Chapter 5

Shallow Water Equations

So far we have concentrated on the dynamics of small-scale disturbances in the atmosphere and ocean with relatively simple background flows. In these analyses we have paid a great deal of attention to the vertical structure of the flow. We now shift our attention to a type of flow with simple vertical structure. This simplification allows us to explore certain large-scale phenomena in which the curvature of the earth and complex horizontal variations in structure are manifested.

The prototype fluid system is the flow of a thin layer of water over terrain which potentially varies in elevation. Friction is ignored and the flow velocity is assumed to be uniform with elevation. Furthermore, the slope of terrain is assumed to be much less than unity, as is the slope of the fluid surface. These assumptions allow the vertical pressure profile of the fluid to be determined by the hydrostatic equation. The resulting *shallow water equations* are highly idealized, but they share some essential characteristics with more complex geophysical flows.

5.1 Derivation of shallow water equations

Figure 5.1 provides a definition sketch for shallow water flow. Using a control volume consisting of a vertical column of fluid, we derive the mass continuity equation

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot (h\boldsymbol{v}) = 0 \tag{5.1}$$

where h is the thickness of the fluid layer, ρh is the mass per unit area of fluid, and ρ is the constant density of the fluid. The vertically integrated horizontal mass flux of fluid is $\rho h\boldsymbol{v}$ where $\boldsymbol{v} = (u, v)$ is the horizontal velocity and $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial y)$ is the horizontal gradient operator. Thus, the mass per unit time leaving a column of fluid of unit area $\boldsymbol{\nabla} \cdot (\rho h\boldsymbol{v})$ is equal to minus the time tendency of the mass in this column $-\partial(\rho h)/\partial t$. Putting these facts together yields equation (5.1). The density cancels because it is constant. Equation (5.1) can be rewritten in advective form

$$\frac{dh}{dt} + h\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{5.2}$$

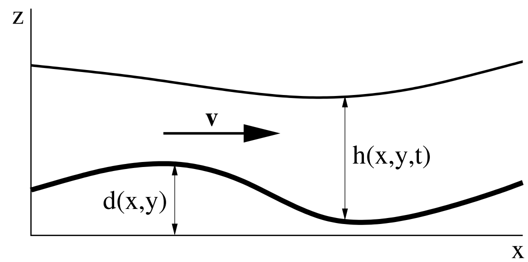

Figure 5.1: Definition sketch for derivation of the shallow water equations. The layer of water has thickness h which is a function of position and time. The flow velocity $\boldsymbol{v}$ varies with position and time, but is assumed to be primarily horizontal and to not vary significantly with height. The terrain over which the water flows has elevation d which is a function of position. The sketch is two-dimensional, but we derive the full equations valid in three dimensions.

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \tag{5.3}$$

as before except that the advective term is two-dimensional.

Rather than do a full control volume analysis for the momentum equation, we take a shortcut and apply Newton's second law to obtain the advective form directly,

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p - f\boldsymbol{k} \times \boldsymbol{v}, \tag{5.4}$$

where we have ignored the vertical component of the Coriolis force. Assuming hydrostatic balance, the pressure as a function of height is

$$p = g\rho(d + h - z) \tag{5.5}$$

where $d(x, y)$ is the terrain elevation and where we assume zero pressure at the upper fluid surface. Thus, $\boldsymbol{\nabla} p = g\rho\boldsymbol{\nabla}(d + h)$ and the momentum equation becomes

$$\frac{d\boldsymbol{v}}{dt} + g\boldsymbol{\nabla}(d + h) + f\boldsymbol{k} \times \boldsymbol{v} = 0. \tag{5.6}$$

Equations (5.2) and (5.6) constitute the full set of shallow water equations – no additional thermodynamic equation is needed.

5.2 Fundamental modes

As previously, we linearize the shallow water equations about a state of rest. The layer thickness h is represented as a constant mean thickness h_0 and a fractional thickness perturbation η:

$$h = h_0(1 + \eta). \tag{5.7}$$

Defining the square of a characteristic speed as

$$c^2 = gh_0, \tag{5.8}$$

the linearized shallow water equations in component form are

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.9}$$

$$\frac{\partial u}{\partial t} + c^2 \frac{\partial \eta}{\partial x} - fv = 0 \tag{5.10}$$

$$\frac{\partial v}{\partial t} + c^2 \frac{\partial \eta}{\partial y} + fu = 0. \tag{5.11}$$

Assuming a plane wave space-time dependence of the form $\exp\left[i\left(kx - \omega t\right)\right]$ results in

$$\begin{pmatrix} -i\omega & ik & 0 \\ ikc^2 & -i\omega & -f \\ 0 & f & -i\omega \end{pmatrix} \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = 0. \tag{5.12}$$

As before, we have a system of linear, homogeneous equations which has a non-trivial solution only when the determinant of the coefficients equals zero, yielding the secular equation

$$\left(\omega^2 - k^2c^2 - f^2\right)\omega = 0. \tag{5.13}$$

The solutions are

$$\omega = \pm\left(k^2c^2 + f^2\right)^{1/2}, \qquad \omega = 0. \tag{5.14}$$

The first two solutions represent the shallow water equivalent of inertia-gravity waves, also called Poincaré waves. As in the real atmosphere, the wave frequency of Poincaré waves is f. Unlike the atmosphere, there is no upper bound on the frequency. This is not suprising, since the upper bound is related to vertical parcel accelerations, which are neglected in the hydrostatic approximation. The third solution, $\omega = 0$, is associated with steady, geostrophically balanced motion in the y direction.

As with inertia-gravity waves in the atmosphere, the velocity vector rotates clockwise in the horizontal plane during wave passage (for $f > 0$). However, unlike the atmosphere, the shallow water system is two-dimensional, so propagation of Poincaré waves is purely horizontal. In the non-rotating case ($f = 0$), the wave phase speed is $\omega/k = c$, so non-rotating waves are not dispersive.

Linearizations of more complex, three-dimensional flows reduce to the linearized shallow water equations when certain assumptions are made about vertical structure. The modes predicted by these reduced equations are mathematically identical to the shallow water modes, but the physical interpretation is different. The parameter linking the two is the characteristic speed c. In the more complex flows the relationship $h_0 = c^2/g$ (see equation (5.8)) does not pertain. Nevertheless, c^2/g is often referred to as the *equivalent depth* of the reduced three-dimensional modes, even though this length scale has nothing to do physically with the fluid depth.

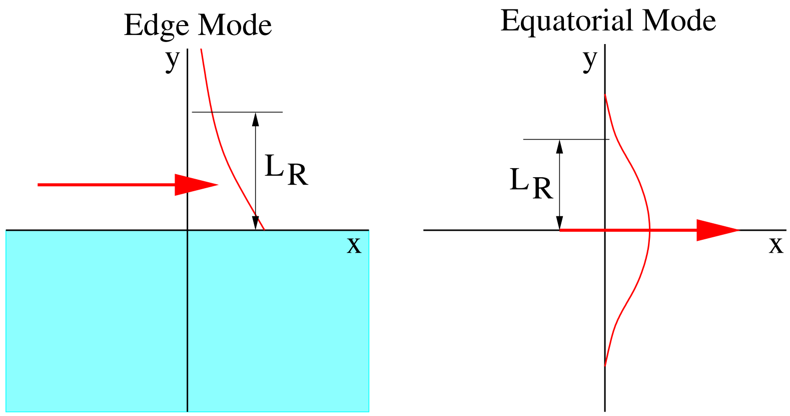

Figure 5.2: Two types of Kelvin wave moving in the $+x$ direction, an edge wave (left) and an equatorial Kelvin wave (right). The transverse dimension of the wave scales with the Rossby radius L_R.

5.3 Kelvin wave

A wave mode related to inertia-gravity waves is the *Kelvin* wave. This type of wave occurs next to a lateral boundary such as a shore line in the ocean or a steep mountain range in the atmosphere. An alternative form, which we will discuss later, is the equatorial Kelvin wave. This wave propagates toward the east on the equator, with rapid decay in the wave amplitude in both directions away from the equator. Kelvin waves play important roles in the ocean and in the atmosphere near lateral boundaries and the equator. (See figure 5.2.)

Let us imagine a free-slip wall at $y = 0$, with fluid occupying the region $y \geq 0$. Kelvin waves decay exponentially in amplitude as a function of distance from the wall, suggesting a trial structure $\exp\left[i\left(kx - \omega t\right) - \sigma y\right]$, where σ is the decay rate away from the wall. At the wall, the free-slip boundary condition requires that $v = 0$ there. Let us assume more generally that $v = 0$ everywhere and substitute our assumed wave structure into equations (5.9)-(5.11):

$$-i\omega\eta + iku = 0 \tag{5.15}$$

$$-i\omega u + ikc^2\eta = 0 \tag{5.16}$$

$$-\sigma c^2\eta + fu = 0. \tag{5.17}$$

Equations (5.15) and (5.16) can be solved by elimination, resulting in the dispersion relation

$$\omega = \pm kc \tag{5.18}$$

and the polarization relation

$$u = \pm c\eta. \tag{5.19}$$

Substitution of the polarization relation into equation (5.17) results in a value for σ:

$$\sigma = \pm\frac{f}{c}. \tag{5.20}$$

In the northern hemisphere where $f > 0$, only the positive solution to this equation makes physical sense; the negative solution causes the wave amplitude to blow up exponentially with distance from the wall. The reverse is true in the southern hemisphere. The positive solution corresponds to a positive phase speed in the x direction. More generally, Kelvin waves in the northern hemisphere move such that the wall is to the right of the wave motion. The phase speed is equal to that of gravity waves in a non-rotating environment. This behavior is possible in a rotating environment because of the constraint imposed on motion normal to the direction of wave propagation by the wall. Kelvin waves are sometimes referred to as *edge* waves. The lateral dimension of a Kelvin wave scales with the *Rossby radius* $L_R = 1/\sigma = c/f$.

5.4 Equations on a sphere

In this section we derive the shallow water equations on a sphere as an approximation to flow on the earth's surface. Contours of constant geopotential (and hence constant elevation) aren't really spherical, but are somewhat ellipsoidal in shape. However, the difference is small enough to be ignored for most purposes.

We first develop the continuity and momentum equations. We then introduce common approximations in which a small patch of a spherical surface can be treated in a simple way. Figure 5.3 shows the spherical coordinate system we use. Note that the longitude λ is the azimuthal coordinate and the latitude ϕ is the elevation coordinate. This system differs from the usual spherical coordinates in which the elevation angle is the co-latitude or $\pi/2 - \phi$.

5.4.1 Mass continuity

Figure 5.4 shows a "rectangular" region of fluid in latitude ϕ and longitude λ on a sphere. The depth of the fluid h itself is a function of λ and ϕ. If the earth's radius is a, then the linear dimensions of this region are given by

$$\Delta x_\lambda = a\cos\phi\Delta\lambda \qquad \Delta x_\phi = a\Delta\phi. \tag{5.21}$$

The volume of the region of fluid is therefore $a^2 h\cos\phi\Delta\lambda\Delta\phi$.

If u is the fluid velocity component in the direction of increasing longitude λ and v is the component in the direction of increasing latitude ϕ, then the rate at which fluid volume enters the region from the sides is

$$\begin{aligned} a^2\cos\phi\Delta\lambda\Delta\phi\frac{\partial h}{\partial t} &= [u(\lambda)h(\lambda) - u(\lambda')h(\lambda')]a\Delta\phi \\ &+ [v(\phi)h(\phi)\cos(\phi) - v(\phi')h(\phi')\cos(\phi')]a\Delta\lambda \end{aligned} \tag{5.22}$$

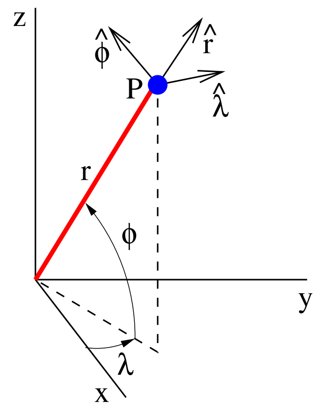

Figure 5.3: Spherical coordinate system used in this chapter. The longitude is λ and the latitude is ϕ. A local Cartesian coordinate system is defined at point P with eastward, northward, and upward unit vectors defined as $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{r}}$.

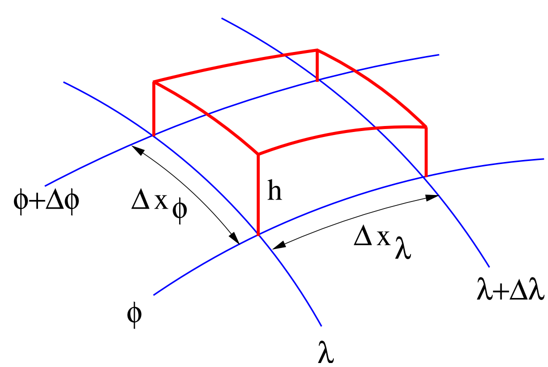

Figure 5.4: Sketch used to obtain shallow water mass continuity equation on a sphere.

where $\lambda' = \lambda + \Delta\lambda$ and $\phi' = \phi + \Delta\phi$. Dividing by $a^2 \cos\phi \Delta\lambda\Delta\phi$, bringing all terms to the left side, and taking the limit of small $\Delta\lambda$ and $\Delta\phi$, we get the mass continuity equation on a sphere:

$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\phi}\left[\frac{\partial}{\partial\lambda}(hu) + \frac{\partial}{\partial\phi}(hv\cos\phi)\right] = 0. \tag{5.23}$$

5.4.2 Momentum equations

The difficult part of deriving the momentum equations is calculating the total time derivative of the horizontal velocity. The difficulty is that the orientation of the unit vectors defining the local east-north-up coordinate system changes for different locations on the sphere. Thus, the variability in these unit vectors needs to be taken into account when taking spatial derivatives.

If $\boldsymbol{v}(\lambda, \phi, t) = u\hat{\boldsymbol{\lambda}} + v\hat{\boldsymbol{\phi}}$ is the horizontal flow velocity, then

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{d\lambda}{dt}\frac{\partial \boldsymbol{v}}{\partial\lambda} + \frac{d\phi}{dt}\frac{\partial \boldsymbol{v}}{\partial\phi}. \tag{5.24}$$

It is possible to show that

- $\partial\hat{\boldsymbol{\lambda}}/\partial\lambda = \hat{\boldsymbol{\phi}}\sin\phi - \hat{\boldsymbol{r}}\cos\phi$;
- $\partial\hat{\boldsymbol{\phi}}/\partial\lambda = -\hat{\boldsymbol{\lambda}}\sin\phi$;
- $\partial\hat{\boldsymbol{\lambda}}/\phi = 0$;
- $\partial\hat{\boldsymbol{\phi}}/\partial\phi = -\hat{\boldsymbol{r}}$.

Furthermore

$$\frac{d\lambda}{dt} = \frac{u}{a\cos\phi} \qquad \frac{d\phi}{dt} = \frac{v}{a} \tag{5.25}$$

where a is the radius of the earth as before. Finally

$$\boldsymbol{\nabla} h = \frac{\partial h}{\partial x_\lambda}\hat{\boldsymbol{\lambda}} + \frac{\partial h}{\partial x_\phi}\hat{\boldsymbol{\phi}} = \frac{1}{a\cos\phi}\frac{\partial h}{\partial\lambda}\hat{\boldsymbol{\lambda}} + \frac{1}{a}\frac{\partial h}{\partial\phi}\hat{\boldsymbol{\phi}}. \tag{5.26}$$

Putting all of this together and splitting into longitudinal and latitudinal components, we get

$$\frac{\partial u}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\phi} - \frac{uv\tan\phi}{a} + \frac{g}{a\cos\phi}\frac{\partial h}{\partial\lambda} - fv = 0 \tag{5.27}$$

$$\frac{\partial v}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial v}{\partial\lambda} + \frac{v}{a}\frac{\partial v}{\partial\phi} + \frac{u^2\tan\phi}{a} + \frac{g}{a}\frac{\partial h}{\partial\phi} + fu = 0 \tag{5.28}$$

where the Coriolis parameter $f = 2\Omega\sin\phi$ now varies with latitude. We have dropped terms pointing in the $\hat{\mathbf{r}}$ direction. These terms actually enter the vertical momentum equation.

However, they are small compared to the other terms in this equation for ordinary velocities, and therefore they don't significantly perturb hydrostatic balance. The geostrophic wind is obtained by dropping all terms related to $d\boldsymbol{v}_h/dt$:

$$v_g = \frac{g}{fa\cos\phi}\frac{\partial h}{\partial\lambda} \qquad u_g = -\frac{g}{fa}\frac{\partial h}{\partial\phi}. \tag{5.29}$$

This treatment breaks down near the north and south poles where $\cos\phi \to 0$, and an alternate coordinate system needs to be used. Global numerical models deal with the problem of the poles in a variety of ways which will not be discussed here.

5.4.3 Beta-plane approximation

Equations (5.23), (5.27), and (5.28) are difficult to solve due to the sines and cosines of latitude which enter. A useful approximation is to treat a region of the earth's surface as being locally flat in all respects except in the latitudinal variation in the Coriolis parameter, for which a Taylor series expansion is made about the central latitude ϕ_0 of the region of interest:

$$f \approx 2\Omega\sin\phi_0 + (2\Omega\cos\phi_0/a)y \equiv f_0 + \beta y. \tag{5.30}$$

where Ω is the angular rotation rate of the earth and a is the earth's radius. A local Cartesian coordinate system centered on the region is employed with x increasing to the east and y increasing to the north. This approximation can be justified if the diameter of the region of interest is much less than the diameter of the earth. This is called the *beta-plane* approximation.

A special case of this approximation obtains when $\phi_0 = 0$. This is called the *equatorial beta-plane* approximation. In this case

$$f \approx 2\Omega\phi = (2\Omega/a)y = \beta y. \tag{5.31}$$

The equatorial beta-plane approximation is valid for a larger domain in the east-west direction, i. e., for the entire equatorial strip around the globe as long as the north-south width of the strip is much less than the earth's diameter.

The *f-plane approximation* is like the beta-plane approximation except that the region is assumed to be small enough that the latitudinal variation in the Coriolis parameter can be ignored as well; f is replaced by a constant representative value.

5.5 References

Many textbooks treat the shallow water equations, among them:

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 745 pp. The material here is covered in chapter 3 of Vallis.

Pedlosky, J., 1979: *Geophysical fluid dynamics.* Springer-Verlag, 624 pp. Pedlosky makes extensive use of the shallow water equations.

5.6 Questions and problems

1. Consider non-rotating, steady, shallow water flow in the x direction.

   (a) Show that the mass continuity and momentum equations may be integrated in x to yield $uh = M$ and $u^2/2 + g(h + d) = B$ where M and B are constants.

   (b) Assume that $h = h_0(1 + \eta)$, $d = \delta h_0$, and $u = U(1 + \mu)$. Assume that η, δ, and μ all have magnitudes much less than unity and linearize the above equations in these quantities. Recall that the original equations must be satisfied when $\eta = \delta = \mu = 0$.

   (c) Defining $gh_0 = c^2$, solve for μ and η in terms of δ, which is an arbitrary function of x except that it is small in amplitude.

   (d) How does the thickness of the flow vary with δ for ridges ($\delta > 0$) and valleys ($\delta < 0$) for the two cases $U < c$ and $U > c$? These two cases are called respectively *subcritical* and *supercritical* flow.

2. Boussinesq equations and shallow water equations:

   (a) Linearize the hydrostatic, inviscid Boussinesq equations about a state of rest with constant ambient Brunt-Väisälä frequency.

   (b) Assume that the perturbation dependent variables have the form $w', b' \propto \sin(mz)$ and $u', v', \pi' \propto \cos(mz)$ where m is a constant. Substitute these into the linearized Boussinesq equations and eliminate w' and b'. Eliminate π' in favor of a variable which makes the resulting equations look like the linearized shallow water equations.

   (c) Determine the characteristic speed of the system along with the equivalent depth.

3. Cylindrical coordinates with $r = (x^2 + y^2)^{1/2}$ and $\phi = \tan^{-1}(y/x)$:

   (a) Derive the shallow water equations in cylindrical coordinates. Hint: $\partial\hat{\boldsymbol{r}}/\partial\phi = \hat{\boldsymbol{\phi}}$ etc., where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are the unit vectors in the radial and azimuthal directions with $\boldsymbol{v} = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\phi}}$; $\boldsymbol{\nabla} = (\partial/\partial r, r^{-1}\partial/\partial\phi)$.

   (b) Determine the balance in the radial momentum equation in the steady, axisymmetric case with zero radial velocity and explain the physical meaning of each term. This balance is called *gradient balance*, and it is a generalization of geostrophic balance for rapidly rotating flows.

(c) The specific angular momentum of a shallow water parcel about the origin in this coordinate system is $L = rv + r^2 f/2$. (Show this.) Show that conservation by parcels of this quantity, $dL/dt = 0$, yields the azimuthally symmetric form of the azimuthal momentum equation.