Chapter 2

Tools and Tricks of Fluid Dynamics

Now that we have the governing equations of geophysical fluid dynamics, we need to "make friends" with these equations. We do this by examining a number of tools and tricks which are commonly used in the field.

2.1 Exterior boundary conditions

Estimates of the typical magnitudes of the acceleration and frictional terms in the momentum equation yield $V/T = V^2/L$ for the former and $\nu V/L^2$ for the latter, where V, T, and L are typical velocity, time, and length scales. For the acceleration term we assume that T is the time required for a parcel to move a distance L at velocity V, so that $T = L/V$. For the viscous term we assume that $\nabla^2 \approx 1/L^2$. The ratio of the acceleration term to the friction term is called the *Reynolds number*

$$\mathrm{Re} = \frac{VL}{\nu}. \tag{2.1}$$

The kinematic viscosities of water and air under typical geophysical conditions are respectively of order 10^{-6} m^2 s^{-1} and 2×10^{-2} m^2 s^{-1}. Both vary significantly with temperature and the kinematic viscosity of air is inversely proportional to density.

Typically, the Reynolds number is very much greater than unity in atmospheric flows, suggesting that the friction term can be ignored in the momentum equation. This assumption is valid in the free atmosphere, but is most definitely invalid near boundaries. This is because friction at a boundary almost always gives rise to turbulent motion near the boundary. This turbulence transfers the effects of friction into the interior of the fluid to a degree far in excess of that produced by molecular transfer. This subject is discussed later in the chapter on boundary layers.

Mathematically, the inclusion or exclusion of the frictional and heat transfer terms changes the boundary conditions to which the governing equations are subject. If these molecular transfer terms are omitted, the boundary condition on the momentum equation is free-slip, i.e., the only restriction on the flow field is that the component of the fluid velocity normal to the (stationary) surface is zero. For the buoyancy (or heat) equation, the temperature of the fluid is unaffected by the temperature of the adjacent surface if molecular transfer is excluded. If molecular transfer terms are included, the fluid velocity adjacent to the surface must be zero and the temperature of the fluid adjacent to the surface must equal the temperature of the surface.

More generally, if the bounding surface is moving, then the fluid adjacent to the boundary must move at the same velocity as the boundary surface if molecular transfer is included. If not, the free slip condition demands only that the velocity components of the fluid and surface normal to the surface be the same.

2.2 Interior boundary conditions

Often we divide fluid flows into different domains with different fluid properties. The most obvious example of this is the division of the global geophysical fluid into the ocean and the atmosphere. The interface between these fluids is generally moving, so the free-slip or no-slip condition (depending on whether molecular transfer is included) must be applied to each fluid in a mutually consistent manner. In addition, the traction must be continuous across the boundary. If a discontinuity in the traction existed, this would be tantamount to a violation of Newton's third law, since the force per unit area of fluid region A on fluid region B would not be the negative of the force per unit area of fluid region B on fluid region A. In addition, the component of the heat or buoyancy flux normal to the interface must be continuous, or there will be a delta function heat or buoyancy source at the interface.

If viscosity and heat conduction are not included, then the only component of the traction is the pressure force. As with fluid adjacent to an external boundary, the buoyancy or temperature is unaffected by an adjacent interface.

2.3 Energy equation

We now develop an equation for the energy of a geophysical fluid, starting with the momentum equation

$$\frac{d\boldsymbol{v}}{dt} + \frac{1}{\rho}\boldsymbol{\nabla} p + g\boldsymbol{k} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = \nu \nabla^2 \boldsymbol{v} \tag{2.2}$$

Dotting both sides of this equation by $\boldsymbol{v}$ and solving for the time derivative term results in

$$\frac{dv^2/2}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla} \cdot (p\boldsymbol{v}) + p\frac{d\alpha}{dt} - \frac{d\Phi}{dt} + \nu \boldsymbol{v} \cdot \nabla^2 \boldsymbol{v} \tag{2.3}$$

which may be interpreted as the time rate of change of kinetic energy per unit mass of a parcel of fluid. The right side of the equation consists of the specific work done by the various forces acting on the parcel. Note that since the Coriolis force acts in a direction normal to the velocity, it does not contribute to this work.

The pressure contribution to the work deserves particular comment. We have invoked the product rule to write

$$-\frac{1}{\rho}\boldsymbol{v}\cdot\boldsymbol{\nabla}p = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot(p\boldsymbol{v}) + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot(p\boldsymbol{v}) - \frac{p}{\rho^2}\frac{d\rho}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot(p\boldsymbol{v}) + p\frac{d\alpha}{dt} \tag{2.4}$$

where we have used the mass continuity equation $d\rho/dt = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}$ and have defined the *specific volume* $\alpha = 1/\rho$. Using the differential form of the *specific internal energy* equation $de = Tds - pd\alpha$, the pressure term may finally be written

$$-\frac{1}{\rho}\boldsymbol{v}\cdot\boldsymbol{\nabla}p = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot(p\boldsymbol{v}) - \frac{de}{dt} + T\frac{ds}{dt} \tag{2.5}$$

where T is the *temperature* and ds is the differential of *specific entropy.* The viscous term is treated as follows,

$$\boldsymbol{v}\cdot\nabla^2\boldsymbol{v} = v_i\frac{\partial^2 v_i}{\partial x_j\partial x_j} = \frac{\partial}{\partial x_j}\left(v_i\frac{\partial v_i}{\partial x_j}\right) - \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} = \nabla^2(v^2/2) - |\boldsymbol{\nabla}\boldsymbol{v}|^2 \tag{2.6}$$

and the geopotential term comes from setting $g\boldsymbol{v}\cdot\boldsymbol{k} = gw = g(dz/dt) = d\Phi/dt$.

Putting all this together results in

$$\frac{d}{dt}\left(v^2/2 + e + \Phi\right) + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\left[p\boldsymbol{v} - \mu\boldsymbol{\nabla}\left(v^2/2\right)\right] = T\frac{ds}{dt} - \nu\left|\boldsymbol{\nabla}\boldsymbol{v}\right|^2 = T\frac{ds_X}{dt} \tag{2.7}$$

where we recall that the kinematic viscosity $\nu = \mu/\rho$ and that μ can be treated as constant. The first term on the left is the time derivative of the total (kinetic + internal + potential) energy per unit mass in a parcel. The second is the divergence of the energy flux due to pressure and viscous forces. On the right side we have the addition of energy via heating (recall that Tds is the heat added per unit mass) and the dissipation of mechanical energy by friction. There is some cancellation between these two terms, as dissipated mechanical energy is a heat source which is just balanced by the part of the entropy increase due to this heat source. We account for this cancellation by replacing the two terms on the right with a single term representing the addition of heat due only to external sources, $T(ds_X/dt)$.

We can use the mass continuity equation to convert the advective form of the energy equation into its flux form in the usual manner,

$$\frac{\partial}{\partial t}\left[\rho\left(v^2/2 + e + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\left(v^2/2 + h + \Phi\right)\boldsymbol{v} - \mu\boldsymbol{\nabla}\left(v^2/2\right)\right] = \rho T\frac{ds_X}{dt}, \tag{2.8}$$

where we have used the definition of specific enthalpy $h = e + p/\rho$. This equation is in a form useful for demonstrating the energy budget in a control volume. Integrating over this volume and using the divergence theorem results in

$$\frac{d}{dt}\int\rho\left(v^2/2 + e + \Phi\right)dV + \oint\left[\rho\left(v^2/2 + h + \Phi\right)\boldsymbol{v} - \mu\boldsymbol{\nabla}\left(v^2/2\right)\right]\cdot\boldsymbol{n}dV = \int\rho T\frac{ds_X}{dt}dV, \tag{2.9}$$

which states that the time rate of change of energy in the volume is minus the flux out of the volume $[\rho\,(v^2/2 + h + \Phi)\,\boldsymbol{v} - \mu\boldsymbol{\nabla}\,(v^2/2)] \cdot \boldsymbol{n}$ integrated over the volume surface plus a term equal to the volume integral of the externally imposed heating. The energy flux includes contributions from the stress tensor $(p\boldsymbol{v} - \mu\boldsymbol{\nabla}\,(v^2/2))$ which are actually equal to the work done per unit time on the fluid in the control volume by stress forces on the control volume boundary.

Note that nothing in this analysis is specific to the type of fluid as long as it is Newtonian and effectively incompressible with respect to the viscosity terms. It applies equally to the ocean and the atmosphere.

2.4 Bernoulli equation

The Bernoulli equation is related to but distinct from the energy equation. It is usually derived for the case of an incompressible fluid only. We extend the analysis to the case of an arbitrary Newtonian fluid subject to conservative forces in which viscosity and external heat sources are neglected. As in the derivation of equation (2.3), we dot the momentum equation (with viscosity excluded) with $\boldsymbol{v}$. However, the pressure term is treated differently; using the definition of specific enthalpy $dh = Tds + dp/\rho$, the result is

$$\frac{\partial v^2/2}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}(v^2/2) + \boldsymbol{v} \cdot \boldsymbol{\nabla} h - T\boldsymbol{v} \cdot \boldsymbol{\nabla} s + \frac{\partial \Phi}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\Phi = 0, \tag{2.10}$$

where we have expanded the material derivative of both $v^2/2$ and Φ. Combining terms and dropping the time derivatives results in

$$\boldsymbol{v} \cdot \boldsymbol{\nabla}\left(v^2/2 + h + \Phi\right) = T\boldsymbol{v} \cdot \boldsymbol{\nabla} s. \tag{2.11}$$

The entropy equation with no heat source or heat conduction is

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0, \tag{2.12}$$

which in the steady state case tells us that the right side of equation (2.11) is zero.

The quantity inside the parentheses on the left side of equation (2.11) is called the *Bernoulli constant*:

$$B = v^2/2 + h + \Phi. \tag{2.13}$$

The equation $\boldsymbol{v} \cdot \boldsymbol{\nabla} B = 0$ tells us that B is constant along parcel trajectories.

In the case of an ideal gas, $h = C_pT$, whereas for an incompressible fluid, $h = CT + p/\rho$ where C is the specific heat of the fluid material. Since the temperature T does not change with pressure for an incompressible material, it is constant along parcel trajectories in the steady state with no heat sources. Thus, it may be dropped from the definition of the Bernoulli constant for an incompressible fluid, resulting in $h = p/\rho$, which yields the conventional form of the Bernoulli equation. For an ideal gas, $C_pT = \theta\Pi$, which leads to the most useful form of the Bernoulli constant for the atmosphere:

$$B = v^2/2 + \theta\Pi + \Phi. \tag{2.14}$$

Given the steady state assumption, the potential temperature θ is constant along parcel trajectories and the Exner function Π depends only on pressure.

2.5 Three-dimensional flow

As with any vector field, the velocity can be decomposed into irrotational and solenoidal parts,

$$\boldsymbol{v} = \boldsymbol{v}_I + \boldsymbol{v}_S. \tag{2.15}$$

The solenoidal part can be written as the curl of a vector potential $\boldsymbol{\Psi}$:

$$\boldsymbol{v}_S = \boldsymbol{\nabla} \times \boldsymbol{\Psi}. \tag{2.16}$$

The three-dimensional divergence of the velocity field in the anelastic case is just

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v}_I = -\frac{d \ln \rho_0}{dz} w \tag{2.17}$$

where w is the vertical velocity and $\rho_0(z)$ is the ambient density profile. In the Boussinesq case, the divergence is zero.

The curl of the velocity field, which is called the *vorticity* $\boldsymbol{\zeta}$, is

$$\boldsymbol{\zeta} = \boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v}_S = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Psi}) = -\nabla^2 \boldsymbol{\Psi}. \tag{2.18}$$

In the last step of equation (2.18) we have applied the auxiliary condition on the vector potential that $\boldsymbol{\nabla} \cdot \boldsymbol{\Psi} = 0$. This is analogous to the Coulomb gauge condition on the vector potential of electromagnetism.

The vorticity plays an important role in geophysical fluid dynamics and it obeys a useful equation of its own, as we now demonstrate. To derive this equation we first rewrite the momentum equation using the vector identity $\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \boldsymbol{\nabla}(v^2/2) - \boldsymbol{v} \times \boldsymbol{\zeta}$. Ignoring viscosity, writing the pressure gradient in terms of the Exner function, and gravity in terms of the geopotential, we get

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}(v^2/2 + \Phi) + \boldsymbol{\zeta}_a \times \boldsymbol{v} + \theta \boldsymbol{\nabla} \Pi = 0. \tag{2.19}$$

The quantity $\boldsymbol{\zeta}_a = \boldsymbol{\zeta} + 2\boldsymbol{\Omega}$ is called the absolute vorticity, and is the vorticity of the flow calcuated in an inertial reference frame.

Taking the curl of this equation gives us the vorticity equation,

$$\frac{\partial \boldsymbol{\zeta}_a}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\boldsymbol{\zeta}_a - \boldsymbol{\zeta}_a \boldsymbol{v}) + \boldsymbol{\nabla}\theta \times \boldsymbol{\nabla}\Pi = 0, \tag{2.20}$$

where we have invoked the vector identity $\boldsymbol{\nabla} \times (\boldsymbol{\zeta}_a \times \boldsymbol{v}) = \boldsymbol{\nabla} \cdot (\boldsymbol{v}\boldsymbol{\zeta}_a - \boldsymbol{\zeta}_a\boldsymbol{v})$, noted that the curl of a gradient is zero, and used the fact that $\boldsymbol{\Omega}$ doesn't change with time, which means that $\partial\boldsymbol{\zeta}_a/\partial t = \partial\boldsymbol{\zeta}/\partial t$. From this we see that the flux of vorticity is the antisymmetric tensor $\mathbf{F}_\zeta = \boldsymbol{v}\boldsymbol{\zeta}_a - \boldsymbol{\zeta}_a\boldsymbol{v}$. The absolute vorticity changes with time as a result of two terms, minus the divergence of this flux, and minus the cross product of the gradients of potential temperature and the Exner function. The last term in equation (2.20) is called the *baroclinic generation* term.

If the vorticity is everywhere zero in a flow, then the velocity field is purely irrotational. Such a velocity field can be represented by a velocity potential χ:

$$\boldsymbol{v} = -\boldsymbol{\nabla}\chi. \tag{2.21}$$

If in addition we are using the Boussinesq equations with $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, the velocity potential satisfies Laplace's equation,

$$\nabla^2\chi = 0. \tag{2.22}$$

The flow is then determined uniquely by the boundary conditions on χ; nothing interesting happens in the interior of the fluid. If the flow satisfies the anelastic conditions, the governing equation for χ is only slightly more complex. Invoking equation (2.17), we have

$$\nabla^2\chi + \frac{d\ln\rho_0}{dz}\frac{\partial\chi}{\partial z} = 0. \tag{2.23}$$

This *potential flow* as it is called, is a staple of classical fluid dynamics, primarily because potential flow problems can be solved by well-known analytical techniques. However, it is not of much interest in geophysical fluid dynamics, because even if the vorticity is initially zero, it is unlikely to stay that way due to the baroclinic generation term in the vorticity equation. At that point obtaining a solution becomes much more complex.

2.6 Two-dimensional slab symmetry

Much can be learned from reduced dimensionality calculations in geophysical fluid dynamics. Here we consider one case of reduced dimensionality, slab symmetry, in which independence of one horizontal space dimension (typically the y dimension) is assumed.

Assuming $\partial/\partial y = 0$, the flow in the $x - z$ plane can be represented by a single function in the case of anelastic or Boussinesq flows. In the anelastic case the mass continuity equation reduces to

$$\frac{\partial\rho_0 v_x}{\partial x} + \frac{\partial\rho_0 v_z}{\partial z} = 0 \tag{2.24}$$

where $\rho_0 = \rho_0(z)$ is the mean density profile. This allows a *streamfunction* ψ to be defined such that

$$v_x = -\frac{1}{\rho_0}\frac{\partial\psi}{\partial z} \qquad v_z = \frac{1}{\rho_0}\frac{\partial\psi}{\partial x}; \tag{2.25}$$

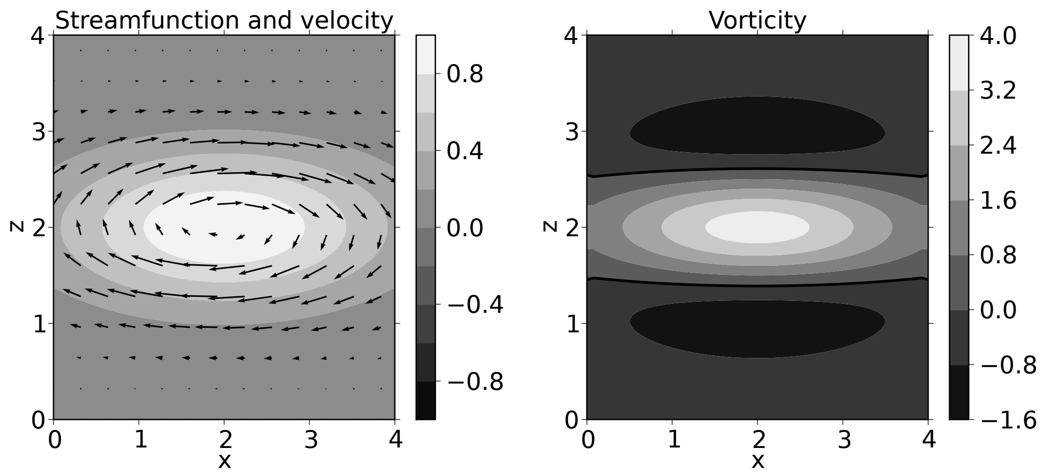

Figure 2.1: Example of Boussinesq streamfunction, velocity, and vorticity fields in two-dimensional slab symmetry. The thick line in the vorticity plot indicates the contour of zero vorticity.

direct substitution into equation (2.24) verifies that this definition is consistent with mass continuity. Defining a quantity called the *vorticity* ζ as

$$\zeta = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \tag{2.26}$$

we note that

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial \psi}{\partial z}\right) = -\zeta, \tag{2.27}$$

which is a Poisson-like elliptic equation for the streamfunction in terms of the vorticity. Given appropriate boundary conditions, the streamfunction can be obtained if the vorticity is known. The velocity components can then be obtained from equation (2.25). Thus, two dependent variables, v_x and v_z, are collapsed into one, ψ. Note that in the Boussinesq approximation the mean density profile is constant and therefore can be set to unity in the subsequent equations, resulting in even more simplification. In the context of section 2.5, $\psi = \Psi_y$, with $\Psi_x = \Psi_z = 0$.

The streamfunction has advantages for visualization of the flow. Figure 2.1 shows a plot of a sample Boussinesq streamfunction as well as the associated velocity and vorticity fields. Note that the velocity vectors are everywhere parallel to contours of constant streamfunction and that the magnitude of the velocity is inversely proportional to the spacing of the contours. (For the more general anelastic case, the spacing of the contours is inversely proportional to

$|\rho_0 \boldsymbol{v}|$.) The free slip boundary condition for a stationary external boundary is simply that the streamfunction is constant on the boundary.

Note that assuming $\partial/\partial y = 0$ does not imply that the y component of the velocity v is zero. In the case of a non-rotating flow, v decouples from the rest of the problem. However, even this is not true in the rotating case.

2.7 Pressure coordinates

Use of pressure as the vertical coordinate in place of geometric height is frequent in meteorology. This is generally done only in the hydrostatic limit, in which case the hydrostatic equation becomes

$$\frac{d\Phi}{dp} = -\frac{RT}{p}. \tag{2.28}$$

The geopotential $\Phi = gz$ becomes a dependent variable while the pressure is an independent variable. The density in pressure coordinates is the amount of mass ΔM in a volume in pressure coordinates, $\Delta V = \Delta x \Delta y \Delta p$, divided by this volume. In hydrostatic equilibrium, the weight of material in the volume is just ΔV. Dividing this by the acceleration of gravity g yields the mass, which means that the density in pressure coordinates is just

$$\rho_p = -\frac{1}{g}. \tag{2.29}$$

The minus sign arises from the fact that pressure decreases upward. The vertical velocity in pressure coordinates is

$$\omega = \frac{dp}{dt}. \tag{2.30}$$

As pressure decreases upward, this has the peculiar property of being negative for upward motion. The material derivative in pressure coordinates is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}. \tag{2.31}$$

Partial derivatives with respect to x, y, and t hold the pressure constant.

In pressure coordinates, constant pressure surfaces are the "horizontal" surfaces. However, these surfaces are generally not horizontal in geometric coordinates. This has two implications: (1) Pressure coordinates are not strictly Cartesian, even on a flat earth. This makes the exact pressure coordinate equations extremely complex. However, the deviation from the horizontal of constant pressure surfaces is small enough that this effect is generally ignored. (2) There is no pressure gradient force along constant pressure surfaces. However, a gravitational force does exist in the "horizontal" equations of motion. The force per unit mass is just minus the slope of the pressure surface times the acceleration of gravity, or $-g\boldsymbol{\nabla}_h z = -\boldsymbol{\nabla}_h \Phi$, where $\boldsymbol{\nabla}_h$ is the two-dimensional gradient along the pressure surface.

The net result is the following set of governing equations for mass, "horizontal" momentum, geopotential, and potential temperature:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{2.32}$$

$$\frac{du}{dt} + \frac{\partial \Phi}{\partial x} - fv = 0 \tag{2.33}$$

$$\frac{dv}{dt} + \frac{\partial \Phi}{\partial y} + fu = 0 \tag{2.34}$$

$$\frac{\partial \Phi}{\partial p} + \frac{\kappa \theta \Pi}{p} = 0 \tag{2.35}$$

$$\frac{d\theta}{dt} = 0. \tag{2.36}$$

The temperature has been eliminated in favor of the potential temperature and the Exner function in the hydrostatic equation. Recall that the latter is a function only of the pressure. All source and viscous terms have been neglected, but these can be added as needed.

A disadvantage of pressure coordinates is that the lower boundary is not flat and generally changes with time. The pressure vertical velocity is not necessarily zero there either.

2.8 References

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 745 pp. The material here is mostly covered in chapters 1, 2, and 4 of Vallis.

2.9 Questions and problems

1. Develop a total energy equation for the Boussinesq equations, ignoring viscosity and the buoyancy source term. Hint: The internal energy does not enter!

2. The quantity $C_pT + \Phi$ is called the dry static energy.

   (a) Assuming that the entropy is constant, obtain a relationship between dp and dT in a parcel of air.

   (b) Use the hydrostatic equation to eliminate dp in favor of dz.

   (c) Use in addition the ideal gas law to infer that $d(C_pT + gz) = d(C_pT + \Phi) = 0$ for a parcel, assuming that the parcel entropy is conserved.

   (d) Can you reconcile this result with the Bernoulli equation? Explain.

3. Derive the vorticity equation as in section 2.5, but for the Boussinesq equations.

4. Show that the flux of the vertical component of absolute vorticity is purely horizontal. Hint: $F_{\zeta ij} = v_i \zeta_{aj} - v_j \zeta_{ai}$ is the flux of the jth component of the vorticity in the i direction.