) \quad \nabla \cdot (\mathbf{r}/r^3) = 4\pi\delta(\mathbf{r})$$

$$(26) \quad \nabla \mathbf{r} = I$$

If V is a volume enclosed by a surface S and $d\mathbf{S} = \mathbf{n}dS$, where \mathbf{n} is the unit normal outward from V ,

$$(27) \quad \int_V dV \nabla f = \int_S d\mathbf{S} f$$

$$(28) \quad \int_V dV \nabla \cdot \mathbf{A} = \int_S d\mathbf{S} \cdot \mathbf{A}$$

$$(29) \quad \int_V dV \nabla \cdot T = \int_S d\mathbf{S} \cdot T$$

$$(30) \quad \int_V dV \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times \mathbf{A}$$

$$(31) \quad \int_V dV (f\nabla^2 g - g\nabla^2 f) = \int_S d\mathbf{S} \cdot (f\nabla g - g\nabla f)$$

$$(32) \quad \int_V dV (\mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A}) \\ = \int_S d\mathbf{S} \cdot (\mathbf{B} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \mathbf{B})$$

If S is an open surface bounded by the contour C , of which the line element is $d\mathbf{l}$,

$$(33) \quad \int_S d\mathbf{S} \times \nabla f = \oint_C d\mathbf{l} f$$

$$(34) \int_S d\mathbf{S} \cdot \nabla \times \mathbf{A} = \oint_C d\mathbf{l} \cdot \mathbf{A}$$

$$(35) \int_S (d\mathbf{S} \times \nabla) \times \mathbf{A} = \oint_C d\mathbf{l} \times \mathbf{A}$$

$$(36) \int_S d\mathbf{S} \cdot (\nabla f \times \nabla g) = \oint_C f dg = - \oint_C g df$$