# Chapter 4

# Gravity Waves in Shear

## 4.1 Non-rotating shear flow

We now study the special case of gravity waves in a non-rotating, sheared environment. Rotation introduces additional complexities in the already complex mathematics of shear, and consideration of this case will be delayed. The case of sheared, non-rotating gravity waves is quite interesting in its own right. In order to simplify the analysis, we use the linearized Boussinesq equations with $f = 0$.

We assume an ambient horizontal wind in the $x$ direction $U$ which is a function of height alone, so that the total $x$ wind is $u = U(z) + u'$ where $u'$ is a small perturbation about this base state. As rotation is neglected, we need not consider the $y$ component of the wind, as it is completely decoupled from the dynamics of a non-rotating wave exhibiting slab symmetry in the $x - z$ plane. We also assume that the buoyancy takes the form of a base state plus perturbation

$$b = b_0(z) + b' = \int N^2 dz + b' \tag{4.1}$$

with a similar assumption for the kinematic pressure $\pi = \pi_0(z) + \pi'$. The base state kinematic pressure $\pi_0$ is in hydrostatic balance with $b_0$. Note that the Brunt-Väisälä frequency $N(z)$ is not necessarily constant in this case. Since the vertical velocity has no base part, there is no need to separate it into base plus perturbation.

The $z$ dependence of $U$ introduces complications into the linearization of the velocity:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = (U\boldsymbol{i} + \boldsymbol{v}') \cdot \boldsymbol{\nabla}(U\boldsymbol{i} + \boldsymbol{v}') \approx \left( U\frac{\partial u'}{\partial x} + \frac{dU}{dz}w, U\frac{\partial w}{\partial x} \right). \tag{4.2}$$

The advection term in the buoyancy equation is somewhat simpler:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} b = (U\boldsymbol{i} + \boldsymbol{v}') \cdot \boldsymbol{\nabla}(b_0(z) + b') \approx U\frac{\partial b'}{\partial x} + N^2 w. \tag{4.3}$$

(Verify these expansions to your satisfaction.) The resulting linearized governing equations are

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{dU}{dz}w + \frac{\partial \pi'}{\partial x} = 0 \tag{4.4}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + \frac{\partial \pi'}{\partial z} - b' = 0 \tag{4.5}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4.6}$$

$$\frac{\partial b'}{\partial t} + U\frac{\partial b'}{\partial x} + N^2 w = 0. \tag{4.7}$$

Assuming that all dependent variables exhibit $x$ and $t$ dependence in the form of a plane wave $\exp\left[i(kx - \omega t)\right]$, some algebra produces a single, second-order differential equation in $w$

$$\frac{\partial^2 w}{\partial z^2} + m^2 w = 0, \tag{4.8}$$

where

$$m^2(z) = \frac{1}{c-U}\frac{d^2U}{dz^2} + \frac{N^2}{(c-U)^2} - k^2 \tag{4.9}$$

is the square of the spatially varying vertical wavenumber $m$, and where $c = \omega/k$. This is called the *Taylor-Goldstein* equation. The fact that $m$ depends on $z$ makes this equation analytically soluble in exact form in only a few special cases.

## 4.2 WKB approximation

As $m$ depends on $z$, analytic solutions to equation (4.8) are only possible in special cases. The WKB method provides approximate solutions which are valid in regions where the fractional change in $m$ over a vertical wavelength $2\pi/m$ is small. In this method we try solutions of the form

$$w(z) = A(z)\exp\left(\pm i\int m dz\right) \tag{4.10}$$

where $A(z)$ is determined by substitution into equation (4.8), which yields after some manipulation

$$\frac{d^2A}{dz^2} \pm i\left(2m\frac{dA}{dz} + \frac{dm}{dz}A\right) = 0. \tag{4.11}$$

If $A$ and $m$ vary significantly on a vertical scale of $Z$, then the first term in this equation scales as $A/Z^2$, whereas the second and third scale as $mA/Z$. The ratio of the latter to the former is

$$\frac{mA/Z}{A/Z^2} = mZ. \tag{4.12}$$

$Z \gg 2\pi/m$ is the assumed condition for the validity of the WKB approximation, which (neglecting the numerical factor $2\pi$) is equivalent to $mZ \gg 1$, i.e., the second and third terms of equation (4.11) greatly exceed the first term. Dropping the first term results in a simple solution for equation (4.11):

$$A = \frac{C}{m^{1/2}} \tag{4.13}$$

where $C$ is a constant. Thus, we have

$$w(x, z, t) = \frac{C}{m^{1/2}} \exp\left[\pm i \int m dz + i(kx - \omega t)\right]. \tag{4.14}$$

By convention, if $m^2$ is real and positive, we take the positive root for $m$, which implies an upward trace velocity for the wave. If $m^2$ is real and negative, a positive imaginary part of $m$ means that $w$ decays in amplitude as $z$ increases, whereas if the imaginary part of $m$ is negative, $w$ decays downward. The WKB approximation fails where $m^2$ changes sign. This typically occurs where $|U - c|$ and $k^2$ are both large, i.e., for short horizontal wavelengths and large differences between the horizontal trace speed and the wind. This type of failure is related to the existence of reflections of upward-moving waves into downward-moving waves and vice versa. WKB does not handle wave reflection correctly. Surprisingly, WKB yields an adequate solution in many cases near the singularity which occurs in the Taylor-Goldstein equation when $|U - c| = 0$. We discuss this case later.

## 4.3 Non-interaction theorem (part 1)

Eliassen and Palm (1961) showed that the vertical flux of horizontal momentum by steady, vertically propagating gravity waves is constant with height. This so-called non-interaction theorem is an important predecessor to a large class of theorems applicable to a broad range of atmospheric disturbances. In the limited case of non-rotating gravity waves it is relatively easy to prove and provides significant insight into the dynamics of these waves.

The vertical flux of the $x$ component of the momentum due to gravity waves in the Boussinesq approximation is

$$F_p = \frac{\rho_R}{L} \int_0^L \mathrm{Re}(\tilde{u}')\mathrm{Re}(\tilde{w})dx \tag{4.15}$$

where $\rho_R$ is the usual constant reference density, Re indicates the real part of a complex function, and $L = 2\pi/k$ is the horizontal wavelength of the gravity wave in question. The dependent variables with the tilde indicate the inclusion of the full space and time dependence, i.e., $\tilde{u}' = u'(z)\exp[i(kx - \omega t)]$ and $\tilde{w} = w(z)\exp[i(kx - \omega t)]$. Using the fact that $\mathrm{Re}(\tilde{w}) = (\tilde{w} + \tilde{w}^*)/2$, etc., where the superscripted asterisk indicates the complex conjugate, and carrying out the integration, we find that

$$F_p = \frac{\rho_R}{4}\left(u'w^* + u'^*w\right). \tag{4.16}$$

To proceed further, we use the continuity equation (4.6), which tells us that

$$u = \frac{i}{k}\frac{\partial w}{\partial z}. \tag{4.17}$$

Substitution into equation (4.16) results in

$$F_p = \frac{i\rho_R}{4k}\left(\frac{\partial w}{\partial z}w^* - w\frac{\partial w^*}{\partial z}\right). \tag{4.18}$$

Finally, differentiation with respect to $z$ leads to

$$\begin{aligned}\frac{\partial F_p}{\partial z} &= \frac{i\rho_R}{4k}\left(\frac{\partial^2 w}{\partial z^2}w^* + \frac{\partial w}{\partial z}\frac{\partial w^*}{\partial z} - \frac{\partial w}{\partial z}\frac{\partial w^*}{\partial z} - w\frac{\partial^2 w^*}{\partial z^2}\right)\\ &= \frac{i\rho_R}{4k}\left(-m^2 ww^* + m^{*2}ww^*\right)\\ &= \frac{i\rho_R}{4k}\left(m^{*2} - m^2\right)|w|^2. \end{aligned} \tag{4.19}$$

If $m^2$ is real, then we conclude that $\partial F_p/\partial z = 0$.

This is called the non-interaction theorem because steady, non-damped waves pass vertically through a sheared environment without extracting or depositing momentum. However, they remove momentum from levels where they are formed (by whatever mechanism) and deposit it where they are absorbed. A similar but simpler analysis shows that steady, non-damped gravity waves transport no buoyancy.

## 4.4 Shear instability

If the wind shear is sufficiently strong, the ambient profile may break down into an instability. An easy way to get a physical feel for this process is to consider the energetics of a parcel displaced vertically from its equilibrium level. The energy required per unit mass to lift a parcel from from its equilibrium level a distance $\delta z$ is just the integrated work against the negative buoyancy force, or $N^2\delta z^2/2$. However, in an environment with environmental shear $dU/dz$, the parcel will acquire a velocity relative to the flow at its new level of $(dU/dz)\delta z$, assuming that its absolute velocity is unchanged in the lifting process. The kinetic energy relative to the flow at its new level, which perhaps might be available to effect the lifting, is $(dU/dz)^2\delta z^2/2$. The ratio of the energy required for lifting to the kinetic energy made available by the lifting is a dimensionless number called the Richardson number:

$$\mathrm{J} = \frac{N^2}{(dU/dz)^2}. \tag{4.20}$$

If $\mathrm{Ri} < 1$, the available kinetic energy exceeds the potential energy which needs to be overcome, and the lifting might happen spontaneously.

This is a very crude argument. However, Miles (1961) and Howard (1961) developed a necessary (but not sufficient) condition for instability to occur. (See also Kundu 1990.) Their argument, which unsurprisingly is called the *Miles-Howard* theorem, follows from the Taylor-Goldstein equation (4.8). It is first necessary to make a change of variables $w = (c-U)^{1/2}\chi$. Introducing subscript notation for vertical derivatives, e.g., $U_z = dU/dz$, $U_{zz} = d^2U/dz^2$, etc., we find that

$$w_{zz} = (c-U)^{1/2}\chi_{zz} - (c-U)^{-1/2}U_z\chi_z - \left(\frac{U_z^2}{4(c-U)^{3/2}} + \frac{U_{zz}}{2(c-U)^{1/2}}\right)\chi. \qquad (4.21)$$

Substituting this and equation (4.9) in equation (4.8), multiplying by $(c-U)^{1/2}$, and simplifying results in

$$[(c-U)\chi_z]_z + \left[\frac{N^2 - U_z^2/4}{c-U} + \frac{U_{zz}}{2} - k^2(c-U)\right]\chi = 0 \qquad (4.22)$$

where we have combined the two terms $(c-U)\chi_{zz} - U_z\chi_z$ to form the first term of this equation.

The next step is to multiply by the complex conjugate of $\chi$ and integrate vertically over the domain, assuming that $w$, and hence $\chi$, is zero at the top and bottom of the domain. This is necessary in order to perform an integration by parts on the first term:

$$\int \chi^* \left[(c-U)\chi_z\right]_z dz = -\int (c-U)|\chi_z|^2 dz. \qquad (4.23)$$

The result is

$$-\int (c-U)|\chi_z|^2 dz + \int \left[(c^*-U)\frac{N^2 - U_z^2/4}{|c-U|^2} + \frac{U_{zz}}{2} - k^2(c-U)\right]|\chi|^2 dz = 0, \qquad (4.24)$$

where we have also written $(c-U)^{-1} = (c^*-U)/|c-U|^2$.

Finally, we take the imaginary part of equation (4.24), splitting $c$ into real and imaginary parts, $c = c_r + ic_i$, and noting that $c^* = c_r - ic_i$:

$$-c_i \int \left[|\chi_z|^2 + \left(\frac{N^2 - U_z^2/4}{|c-U|^2} + k^2\right)|\chi|^2\right] dz = 0. \qquad (4.25)$$

If $N^2 \geq U_z^2/4$, then the integral is positive definite and we must have $c_i = 0$. Instability exists when the imaginary part of $\omega$ is positive, resulting in exponential growth in the amplitude of a disturbance. This is precluded if $c_i = 0$. Hence no instability can occur if $N^2 \geq U_z^2/4$. Expressing this in terms of the Richardson number, we find that

$$\mathrm{J} = \frac{N^2}{U_z^2} < \frac{1}{4} \qquad (4.26)$$

somewhere in the vertical domain is necessary (but not sufficient) for instability to occur. In this case the integrand in equation (4.25) is not positive definite and a positive value for the integral cannot be guaranteed.

In practice, the Richardson number must be less than 1/4 in a region where $c_r - U$ is small in order to amplify the effect of this region of negative integrand on the integration. A level where $c_r = U$ is called a *critical level*, and it is clear from this analysis that critical levels are likely to play an important role in unstable modes of a sheared mean flow.

## 4.5 References

**Eliassen,** A. and E. Palm, 1961: On the transfer of energy in stationary mountain waves. *Geofys. Publ.*, **22**, 1-23. Oddly, the non-interaction theorem is a minor side result of this paper, which is more concerned about energy transfer, a much more difficult subject.

**Howard,** L. N., 1961: Note on a paper of John W. Miles. *J. Fluid Mech.*, **10**, 509-512. Howard is responsible for the form of the Miles-Howard theorem presented here. His paper is a short note on the paper by Miles, in which he presents a more elegant derivation of Miles' result.

**Kundu,** P. K., 1990: *Fluid Mechanics*, Academic Press, San Diego, 638 pp. This text has discussions of a number of theorems related to the stability of parallel shear flow.

**Miles,** J. W., 1961: On the stability of heterogeneous shear flows. *J. Fluid Mech.*, **10**, 496-508.

## 4.6 Questions and problems

1. Derive equation (4.8) from equations (4.4)-(4.7).

2. Prove that steady, non-damped gravity waves transport no buoyancy vertically.

3. Show by direct substitution in equation (4.16) that the WKB solutions obey the non-interaction theorem.

4. Use the results of your solution for upward-propagating gravity waves from the last chapter to determine the upward flux of horizontal momentum in these waves. Where does this momentum come from?

5. Consider a parallel, stratified shear flow with $N$ constant and $U = Sz$ with the shear $S$ constant. Note that the Richardson number in this case is $\mathrm{J} = N^2/S^2$.

(a) Obtain solutions to the Taylor-Goldstein equation (4.8) for the limit in which $k^2$ is small enough to be ignored. Don't worry about boundary conditions. Hint: Try $w = (c - Sz)^X$ where $X$ is to be determined.

(b) Determine how the character of the solutions change as the Richardson number changes from J > 1/4 to J < 1/4.

(c) Compare your solutions to WKB solutions to this problem. When are they approximately the same?