Manual for Determining G with a Cavendish Balance

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Background and History

Here will be given a brief account of the history of the Cavendish Experiment and its significance. A detailed discussion of the theory, due to its extensive length and the fact that the details of how the experiment is conducted have changed since Cavendish's time, will be discussed in later parts of this document.

Nowadays, most people think of the Cavendish Experiment as the experiment to determine the universal gravitational constant G. However, this is not the purpose for which the experiment was originally designed. The experiment was originally conceived and built by the geologist John Michell for the purpose of determining the mean density of the Earth. The value of G was never an objective of the experiment, and, in fact, the results of the Cavendish Experiment was not used to determine a value for G until nearly 100 years later when C.V. Boys used it in a paper he presented in 1892.[1] Why was the Cavendish Experiment used for what seems today such a banal purpose? It may be illuminating to consider the fact that in the *Principia*, Isaac Newton never wrote his Law of Universal Gravitation in terms of an equation that included G. Instead he performed his calculations with ratios.[2] But still, why was the density of Earth so interesting? Scientists wished to know the density of the Earth because Newton had calculated the densities of the Sun, Jupiter, and Saturn as proportions of the density of Earth. Hence, if Earth's density was known, so were the other three.[2]

John Michell died in 1793 before he had an opportunity to perform his experiment. As a result, the balance passed to Francis Wallaston and then to Michell's longtime friend, with whom he had kept almost constant correspondence, Henry Cavendish.[1,2] Cavendish, perhaps as a way of honoring his friend's memory decided to proceed with the experiment much as Michell had first conceived of it. However, Cavendish was an exceptionally careful experimenter and ended up remaking large parts of the apparatus in order to eliminate everything from air currents to magnetic forces that could possibly affect the result.[3] A drawing of the apparatus from Henry Cavendish's paper is shown below in Figure 1. As an aside, it is interesting to note that Coulomb used a similar torsion balance in his studies of electrostatic forces. However, as Cavendish points out in his paper, John Michell conceived of the torsion balance design independently of Coulomb since Coulomb had not yet published his results when Michell was designing this experiment.[3]

Cavendish published his results in 1789 in a paper entitled: "Experiments to Determine the Density of the Earth", and they were astounding. Since the force of the Earth on the smaller lead spheres was known (their weight), and their densities, sizes, and separations were also known, it was possible to use a ratio of the forces on the smaller spheres due to the Earth and due to the larger lead spheres to find the mean density of the Earth. The number thus obtained was 5.48 ± 0.038 times the density of water, which is within one percent of the currently accepted value of 5.52.[1] Deriving G from these results gives a value of $G = 6.74 \times 10^{-11} kg^{-1} \cdot m^3 \cdot s^{-2}$, which is again very close to the currently accepted value of $G = 6.73 \times 10^{-11} kg^{-1} \cdot m^3 \cdot s^{-2}$ (Think you can do better?).

In this experiment, you will perform a version of the Cavendish Experiment, except with a modern (and much smaller) balance, with the objective of obtaining a value for G. If you follow the instructions set forth in this manual, you should be able to get a value for G within 3% of the accepted value in a reasonable amount of time.

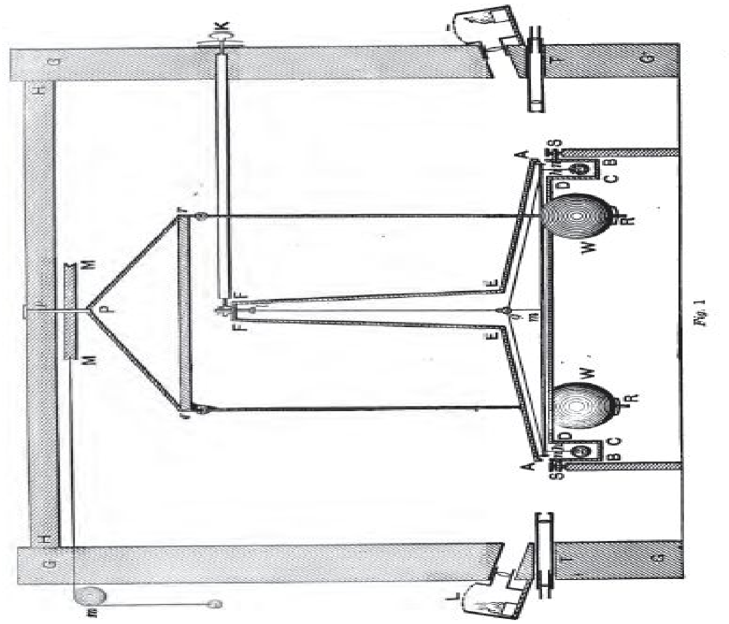

Figure 1: This diagram is taken from Cavendish's 1798 paper[3] and shows in detail the experimental setup he used to conduct his measurements. A description of each component would be too space consuming, but you probably get the general idea. However, if you are interested, Cavendish describes every piece of the apparatus at great length in his paper.

Procedure

These instructions follow the method that I used when I ran this experiment. For alternate approaches or more information, please refer to the TEL-Atomic users' manual.[4]

<u>Setting up</u>

A picture of the balance with parts labeled is given for reference as Figure 2 below.

Cavendish balance should be set up opposite a piece of white posterboard such that the glass on the outside of the balance is parallel to the posterboard. The farther away the balance is from the posterboard, the better; but a distance of about 2.5 meters should be fine.

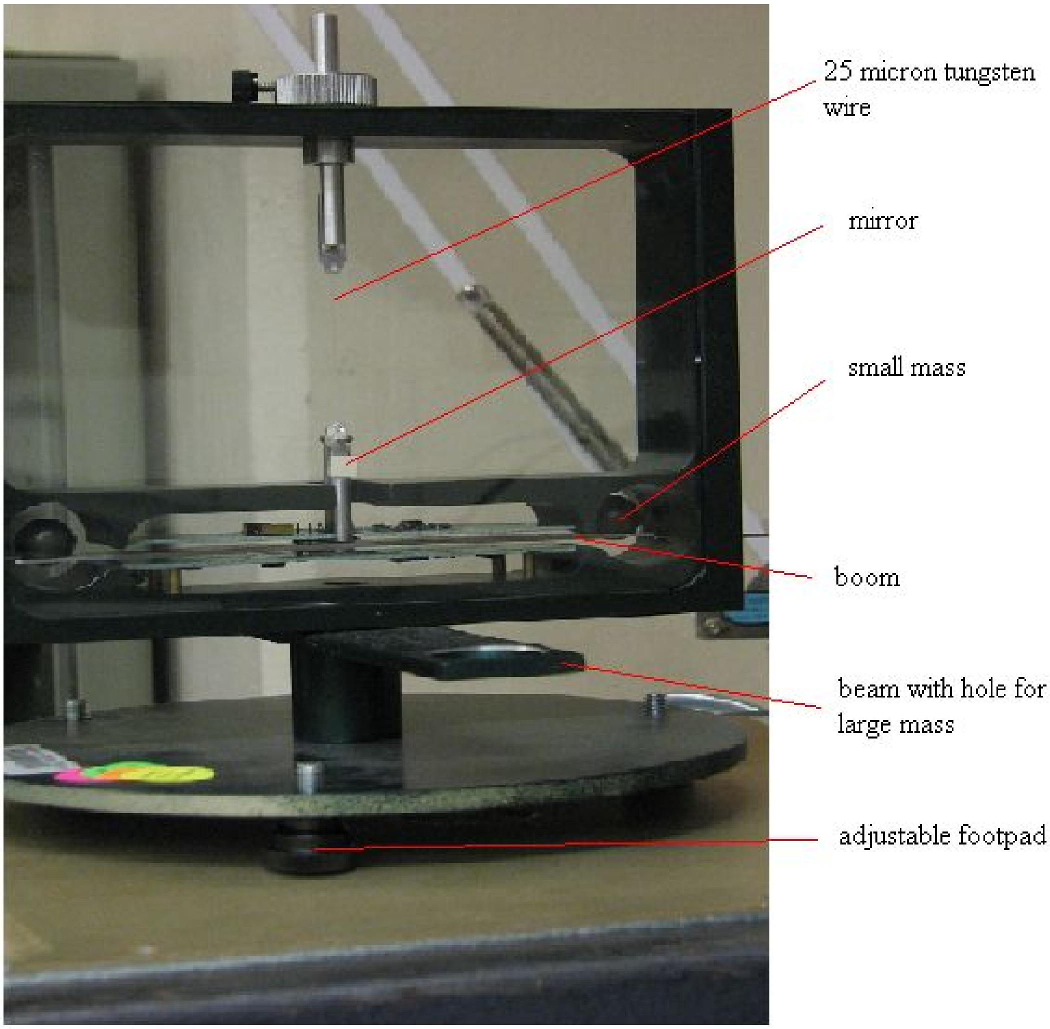

Figure 2: This labeled image of the Cavendish balance should be referred to while following the instructions in this manual.

Set up the neon laser about halfway between the posterboard and the balance. The aperture of the laser should be level with the mirror inside the Cavendish balance(you will probably need to place books you find in the lab under the laser to accomplish this) and there should be a small angle between the laser beam and the normal to the surface of the balance.

Now, turn on the laser and make sure that it is shining on the mirror attached to the boom inside the balance. Look for the reflection. If the reflection is too far to one side of the posterboard (or off the posterboard altogether), move it towards the center by turning the balance. Do this slowly.

Chances are that you excited some small oscillations while turning the balance. Does the reflection of the laser beam move perfectly along the horizontal? If not, adjust the screw pads on the base of the balance to correct for any tilt. Also, if the reflection is too high or too low, the footpads may be adjusted to fix that. (*Note from KM: the* *footpads should be adjusted to keep the boom level and centered between the small openings on either end; the most important thing is to keep the boom and connecting rod from rubbing against the frame*.)

Make sure the cross-bar (labeled "beam with hole" in Fig 2) is in the neutral position, perpendicular to the face of the balance, and wait for the oscillations to damp out. Once the reflection has stopped moving, tape a ruler horizontally to the posterboard so that the lower half of the reflection lies on the ruler proper, and the reflection is in the horizontal center of the ruler (15 cm mark, or thereabouts). See Figure 3. Note this equilibrium position of the reflection and estimate your uncertainty.

Measure the horizontal distance from the center of the balance to the posterboard and estimate your uncertainty.

Plug in the power cord for the box labeled "TEL-Atomic". Make sure the chart recorder paper is properly spooled (the edge of the paper should be parallel to the paper guide; see Figure 3). Use the lever on the right side of the chart recorder to raise the pen holder into the up position. Place a pen in the pen holder, uncap the pen, and use the same lever to move the pen holder back into the down position. Now set the speed on the chart recorder. 60 cm/hr usually works well, but be sure to record in your lab book the value you select. Set the voltage range on the chart recorder to around 5V.

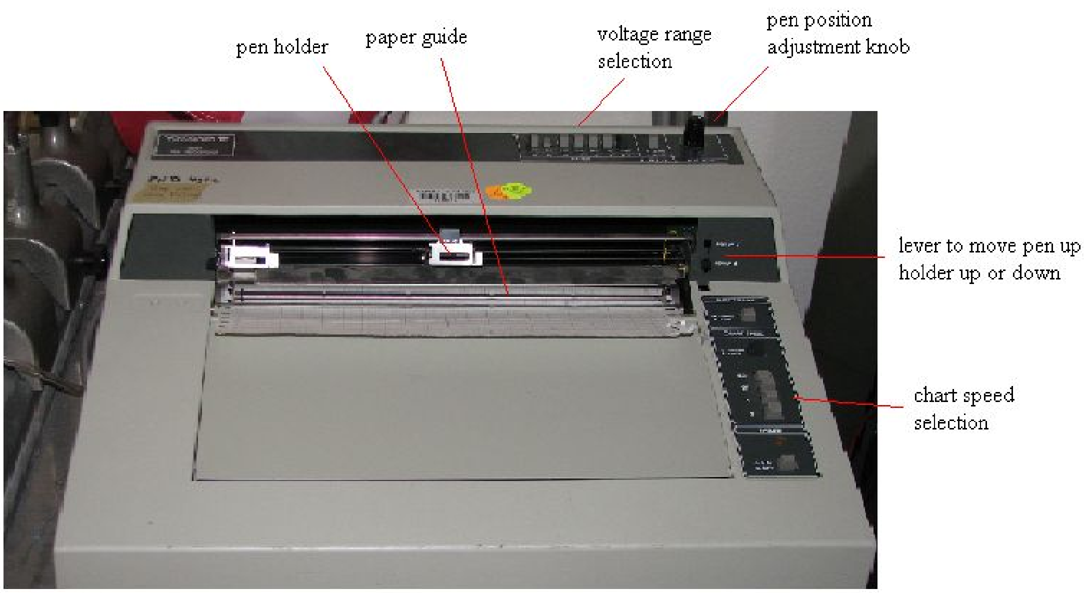

Figure 3: This is the strip-chart recorder that you will be using during the experiment. You can play around with the range and speed settings to see what works best.

<u>Doing the Experiment</u>

Turn on the chart recorder and use the position knob to place the pen on the center-line of the paper. Place the two large spherical masses on the holes in the crossbar. Watch the chart recorder and wait for any resulting oscillations to damp out. You are now ready to start the actual experiment.

Gently swing the masses to one side. Get them as close to the glass as you can without striking it. If you accidentally tap the glass, you're probably still okay as long as you didn't hit it too hard, but don't get cavalier! After you've moved the masses, watch the reflected laser spot on the cardboard. It should be slowly moving to one side. Once the movement of the spot stops, swing the masses to the other side and record the position (read it from the ruler that you taped to the posterboard) where the motion of the spot stopped (you should do this for every turn-around point during this part of the experiment). Remember to estimate your uncertainty. Continue this process until the magnitude of the oscillation (as measured on the ruler) stops growing with each cycle. Once this happens, move the masses to the neutral position and wait for the oscillations to damp out. Continue recording the linear deflection of the turning points until the oscillations have damped out. Now take your data from the chart recorder and return all of the equipment to the way you found it.

Tips: The oscillation-forcing part of this lab is easier with two people. One can stand near the posterboard to say when the spot's motion has reached its max and to record the displacement of the spot at that time while the other moves the masses from one side to the other. This technique eliminates the latency between noting the maximum of the oscillation and moving the masses, which can lead to "blips" in the data on the strip-chart. If you do not have a second person, I suggest slightly anticipating the oscillation reaching its peak (say, to about a millimeter or two), going to switch the masses, and writing down the value where you think (from observations of the spot prior to going back to move the masses) the oscillation stopped. When I did this experiment, the oscillations had a period of about 186 seconds. So, while your period won't necessarily be exactly the same, you will probably need to move the masses every 93 seconds or so (except for the first peak, which should be reached in about 46 seconds).

Data Analysis

Measure the distance of each peak on your chart from the centerline. I suggest using a pair of good calipers to do this. Estimate the uncertainty in your measurements. You may notice that there is a slow drift in your chart towards more positive or negative values. At some point, this drift may even overwhelm the growth of the oscillation in one direction. I am not sure why the drift occurs, but it may be due to charge traps or some other defect in the balance or peripheral equipment. I suggest that for the next section you use only data taken early enough that it is not catastrophically affected by this drift.

Now, from your displacement data, calculate your angular displacement for each data point. The equation for converting tangential displacements to angles is:

$$\theta = \frac{1}{2}\arctan\left(\frac{d_1 - d_0}{L}\right) \quad (1)$$

where d_1 is the displacement measured on the ruler, d_0 is the equilibrium position of the reflected laser spot on the ruler, and L is the distance from the balance to the posterboard. The factor of ½ is there because the angular deflection of the laser beam will be twice the amount that the mirror is deflected. As you can see, for very small displacements (which is the regime we should be working in) θ will be linearly related to the displacement d_l by the small angle approximation.

Once you have your angular displacements (in radians), plot the angular displacement versus the displacement that you measured from the strip-chart. Run a linear regression on this graph, the fit should be very good if your data is reliable. The equation that the regression returns is the equation to turn the displacements that you measured on your strip-chart into angular displacements. You may be questioning why you would want to do this step. After all, can't you just get the angular displacements from equation (1) and your displacement data? The answer is that you do this step in order to linearize your data and help beat down uncertainties. Speaking of uncertainties, you now need to calculate the uncertainty in the slope and intercept of your regression line. Use the following equations to do this (See John Taylor, An Introduction to Error Analysis, pages 184-188 for more details).[5]

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}\left(y_i - A - Bx_i\right)^2} \qquad (2)$$

$$\Delta = N\sum_{i=1}^{N} x_i^{\,2} - \left(\sum_{i=1}^{N} x_i\right)^2 \qquad (3)$$

$$\sigma_A = \sigma_y\sqrt{\frac{\sum_{i=1}^{N} x_i^{\,2}}{\Delta}} \qquad (4)$$

$$\sigma_B = \sigma_y\sqrt{\frac{N}{\Delta}} \qquad (5)$$

where A and B are from the equation y = A + Bx, if y is the (raw) angular displacement and x is the corresponding displacement on the strip-chart. Call the A and B that you obtain for the growing case A_g and B_g. I strongly suggest that you use a computer to find A and B. However, if you insist, you may find them with the following equations:

$$A = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta} \qquad (6)$$

$$B = \frac{N\sum xy - \sum x \sum y}{\Delta} \qquad (7)$$

For most of the error analysis in the rest of this lab, you will be using the standard error propagation equation from Taylor[6], which is given as equation (8) below:

$$\delta f = \sqrt{\left(\frac{\partial f}{\partial a}\delta a\right)^2 + \ldots + \left(\frac{\partial f}{\partial z}\delta z\right)^2} \qquad (8)$$

Now we can find the uncertainty in θ by using equation (8):

$$\delta\theta_i = \sqrt{\sigma_A{}^2 + (x_i\sigma_B)^2 + (B\delta x)^2 + \left\{\frac{1}{2}\left[\frac{1}{1+\left(\frac{d_1-d_0}{L}\right)^2}\right]\left(\frac{d_1-d_0}{L^2}\right)\delta L\right\}^2} \qquad (9)$$

Actually, only the first three terms in the square root in (9) come from (8); the fourth term is extra and accounts for possible systematic error due to uncertainties in your balance-posterboard distance measurement (don't worry, you don't need to include any extra terms in your analysis for the rest of the lab). Use Microsoft Excel or some other spreadsheet program to calculate and average your uncertainties.

$$\delta\theta = \frac{\sum_{i=1}^{N}(\delta\theta_i)}{N} \qquad (10)$$

Call the average fractional uncertainty calculated this way for the growing oscillation case $\delta\theta_g$.

Now repeat the above analysis but for the case where the oscillator is damping out. Only use turning points with relatively large amplitudes. You should have N~ 11 points. Give the values a subscript d (so: A_d, B_d, etc.). Note: you will need to repeat the error analysis as well (ie. calculate a new σ_A, Δ, etc.).

Before we can go further in our analysis, we are going to want to find the amount that the oscillations of the boom decay over one-half oscillation period during free damping; call this value h. h is related to other constants by:

$$h \equiv e^{-bT/2} \qquad (11)$$

where b is the decay constant and T is the period of the oscillation.

Convert the displacements from your stripchart for the damping oscillation into angles using the equation:

$$\theta_i = A_d + B_d x_i \qquad (12)$$

Now, pick a peak on your stripchart after you began letting the oscillation damp out and label it θ_1. Number each peak after it in the same manner up to θ_N, where N should be an odd number ≥ 11. h can then be calculated with the following equations:

$$h = 1 - \frac{\theta_1 - \theta_N}{\theta_1 - \theta_2 + \theta_3 - \theta_4 + \ldots - \theta_{N-1}} \quad (13)$$

$$h' = 1 - \frac{\theta_2 - \theta_{N-1}}{\theta_2 - \theta_3 + \theta_4 - \theta_5 + \ldots - \theta_{N-2}} \quad (14)$$

Average the values thus obtained for h and h' to get the value of h that you will use for the rest of this lab. The reason for averaging h and h' is that the balance can sometimes act slightly asymmetrically, so, depending on whether you start on a positive or negative turning point, you can get different decay constants.

The error in h can be calculated as shown below.

$$\delta h = \frac{\delta\theta(1-h)[(N-1)(1-h)^2 + 2h]^{1/2}}{\left|\theta_1 - \theta_N\right|} \quad (15)$$

As it is not immediately clear how equations (11), (13), and (14) came about, the derivations are given in Appendix A.1.

The effect of the spherical masses, when they are rotated to either of their extreme positions, is to change the equilibrium angle of the balance (to the point where the torque from the wire equals the torque due to gravity). Now, choose a turning point during the forced oscillation to be your starting point. Label that point θ_1 and, just as with the decay, label each following turning point up to θ_N where N is an odd number. For this part, you want N to be as large as possible, but you do not want to include points where the growth is overwhelmed by drift or where the growth has ceased. Convert the displacements of your turning points into angular displacements with the equation:

$$\theta_i = A_g + B_g x_i \quad (16)$$

Note that the subscript on the A and B has changed from that in equation (12). Now, calculate the change in the equilibrium angle θ_D with the following equation:

$$\theta_D = \frac{[(1-h)(\theta_1 - \theta_2 + \theta_3 - \theta_4 + \ldots\theta_N) - \theta_1 + h\theta_N]}{(N-1)(1+h)} \quad (17)$$

The error in θ_D comes from two sources, error in h and error in θ. The contributions from these two components can be calculated as shown below.

$$\delta\theta_{D_\theta} = \frac{\delta\theta[(N-1)(1-h)^2 + 2x]^{1/2}}{(N-1)(1+h)} \quad (18)$$

$$\delta\theta_{D_h} = \frac{\delta h[2(\theta_1 - \theta_2 + \theta_3 - ... - \theta_{N-1}) + (\theta_N - \theta_1)]}{(N-1)(1+x)^2} \qquad (19)$$

Add these two errors in quadrature to get $\delta\theta_D$. The derivation of equation (17) is given in Appendix A.2.

The final value that you must measure is the oscillation period T. Go to the part of your strip-chart where the oscillator is in free decay. Choose two positive (or negative) turning points and measure the horizontal (time axis) distance between them with your calipers. Convert your measurement to a time by using the spooling speed of the strip-chart recorder and divide that by the number of periods between your selected turning points in order to get the oscillators natural period. Repeat this procedure using several different combinations of turning points and average them to get your best value. Remember not to use points on the graph whose amplitude is very low, as the period can get distorted in this regime. The error in your value for T may be estimated by the standard deviation of your individual measurements:

$$\delta T = \sqrt{\frac{\sum_{i=1}^{N}(\bar{T} - T_i)^2}{N-1}} \qquad (20)$$

where $\bar{T}$ is the mean of your period measurements and T_i is an individual period measurement.

From the data you have measured, you may calculate the torsion constant K of the tungsten wire.

$$K = \left(\frac{4\pi^2}{T^2} + b^2\right)I \qquad (21)$$

where I is the moment of inertia of the boom plus the small spheres and b is the decay constant, which can be calculated from h (see equation(11)). Equation (21) is derived in Appendix A.3.

$$b = -\frac{2}{T}\ln(h) \qquad (22)$$

The uncertainty in this measurement is then, from (8):

$$\delta b = \sqrt{\left(\frac{2}{T^2}\ln(h)\delta T\right)^2 + \left(-\frac{2}{hT}\delta h\right)^2} \qquad (23)$$

I strongly suggest that you use the value listed in Appendix B for I since to do otherwise would entail taking apart the balance, which risks damage to the apparatus. However, since it is the instructor's discretion whether or not this is done, I may be calculated as the sum of the moments of inertia of the two small spheres (I_s) and the boom (I_b).

$$I_s = 2\left(m_s d^2 + \frac{2}{5} m_s r^2 \right) \qquad (24)$$

and

$$I_b = m_b \frac{l_b^{\;2} + w_b^{\;2}}{12} \qquad (25)$$

where m_s is the mass of a small sphere, d is the distance from the center of the boom to the center of a small sphere, r is the radius of a small sphere, m_b is the mass of the boom, and w_b and l_b are the boom's width and length, respectively. Equations (23) and (24) can be derived from standard mechanics, and the error may be gotten by using equation (8**).** The error in I would then be the errors in I_s and I_b added in quadrature. Again using (8), we can find the uncertainty in K:

$$\delta K = \sqrt{\left(\frac{-8\pi^2}{T^3} I \delta T \right)^2 + \left(2bI\delta b\right)^2 + \left[\left(\frac{4\pi^2}{T^2} + b^2 \right)\delta I\right]^2} \qquad (26)$$

An equation for G can be found by equating the torque on the balance due to the large masses (when they are located in one of the extreme positions) to the torque from the tungsten wire when the boom is displaced by the change in equilibrium angle θ_D. However, two corrections must be made to this basic equation. One is a correction for the attraction of the aluminum boom to the large mass. The second is a correction for the fact that each large mass acts not only on the small mass that it is placed next to, but also on the small mass on the other side of the balance. The resulting equation is given below:

$$G = \frac{K\theta_D R^2}{2M[(m - m_h)(1 - f_d) + m_b f_b]d} \qquad (27)$$

where m_h is the mass of the hole in the boom where the small mass sits, M is the mass of the large sphere, and f_d and f_b terms are corrections for the attraction of the large masses to the distant small sphere and boom, respectively. With the instructor's permission, you may take the value for the correction from Appendix B. For further discussion of the correction factors, including the derivation, please refer to the TEL-Atomic user's manual.

In order to calculate the uncertainty in G, I suggest turning all of your uncertainties into fractional uncertainties, so that you can just add them instead of needing to use (8) (assuming that you have the uncertainty for the bracketed expression in the denominator as a whole).

[1] S.P. Lally, The Phys. Teacher **37**, 34 (1999).
[2] B.E. Clotfelter, Am. J. Phys. **55** (3), 210 (1987).
[3] H. Cavendish, in *Scientific Memoirs: The Laws of Gravitation*, edited by A.S. MacKenzie (The American Book Company, 1900).
[4] TEL-Atomic Cavendish Balance User's Manual (available in lab).
[5] J.R. Taylor, *An Introduction to Error Analysis*, 2nd Edition (University Science Books, Sausalito, CA, 1997), pp. 184-188.
[6] J.R. Taylor, *An Introduction to Error Analysis*, 2nd Edition (University Science Books, Sausalito, CA, 1997), pp. 75.

Appendix A.1

If the differential equation for your oscillator has the form:

$$\ddot{\theta} + 2b\dot{\theta} + \omega_0^{\ 2} = 0 \quad (28)$$

(that is, if the damping is proportional to the velocity of the oscillator), then the oscillator equation may be written as:

$$\theta = \theta_e + A_0 e^{-bt} \cos(\omega t) \quad (29)$$

if we define the phase to be zero at time zero. Then, h (as defined in the **Data Analysis** section) is the factor by which the oscillation decays over one half period. So,

$$(\theta_{n+1} - \theta_e) = -h(\theta_n - \theta_e) \quad (30)$$

You can add equation (30) with itself over and over for different values of n:

$$
\begin{aligned}
&(\theta_2 - \theta_e) = -h(\theta_1 - \theta_e) \\
&- \\
&(\theta_3 - \theta_e) = -h(\theta_2 - \theta_e) \\
&+ \\
&(\theta_4 - \theta_e) = -h(\theta_3 - \theta_e) \\
&- \\
&(\theta_5 - \theta_e) = -h(\theta_4 - \theta_e) \\
&+ \\
&.... \\
&- \\
&(\theta_N - \theta_e) = -h(\theta_{N-1} - \theta_e)
\end{aligned}
$$

Hence, the θ_e's will cancel out and you will be left with:

$$h = -\frac{\theta_2 - \theta_3 + \theta_4 - \theta_5 + \ldots - \theta_N}{\theta_1 - \theta_2 + \theta_3 - \theta_4 + \ldots - \theta_{N-1}} \quad (31)$$

This can then be rewritten as equation (13). If one begins with the indices such that the left side has odd numbers first, it is possible to end up with equation (14) through the same derivation.

The uncertainty in h (15), though it looks daunting, is actually derived using equation (8) and treating each θ as an independent variable, but all with the same uncertainty δθ.

Appendix A.2

When you move the large masses to one side or the other, what they really do is change the equilibrium angle of the boom. The amount that they change it (θ_D) is the most important quantity to allow you to calculate G. Consider a turning point. When you flip the masses around, it seems to the oscillator as though its amplitude has grown by an amount $(\theta_e \pm \theta_D)$ depending on whether the amplitude is positive or negative. Moreover, since the equilibrium point has moved, the boom won't start to slow down until it reaches the angle $(\theta_e \mp \theta_D)$. Putting this more quantitatively:

$$\theta_{n+1} = [\theta_e + (-1)^n \theta_D] - h[\theta_n - (\theta_e + (-1)^n \theta_D)] \quad (32)$$

Solving this for θ_D, you obtain:

$$\theta_D = \frac{(-1)^n [\theta_{n+1} - \theta_e + h(\theta_n - \theta_e)]}{1+h} \quad (33)$$

Adding this up for different values of n:

$$\theta_D = \frac{\theta_2 - \theta_e + h(\theta_1 - \theta_e)}{1+h}$$

$+$

$$\theta_D = \frac{-\theta_3 + \theta_e + h(-\theta_2 + \theta_e)}{1+h}$$

$+$

$$\theta_D = \frac{\theta_4 - \theta_e + h(\theta_3 - \theta_e)}{1+h}$$

$+$

$$\theta_D = \frac{-\theta_5 + \theta_e + h(-\theta_4 + \theta_e)}{1+h}$$

$+$

......

$+$

$$\theta_D = \frac{\theta_N - \theta_e + h(\theta_{N-1} - \theta_e)}{1+h}$$

As you can see, the θ_e's will cancel out and you will be left with:

$$\theta_D = \frac{\theta_2 - \theta_3 + \theta_4 - \theta_5 + ... - \theta_N + h(\theta_1 - \theta_2 + \theta_3 - \theta_4 + .. - \theta_{N-1})}{(N-1)(1+h)} \quad (34)$$

(34) can be rewritten as (17) with a little algebra. Again, getting the uncertainty for θ_D is just a matter of judiciously applying (8).

Appendix A.3

Equation (21) is actually very simple. If you recall, for an underdamped oscillator ($b < \omega_o$) , the frequency becomes

$$\omega^2 = \omega_0{}^2 - b^2 \quad (32)$$

And, as you should remember,

$$\omega_0 = \sqrt{\frac{K}{I}} \quad (33)$$

We can then put (32) and (33) together to get

$$K = \left(\omega^2 + b^2\right) I \quad (34)$$

which becomes (21) when you remember the fact that $\omega = \frac{2\pi}{T}$.

Appendix B – Values for the Cavendish Balance

Quantity	Value	Uncertainty
M	1.039 kg	0.001 kg
ms	0.014545 kg	0.000001 kg
I	0.000143 kg*m^2	0.000001 kg*m^2
R	0.0461 m	0.00016 m
(m-mh)*(1-fd)+mb*fb	0.01511 kg	0.00004 kg
d	0.06665 m	0.00004 m